DERIVATIVE POLYNOMIALS, EULER POLYNOMIALS, AND ASSOCIATED INTEGER SEQUENCES

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Abstract. Let P_n and Q_n be the polynomials obtained by repeated differentiation of the tangent and secant functions respectively. From the exponential generating functions of these polynomials we develop relations among their values, which are then applied to various numerical sequences which occur as values of the P_n and Q_n. For example, $P_n(0)$ and $Q_n(0)$ are respectively the nth tangent and secant numbers, while $P_n(0)+Q_n(0)$ is the nth André number. The André numbers, along with the numbers $Q_n(1)$ and $P_n(1)-Q_n(1)$, are the Springer numbers of root systems of types A_n, B_n, and D_n respectively, or alternatively (following V. I. Arnol'd) count the number of "snakes" of these types. We prove this for the latter two cases using combinatorial arguments. We relate the values of P_n and Q_n at $\sqrt{3}$ to certain "generalized Euler and class numbers" of D. Shanks, which have a combinatorial interpretation in terms of 3-signed permutations as defined by R. Ehrenborg and M. A. Readdy. Finally, we express the values of Euler polynomials at any rational argument in terms of P_n and Q_n, and from this deduce formulas for Springer and Shanks numbers in terms of Euler polynomials.

1. Introduction. Consider the sequences P_n and Q_n of "derivative polynomials" defined by

$$\frac{d^n}{dx^n}\tan x = P_n(\tan x) \quad\text{and}\quad \frac{d^n}{dx^n}\sec x = Q_n(\tan x)\sec x$$

for integer $n \geq 0$. As shown in [12], their exponential generating functions

$$P(u,t)=\sum_{n=0}^{\infty} P_n(u)\frac{t^n}{n!} \quad\text{and}\quad Q(u,t)=\sum_{n=0}^{\infty} Q_n(u)\frac{t^n}{n!}$$

are given by the explicit formulas

$$P(u,t)=\frac{\sin t+u\cos t}{\cos t-u\sin t} \quad\text{and}\quad Q(u,t)=\frac{1}{\cos t-u\sin t}. \tag{1}$$

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In §2 we obtain from identities in these generating functions some useful relations among values of the polynomials (Theorem 2.2 below), and recall from [12] a result (Theorem 2.3) relating the polynomials to series of reciprocal powers. In §3 we apply results of §2 to the computation of $P_n(u)$ and $Q_n(u)$ for $u = 0, 1, \sqrt{3}$, and $1/\sqrt{3}$, in the process obtaining several integer sequences studied by Glaisher [9,10,11].

In §4 we give a combinatorial interpretation of the values of the derivative polynomials at 0 and 1. These values give the Springer numbers of the irreducible root systems A_n, B_n, and D_n [19], which also count the corresponding types of "snakes" as defined in [3]. (Snakes for the root system A_{n-1} are alternating permutations of $\{1, 2, \ldots, n\}$, whose study dates back to André [2].) This follows from comparison of equations (1) with the generating functions found in [19], but we also give combinatorial proofs using snakes (Theorems 4.2 and 4.3). Results from §3 then give identities for the Springer numbers (e.g., Proposition 4.4).

In §5 we recall the definition of the the "generalized Euler and class numbers" of Shanks [17]. These are arrays of positive integers $c_{a,n}$ and $d_{a,n}$. The first two "rows" (i.e., the $c_{a,n}$ and $d_{a,n}$ with $a = 1, 2$) are the Springer numbers of the preceding paragraph; we show that the third row of Shanks's numbers are given by the values $P_{2n}(\sqrt{3})$ and $Q_{2n-1}(\sqrt{3})$ of the derivative polynomials. We also give a combinatorial interpretation to the numbers $c_{3,n}$ and $d_{3,n}$ in terms of 3-signed alternating permutations as defined by Ehrenborg and Readdy [8].

