# A criterion for unimodality

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**Abstract**

We show that if $P(x)$ is a polynomial with nondecreasing, nonnegative coefficients, then the coefficient sequence of $P(x+1)$ is unimodal. Applications are given.

## 1. Introduction

A finite sequence of real numbers $\{d_0, d_1, \cdots, d_m\}$ is said to be *unimodal* if there exists an index $0 \leq m^* \leq m$, called the *mode* of the sequence, such that $d_j$ increases up to $j = m^*$ and decreases from then on, that is, $d_0 \leq d_1 \leq \cdots \leq d_{m^*}$ and $d_{m^*} \geq d_{m^*+1} \geq \cdots \geq d_m$. A polynomial is said to be unimodal if its sequence of coefficients is unimodal.

Unimodal polynomials arise often in combinatorics, geometry and algebra. The reader is referred to [2] and [3] for surveys of the diverse techniques employed to prove that specific families of polynomials are unimodal.

A sequence of positive real numbers $\{d_0, d_1, \cdots, d_m\}$ is said to be *logarithmically concave* (or *log-concave* for short) if $d_{j+1}d_{j-1} \leq d_j^2$ for $1 \leq j \leq m-1$. It is easy to see that if a sequence is log-concave then it is unimodal [4]. A sufficient condition for log-concavity of a polynomial is given by the location of its zeros: if all the zeros of a polynomial are real and negative, then it is log-concave and therefore unimodal [4]. A second criterion for the log-concavity of a polynomial was determined by Brenti [2]. A sequence of real numbers is said to have *no internal zeros* if whenever $d_i, d_k \neq 0$ and $i < j < k$ then $d_j \neq 0$. Brenti's criterion states that if $P(x)$ is a log-concave polynomial with nonnegative coefficients and with no internal zeros, then $P(x+1)$ is log-concave.

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## 2. The main result

**Theorem 2.1.** *If $P(x)$ is a polynomial with positive nondecreasing coefficients, then $P(x+1)$ is unimodal.*

**Proof.** Observe first that $P_{m,r}(x) := (1+x)^{m+1} - (1+x)^r$ with $0 \le r \le m$ is unimodal with mode at $1 + \lfloor \frac{m}{2} \rfloor$. This follows by induction on $m \ge r$ using $P_{m+1,r}(x) = P_{m,r}(x) + x(1+x)^{m+1}$. For $m$ even, $P_{m+1,r}$ is the sum of two unimodal polynomials with the same mode. For $m = 2t+1$, the modes are shifted by 1, so it suffices to check

$$a_{t+1} + \binom{m+1}{t} \quad \le \quad a_{t+2} + \binom{m+1}{t+1}, \tag{2.1}$$

where $a_{t+1}$ is the coefficient of $x^t$ in $P_{m,r}(x)$. The case $t \ge r$ yields equality in (2.1). If $t \le r-2$ then (2.1) is equivalent to $r \le m+2$. The final case $t = r-1$ amounts to $0 = \binom{m+1}{r-1} - \binom{m+1}{r+1} \le 1$,

Now $P(x+1) = \frac{1}{x}(b_0 P_{m,0}(x) + (b_1 - b_0)P_{m,1}(x) + \cdots + (b_m - b_{m-1})P_{m,m}(x))$, so $P(x+1)$ is a sum of unimodal polynomials with the same mode, and hence unimodal.

We now restate Theorem 2.1 and offer an alternative proof.

**Theorem 2.2.** *Let $b_k > 0$ be a nondecreasing sequence. Then the sequence*

$$c_j \quad := \quad \sum_{k=j}^{m} b_k \binom{k}{j}, \quad 0 \le j \le m \tag{2.2}$$

*is unimodal with mode $m^* := \lfloor \frac{m-1}{2} \rfloor$.*

**Proof.** For $0 \le j \le m-1$ we have

$$(j+1)(c_{j+1} - c_j) \quad = \quad \sum_{k=j}^{m} b_k \binom{k}{j} \times (k - 2j - 1). \tag{2.3}$$

Suppose first that $j \ge m^*$, and let $m$ be odd so that $m = 2m^* + 1$; the case $m$ even is treated in a similar fashion. Every term in (2.3) is negative because, if $j > m^*$, then $k - 2j - 1 \le m - 2j - 1 = 2(m^* - j) < 0$, and for $j = m^*$,

$$(m^* + 1)(c_{m^*+1} - c_{m^*}) \quad = \quad \sum_{k=m^*}^{m-1} b_k \binom{k}{m^*} \times (k - m) < 0. \tag{2.4}$$

Thus $c_{j+1} < c_j$.