In §6 we consider the Euler polynomials $E_n(x)$, defined by

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}. \tag{2}$$

Euler polynomials appear in many classical results (see Chapter 23 of [1]). In [6], the values of these polynomials at rational arguments were expressed in terms of the Hurwitz zeta function. Here we give explicit formulas for the Euler polynomials at rational arguments in terms of the polynomials P_n and Q_n (Theorem 6.1), and use them together with the computations of §3 to find $E_n(p/q)$ for $0 \le p \le q$ and $q = 2, 3, 4$ and 6. We also write the Springer and Shanks numbers in terms of values of the Euler polynomials (Theorem 6.2).

2. Derivative polynomials. From the chain rule it follows that the polynomials P_n satisfy $P_0(u) = u$ and $P_{n+1}(u) = (u^2 + 1)P_n'(u)$, $n \ge 0$, and similarly $Q_0(u) = 1$ and $Q_{n+1}(u) = (u^2+1)Q_n'(u) + uQ_n(u)$, $n \ge 0$. The following result is then clear by induction on n.

Theorem 2.1. *Let $n \ge 0$. Then $P_n(u)$ is a polynomial of degree $n + 1$ consisting of even powers with positive integral coefficients when n is odd and of odd powers with positive integral coefficients when n is even; and $Q_n(u)$ is a polynomial of degree n consisting of even powers with positive integral coefficients when n is even and of odd powers with positive integral coefficients when n is odd.*

In particular, for $n \ge 0$ we have

$$P_n(-u) = (-1)^{n+1} P_n(u) \quad \text{and} \quad Q_n(-u) = (-1)^n Q_n(u).$$

The key properties of the P_n and Q_n come from two sorts of identities in their corresponding generating functions P and Q. First, there are the composition relations

$$P(P(u,t),s) = P(u,t+s) \quad \text{and} \quad Q(P(u,t),s)Q(u,t) = Q(u,t+s); \tag{3}$$

these follow from the representations

$$P(u,t) = \tan(\tan^{-1} u + t) \quad \text{and} \quad Q(u,t) = \frac{\sec(\tan^{-1} u + t)}{\sec(\tan^{-1} u)},$$

equivalent to equations (1) above. Second, there is the functional equation

$$P(u,t) = P\left(\frac{u^2-1}{2u}, 2t\right) + \frac{u^2+1}{2u} Q\left(\frac{u^2-1}{2u}, 2t\right), \tag{4}$$

which follows from equations (1) and the half-angle formula for tangent (cf. Theorem 3.1 of [12]). The composition relations (3) imply the following relations among values of the polynomials P_n and Q_n.

Theorem 2.2. *For nonnegative integers n,*

$$P_n(P(u,s)) - \tan s \sum_{k=0}^{n} \binom{n}{k} P_k(P(u,s))P_{n-k}(u) = P_n(u) + \delta_{0n} \tan s; \tag{i}$$

$$Q_n(P(u,s)) - \tan s \sum_{k=0}^{n} \binom{n}{k} Q_k(P(u,s))P_{n-k}(u) = (1 - u\tan s)Q_n(u). \tag{ii}$$

Proof. The composition relation for P gives

$$P(P(u,s),t) = P(u,s+t) = P(P(u,t),s) = \frac{\sin s + \cos s P(u,t)}{\cos s - \sin s P(u,t)},$$

or $\cos s P(P(u,s),t) - \sin s P(P(u,s),t)P(u,t) = \sin s + \cos s P(u,t)$. Take the coefficient of $t^n/n!$ and divide by $\cos s$ to get (i). The proof of (ii) proceeds similarly, using the composition relation for Q.

In [12] contour integration and expansion into power series were used to obtain closed forms in terms of the P_n and Q_n for certain series. We need the following definitions. Call a function $\psi : \mathbf{Z} \to \mathbf{C}$ periodic mod q if $\psi(0) = 0$ and $\psi(n+q) = \psi(n)$ for all $n \in \mathbf{Z}$, and alternating mod q if $\psi(0) = 0$ and $\psi(n+q) = -\psi(n)$ for all $n \in \mathbf{Z}$. If ψ is periodic or alternating mod q, we call it even if $\psi(q-j) = \psi(j)$ for $0 < j < q$, and odd if $\psi(q-j) = -\psi(j)$ for $0 < j < q$. From [12] we have the following result.

Theorem 2.3. *Let $n \ge 0$ be an integer. If ψ is periodic mod q, then*

$$\sum_{j=1}^{\infty} \frac{\psi(j)}{j^{n+1}} = \frac{\pi^{n+1}}{2q^{n+1}n!} \sum_{p=1}^{q-1} \psi(p) P_n(\cot \tfrac{p\pi}{q})$$

provided n and ψ have opposite parity. If ψ is alternating mod q, then

$$\sum_{j=1}^{\infty} \frac{\psi(j)}{j^{n+1}} = \frac{\pi^{n+1}}{2q^{n+1}n!} \sum_{p=1}^{q-1} \psi(p) \csc \frac{p\pi}{q} Q_n(\cot \tfrac{p\pi}{q})$$

provided n and ψ have the same parity.

3. Particular values of derivative polynomials. By setting $u = 0$ in equations (1) we see that $P_n(0)$ and $Q_n(0)$ are respectively the tangent and secant numbers, i.e. the coefficients of $t^n/n!$ in the Maclaurin series of $\tan t$ and $\sec t$. We shall take these numbers as known. They can be computed from the Euler-Bernoulli triangle as discussed in [4] and [3]; see also [13]. In this section we show how to compute $P_n(u)$ and $Q_n(u)$ for $u = 1, \sqrt{3}$, and $1/\sqrt{3}$ from the tangent and secant numbers.

Set $u = 1$ and consider the coefficient of $t^n/n!$ in the functional equation (4) to get $P_n(1)$:

$$P_n(1) = 2^n(P_n(0) + Q_n(0)) = \begin{cases} 2^n Q_n(0), & n \text{ even}, \\ 2^n P_n(0), & n \text{ odd}. \end{cases} \tag{5}$$

Now compute $Q_n(1)$ using the following result.

Theorem 3.1. *For integers $n \geq 0$,*

$$Q_n(1) = -\sin\frac{n\pi}{2} + \sum_{2k \leq n} \binom{n}{2k} (-1)^k P_{n-2k}(1).$$

Proof. Set $u = 1$ in Theorem 2.2(ii), and then let $s \to +\infty i$ so that $\tan s \to i$ and $P(1, s) \to i$. This gives

$$(1 - i)Q_n(1) = Q_n(i) - i \sum_{k=0}^{n} \binom{n}{k} Q_k(i) P_{n-k}(1).$$

Now $Q(i, t) = e^{it}$, so $Q_n(i) = i^n$. Thus, we have

$$(1 - i)Q_n(1) = i^n - i \sum_{k=0}^{n} \binom{n}{k} i^k P_{n-k}(1).$$

Take the imaginary part to get the conclusion.

Remarks. 1. This result, together with the corresponding one obtained by taking the real part in the proof, is equivalent to the computation of the $Q_n(1)$ from the numbers $P_n(1)$ via Seidel matrices as described in [7].
2. Since the $Q_n(1)$ turn out to be the Springer numbers b_n of the next section, they can be computed via the pair of "boustrophedonic" triangles $L(b)$ and $R(b)$ described in [3] (see also [14]); in fact these triangles are equivalent to Seidel matrices as is explained in [7].
3. The numbers $Q_n(1)$ were extensively studied by Glaisher [9,11], who wrote P_n for $Q_{2n}(1)$ and Q_n for $Q_{2n-1}(1)$.

Set $u = -1/\sqrt{3}$ in equation (4) and examine the coefficient of $t^n/n!$ to get

$$(2^n + (-1)^n)P_n(\frac{1}{\sqrt{3}}) = \frac{2^{n+1}}{\sqrt{3}} Q_n(\frac{1}{\sqrt{3}});$$

then combine this with the equation obtained by setting $u = \sqrt{3}$ to get

$$P_n(\sqrt{3}) = (2^{n+1} + (-1)^n)P_n(\frac{1}{\sqrt{3}}).$$

In view of these equations, to find $P_n(\frac{1}{\sqrt{3}})$ and $Q_n(\frac{1}{\sqrt{3}})$ it is enough to find $P_n(\sqrt{3})$. Our next two results give $P_n(\sqrt{3})$ and $Q_n(\sqrt{3})$.