Now suppose $0 \le j < m^*$ and define

$$T_1 \quad := \quad \sum_{k=j}^{2j} b_k \binom{k}{j} (2j + 1 - k) \tag{2.5}$$

and

$$T_2 \;:=\; \sum_{k=2j+2}^{m} b_k \binom{k}{j} (k-2j-1) \tag{2.6}$$

so that $(j+1)(c_{j+1}-c_j) = T_2 - T_1$. Then

$$T_1 < b_{2j+2} \sum_{k=j}^{2j} \binom{k}{j} (2j+1-k) = b_{2j+2} \binom{2j+2}{j} < T_2.$$

Thus $c_{j+1} > c_j$.

## 3. Examples

**Example 1**. The case $P(x) = x^n$ in Theorem 2.1 gives the unimodality of the binomial coefficients.

**Example 2**. For $0 \le k \le m-1$, define

$$b_k(m) \;:=\; 2^{-2m+k} \binom{2m-2k}{m-k} \binom{m+k}{m} (a+1)^k$$

for $0 \le k \le m-1$. Then

$$\frac{b_{k+1}(m)}{b_k(m)} \;=\; \frac{(m-k)(m+k+1)}{(2m-2k-1)(k+1)} > 1$$

so the polynomial

$$P_m(a) \;:=\; \sum_{k=0}^{m} b_k(m)(a+1)^k$$

is unimodal. We encountered $P_m$ in the integral formula [1]

$$\int_0^\infty \frac{dx}{(x^4+2ax^2+1)^{m+1}} \;=\; \frac{\pi \; P_m(a)}{2^{m+3/2}(a+1)^{m+1/2}}. \tag{3.1}$$

This does not appear in the standard tables.

**Example 3**. For $0 \le k \le m-1$, define

$$b_k(m) \;:=\; \frac{(-m-\beta)_m}{m!} \, \frac{(-m)_k(m+1+\alpha+\beta)_k}{(\beta+1)_k \, k! \, 2^k}.$$

Then, with $\alpha = m+\epsilon_1$ and $\beta = -(m+\epsilon_2)$, we have

$$\frac{b_{k+1}(m)}{b_k(m)} \;=\; \frac{m-k}{m-k+\epsilon_2-1} \times \frac{k-1+m+\epsilon_1-\epsilon_2}{2(k+1)} > 1$$

provided $0 < \epsilon_1 \leq \epsilon_2 < 1$. Therefore the polynomial

$$P_m^{(\alpha,\beta)}(a) \ := \ \sum_{k=0}^{m} b_k(m)(a+1)^k$$

is unimodal. This is a special case of the Jacobi family, where the parameters $\alpha$ and $\beta$ are not standard since they depend on $m$. These polynomials do not have real zeros, so their unimodality is not immediate. The case of Example 2 corresponds to $\epsilon_1 = \epsilon_2 = \frac{1}{2}$.

**Example 4**. Let $n$, $m \in \mathbb{N}$ be fixed. Then the sequences

$$\alpha_j := \sum_{k=j}^{m} n^k \binom{k}{j}, \quad \beta_j := \sum_{k=j}^{m} k^n \binom{k}{j}, \quad \text{and } \gamma_j := \sum_{k=j}^{m} k^k \binom{k}{j}$$

are unimodal for $0 \leq j \leq m$.

**Example 5**. Let $2 < a_1 < \cdots < a_p$ and $n_1, \cdots, n_p$ be two sequences of $p$ positive integers. For $0 \leq j \leq m$, define

$$c_j \ := \ \sum_{k=j}^{m} \binom{a_1 m}{k}^{n_1} \binom{a_2 m}{k}^{n_2} \cdots \binom{a_p m}{k}^{n_p} \binom{k}{j}. \tag{3.2}$$

Then $c_j$ is unimodal.

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