Theorem 3.2. *i. If n is odd, $P_n(\sqrt{3}) = \frac{1}{2}(3^{n+1}-1)P_n(0)$.*
ii. If n is even, $Q_n(\sqrt{3}) = \frac{1}{4}(3^{n+1}+1)Q_n(0)$.

Proof. Note first that $\cos 3t = \cos t(2\cos 2t - 1)$, by the addition formula for cosine and the double-angle formulas for sine and cosine. Then

$$P(\sqrt{3},t) = \frac{\sin t + \sqrt{3}\cos t}{\cos t - \sqrt{3}\sin t} = \frac{\sqrt{3}+2\sin 2t}{2\cos 2t - 1} = \frac{(\sqrt{3}+2\sin 2t)\cos t}{\cos 3t},$$

while on the other hand

$$\begin{aligned} 3P(0,3t) - P(0,t) &= \frac{3(\sin 2t\cos t + \sin t\cos 2t) - (2\cos 2t - 1)\sin t}{\cos 3t} \\ &= \frac{3\sin 2t\cos t + 2\cos^2 t\sin t}{\cos 3t} = \frac{4\sin 2t\cos t}{\cos 3t}. \end{aligned}$$

Thus

$$P(\sqrt{3},t) - \frac{1}{2}(3P(0,3t) - P(0,t)) = \frac{\sqrt{3}\cos t}{\cos 3t}, \tag{6}$$

and (i) follows from consideration of the coefficient of $t^n/n!$, n odd (Note the right-hand side is an even function). A similar argument proves the identity

$$Q(\sqrt{3},t) - \frac{1}{4}(3Q(0,3t) + Q(0,t)) = \frac{\sqrt{3}}{2}\frac{\sin 2t}{\cos 3t}, \tag{7}$$

from which (ii) follows upon the observation that the right-hand side is an odd function.

Theorem 3.3. *i. If $n > 0$ is even, $P_n(\sqrt{3})$ can be computed from the tangent numbers $P_k(0)$ via*

$$P_n(\sqrt{3}) = \frac{\sqrt{3}}{2}\sum_{k \text{ odd}}\binom{n}{k}(3^{k+1}-1)P_k(0)P_{n-k}(0).$$

ii. If n is odd, $Q_n(\sqrt{3})$ can be computed from the tangent numbers $P_k(0)$ and the secant numbers $Q_k(0)$ via

$$Q_n(\sqrt{3}) = \frac{\sqrt{3}}{8}\sum_{k \text{ odd}}\binom{n}{k}(3^{k+1}-1)P_k(0)Q_{n-k}(0).$$

Proof. For (i), set $s = \frac{\pi}{3}$ and $u = 0$ in Theorem 2.2(i) to get

$$P_n(\sqrt{3}) - \sqrt{3}\sum_{k=0}^{n}\binom{n}{k}P_k(\sqrt{3})P_{n-k}(0) = P_n(0) + \sqrt{3}\delta_{0n}.$$

Now suppose $n > 0$ is even; then this reduces to

$$P_n(\sqrt{3}) = \sqrt{3} \sum_{k \text{ odd}} \binom{n}{k} P_k(\sqrt{3}) P_{n-k}(0) = \sqrt{3} \sum_{k \text{ odd}} \binom{n}{k} \frac{3^{k+1}-1}{2} P_k(0) P_{n-k}(0),$$

where we have used Theorem 3.2(i) in the last step. For (ii), proceed similarly after setting $s = \frac{\pi}{3}$ and $u = -\sqrt{3}$ in Theorem 2.2(ii).

Remarks. 1. Comparing equation (6) to the equation at the beginning of §24 in [10], we see the numbers H_n of [9,10] are given by $H_n = \sqrt{3} P_{2n}(\sqrt{3})/2^{2n+1}$.
2. Similarly, equation (7) shows that the numbers T_n of [9] are $Q_{2n-1}(\sqrt{3})/\sqrt{3}$.
3. In Theorem 5.1 below we show that the numbers $P_{2n}(\sqrt{3})$ and $Q_{2n-1}(\sqrt{3})$ are closely related to certain generalized Euler and class numbers as defined in [17].

4. Root systems, values of derivative polynomials at 0 and 1, and the combinatorics of snakes. Let V be a real vector space, R a root system in V, and W the Weyl group of R (for definitions see [5]). Fix a set S of simple roots for R: then any $\alpha \in R$ is either a positive or negative linear combination of elements of S; we write $\alpha > 0$ in the first case and $\alpha < 0$ in the second. For $I \subset S$, denote by $\sigma(I, S)$ the number of elements $w \in W$ such that $w\alpha > 0$ for $\alpha \in I$ and $w\alpha < 0$ for $\alpha \in S - I$. Let $M(R)$ be the maximum value of $\sigma(I, S)$. T. A. Springer [19] computed the quantity $M(R)$ for all irreducible root systems R. Setting aside the exceptional root systems, his results are as follows.
1. If R is of type A_n, $n \geq 1$, then $M(R) = a_n$ satisfies

$$1 + t + \sum_{n \geq 2} \frac{a_{n-1}}{n!} t^n = \tan t + \sec t.$$

2. If R is of type B_n or C_n, $n \geq 2$, then $M(R) = b_n$ satisfies

$$1 + t + \sum_{n \geq 2} \frac{b_n}{n!} t^n = \frac{\cos t + \sin t}{\cos 2t}.$$

3. If R is of type D_n, $n \geq 3$, then $M(R) = d_n$ satisfies

$$t + \frac{1}{2} t^2 + \sum_{n \geq 3} \frac{d_n}{n!} t^n = \frac{1 + \sin 2t - \cos t - \sin t}{\cos 2t}.$$

We shall call $M(R)$ the Springer number of the root system R.

Proposition 4.1. *The Springer numbers a_n, b_n, and d_n as defined above are given by $a_n = P_{n+1}(0) + Q_{n+1}(0)$, $b_n = Q_n(1)$, and $d_n = P_n(1) - Q_n(1)$.*

Proof. This amounts to writing the the generating functions above in terms of P and Q. For example, the formula for a_n follows from observing that $P(0, t) + Q(0, t) = \tan t + \sec t$. Similarly,

$$Q(1, t) = \frac{1}{\cos t - \sin t} = \frac{\cos t + \sin t}{\cos 2t}$$

and

$$P(1,t) - Q(1,t) = \frac{\sin t + \cos t - 1}{\cos t - \sin t} = \frac{1 + \sin 2t - \cos t - \sin t}{\cos 2t}.$$

By describing Springer numbers geometrically in terms of Weyl chambers, Arnol'd [3] showed that the numbers a_n, b_n, and d_n can be thought of as counting various types of snakes (updown sequences). The formal definitions are as follows.

Definition. A snake of type A_n is a sequence $(x_0, x_1, \dots, x_n)$ of integers such that $x_0 < x_1 > x_2 < \cdots x_n$ and $\{x_0, x_1, \dots, x_n\} = \{0, 1, \dots, n\}$. A snake of type B_n is a sequence $(x_1, x_2, \dots, x_n)$ of integers such that $0 < x_1 > x_2 < \cdots x_n$ and $\{|x_1|, |x_2|, \dots, |x_n|\} = \{1, 2, \dots, n\}$. A snake of type D_n is a sequence $(x_1, \dots, x_n)$ of integers such that $-x_2 < x_1 < x_2 > x_3 < \cdots x_n$ and $\{|x_1|, |x_2|, \dots, |x_n|\} = \{0, 1, \dots, n-1\}$.

We shall write A_n for the set of snakes of type A_n and so forth. The geometric argument of [3] shows that $\operatorname{card} A_n$, $\operatorname{card} B_n$, and $\operatorname{card} D_n$ are a_n, b_n, and d_n respectively. On the other hand, it is possible to prove that these cardinalities are given by the formulas of Proposition 4.1 using combinatorial arguments about snakes. This is done for A_n in [2] and [3] (see the remark following Theorem 13): we do it here for B_n and D_n. For this purpose, it is convenient to introduce another type of snake from [3]: an integer sequence $(x_1, \dots, x_n)$ such that $x_1 < x_2 > x_3 < \cdots x_n$ and $\{|x_1|, \dots, |x_n|\} = \{1, \dots, n\}$ is called a snake of type β_n. Let β_n denote the set of snakes of type β_n.

Theorem 4.2. *Let $b(t) = \sum_{n \geq 0} \operatorname{card} B_n t^n/n!$ and $\beta(t) = \sum_{n \geq 0} \operatorname{card} \beta_n t^n/n!$. Then $b(t) = Q(1,t)$ and $\beta(t) = P(1,t)$ (so $\operatorname{card} B_n = Q_n(1)$ and $\operatorname{card} \beta_n = P_n(1)$).*

Proof. We prove the formula for $\beta(t)$ first. Given $(x_1, \dots, x_{n+1}) \in \beta_{n+1}$, let r be the unique element of $\{0, \dots, n\}$ with $|x_{r+1}| = n+1$. Then the sets $\{|x_1|, \dots, |x_r|\}$ and $\{|x_{r+2}|, \dots, |x_{n+1}|\}$ partition $\{1, \dots, n\}$. The sequence $(x_1, \dots, x_r)$ can be shrunk into a snake of type β_r by applying the order-preserving bijection of $\{|x_1|, \dots, |x_r|\}$ onto $\{1, \dots, r\}$; similarly $((-1)^{r+1}x_{r+2}, \dots, (-1)^{r+1}x_{n+1})$ gives a snake of type β_{n-r}. Conversely, given $r \in \{0, \dots, n\}$ and a partition of $\{1, \dots, n\}$ into an r-set and an $(n-r)$-set, together with elements of β_r and β_{n-r}, we can construct a β_{n+1}-snake in a unique way. Hence

$$\operatorname{card} \beta_{n+1} = \sum_{r=0}^{n} \binom{n}{r} \operatorname{card} \beta_r \operatorname{card} \beta_{n-r} + \delta_{n0}.$$

from which follows $\beta'(t) = \beta(t)^2 + 1$. (The Kronecker delta term reflects the fact that there are two β_1-snakes, (1) and (-1).) The unique solution of this differential equation satisfying the initial condition $\beta(0) = \operatorname{card} \beta_0 = 1$ (β_0 consists of the empty snake) is

$$\beta(t) = \tan(t + \tfrac{\pi}{4}) = \frac{\tan t + 1}{1 - \tan t} = P(1,t).$$

Now suppose $(x_1, \dots, x_{n+1}) \in B_{n+1}$, with $r \in \{0, \dots, n\}$ such that $|x_{r+1}| = n+1$. Again the sequences $(x_1, \dots, x_r)$ and $(x_{r+2}, \dots, x_{n+1})$ consist of integers whose absolute values partition $\{1, \dots, n\}$. The sequence $(x_1, \dots, x_r)$ can be shrunk into a B_r-snake, since

$x_1 > 0$; but the shrinkage of the sequence $((-1)^r x_{r+2}, \ldots, (-1)^r x_{n+1})$ is a snake of type β_{n-r}. Hence

$$\operatorname{card} B_{n+1} = \sum_{r=0}^{n} \binom{n}{r} \operatorname{card} B_r \operatorname{card} \beta_{n-r}$$

and we have $b'(t) = b(t)\beta(t)$. Using $b(0) = 1$ and our formula for $\beta(t)$, this gives

$$b(t) = \frac{1}{\sqrt{2}} \sec(t + \tfrac{\pi}{4}) = Q(1,t).$$

Theorem 4.3. $\operatorname{card} \beta_n = \operatorname{card} B_n + \operatorname{card} D_n$ *(so* $\operatorname{card} D_n = P_n(1) - Q_n(1)$*).*

Proof. First note that we have a partition $\beta_n = \beta_n^- \cup \beta_n^+$, where β_n^- and β_n^+ are respectively the sets of β_n-snakes that start with a negative integer and with a positive integer. We shall define bijections $f : \beta_n^- \to B_n$ and $g : \beta_n^+ \to D_n$. Let $f(x_1, \ldots, x_n) = (-x_1, \ldots, -x_n)$: it is easy to see that f is a bijection of β_n^- onto B_n. For g, let $(x_1, \ldots, x_n) \in \beta_n^+$, with $r \in \{1, \ldots, n\}$ such that $|x_r| = 1$. Then $g(x_1, \ldots, x_n) = (x_r \tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n)$, where $\tilde{x}_i = (\operatorname{sgn} x_i)(|x_i| - 1)$. The reader may verify that the image of g is in D_n, and in fact that g has an inverse given by

$$g^{-1}(y_1, \ldots, y_n) = \begin{cases} (1, \hat{y}_2, \ldots, \hat{y}_n), & \text{if } r = 1; \\ (|\hat{y}_1|, \hat{y}_2, \ldots, \hat{y}_{r-1}, \operatorname{sgn} y_1, \hat{y}_{r+1}, \ldots, \hat{y}_n), & \text{otherwise}; \end{cases}$$

for $(y_1, \ldots, y_n) \in D_n$ with $y_r = 0$, and $\hat{y}_i = (\operatorname{sgn} y_i)(|y_i| + 1)$ for $i \neq r$.

We can use the machinery of previous sections to obtain relations among the Springer numbers (and $\operatorname{card} \beta_n$). For example, equation (5) above implies $\operatorname{card} \beta_n = 2^n a_{n-1}$, which has a simple combinatorial interpretation in terms of snakes (cf. Theorem 24 of [3]). Other relations, like the following, appear to be new.

Proposition 4.4. *For positive integers* n,

$$b_n = \frac{(-1)^n + 1}{2} a_{n-1} + \sum_{k \text{ odd}} \binom{n}{k} a_{k-1} b_{n-k}$$

and

$$d_n = (-1)^{n-1} a_{n-1} + \sum_{k \text{ odd}} \binom{n}{k} a_{k-1} d_{n-k}.$$

Proof. Set $s = \frac{\pi}{4}$ and $u = 0$ in Theorem 2.2: then the first identity follows from part (ii) of the theorem, and the second upon subtracting part (ii) from part (i).

Remark. The formula for b_n can be given a combinatorial interpretation. Let $\bar{A}_m$ be the set of sequences $(x_0, \ldots, x_m)$ such that $\{x_0, \ldots, x_m\} = \{0, 1, \ldots, m\}$ and $x_0 > x_1 < x_2 > \cdots x_m$: evidently $\bar{A}_m$ is in 1-1 correspondence with A_m via $(x_0, \ldots, x_m) \to (m - x_0, \ldots, m - x_m)$. Now suppose $(x_1, \ldots, x_n) \in B_n$. If all the x_i are positive, then $(x_1 - 1, \ldots, x_n - 1) \in \bar{A}_{n-1}$. Otherwise, there is a smallest $k \in \{1, \ldots, n\}$ with $x_k < 0$, and it follows from the definition of B_n that k must be even. Then $(x_1, \dots, x_{k-1})$ can be shrunk into an element of $\bar{A}_{k-2}$, and $(-x_k, -x_{k+1}, \dots, -x_n)$ can be shrunk into an element of B_{n-k+1}. Since there are $\binom{n}{k-1}$ ways to choose $\{x_1, \dots, x_{k-1}\} \subset \{1, \dots, n\}$, we have

$$b_n = a_{n-1} + \sum_{2 \le k \le n \text{ even}} \binom{n}{k-1} a_{k-2} b_{n-k+1} = a_{n-1} + \sum_{k \le n-1 \text{ odd}} \binom{n}{k} a_{k-1} b_{n-k},$$

which is equivalent to the first identity above.

5. Values of derivative polynomials at $\sqrt{3}$, generalized Euler and class numbers, and 3-signed permutations. In [17] Shanks defined positive integers $c_{a,n}$ (for integer $a \ge 1$ and $n \ge 0$) and $d_{a,n}$ (for integer $a, n \ge 1$) by

$$L_a(2n+1) = K_a \sqrt{a} \frac{c_{a,n}}{(2n)!} \left(\frac{\pi}{2a}\right)^{2n+1} \quad \text{and} \quad L_{-a}(2n) = K_a \sqrt{a} \frac{d_{a,n}}{(2n-1)!} \left(\frac{\pi}{2a}\right)^{2n},$$

where $K_a = \frac{1}{2}$ if $a = 1$ and 1 otherwise, and

$$L_a(s) = \sum_{k=0}^{\infty} \left(\frac{-a}{2k+1}\right) (2k+1)^{-s};$$

here $(-a/(2k+1))$ is the Jacobi symbol. As noted in [17], the numbers $c_{1,n}$ are just the secant numbers $Q_{2n}(0)$, and the $d_{1,n}$ are the tangent numbers $P_{2n-1}(0)$. Comparison of the tables in [17] and those of [3] reveals that the numbers $c_{2,n}$ and $d_{2,n}$ are Springer numbers: in fact $c_{2,n} = Q_{2n}(1) = b_{2n}$ and $d_{2,n} = Q_{2n-1}(1) = b_{2n-1}$. This can be proved using the recurrences given in [17] together with the generating function for the $Q_n(1)$: see Proposition 6.3 of [16], where b_n is denoted $E_n^{\pm}$. Our next result gives the third row of Shanks's numbers in terms of the numbers $P_{2n}(\sqrt{3})$ and $Q_{2n-1}(\sqrt{3})$ discussed in Theorem 3.3 above.

Theorem 5.1. *i. For $n \ge 0$, $c_{3,n} = \frac{1}{\sqrt{3}} P_{2n}(\sqrt{3})$.*
ii. For $n \ge 1$, $d_{3,n} = \frac{2}{\sqrt{3}} Q_{2n-1}(\sqrt{3})$.

Proof. By substituting into the first of equations (19) of [17] the constants corresponding to the expression for $L_3(s)$ in equations (19) of [18], we have

$$\sum_{n=0}^{\infty} w^{2n} \frac{c_{3,n}}{(2n)!} = \frac{\cos(3w(1-4/3))}{\cos 3w} = \frac{\cos w}{\cos 3w}, \tag{8}$$

and comparison with equation (6) above proves (i). Similarly, substitute into the second of equations (19) of [17] the constants from the expression for $L_{-3}(s)$ in equations (19) of [18] to get

$$\sum_{n=1}^{\infty} w^{2n-1} \frac{d_{3,n}}{(2n-1)!} = \frac{\sin(3w(1-4/12))}{\cos 3w} = \frac{\sin 2w}{\cos 3w}, \tag{9}$$

which on comparison with equation (7) gives (ii).

The notion of an alternating permutation of $\{1, 2, \ldots, n\}$ is generalized in [8] to a "Λ-alternating augmented r-signed permutation" of $\{1, 2, \ldots, n\}$ for any pair (p, r) of positive integers with $p \leq r$. The cases $(1, 1)$, $(1, 2)$ and $(2, 2)$ correspond respectively to the A_{n-1}-snakes, B_n-snakes, and β_n-snakes of the previous section. Here we give a combinatorial interpretation of the numbers $c_{3,n}$ and $d_{3,n}$ using the case $(p, r) = (2, 3)$. Let $S = \{n\omega^m |\ n, m \text{ nonnegative integers}\}$, where $\omega = e^{\frac{2\pi i}{3}}$, with the linear order

$$\omega^2 < 2\omega^2 < 3\omega^2 < \cdots < 0 < \omega < 2\omega < 3\omega < \cdots < 1 < 2 < 3 < \cdots$$

Define an ER_n-snake to be a sequence $(x_1, x_2, \ldots, x_n)$ of elements of S such that $0 < x_1 > x_2 < \cdots x_n$ and $\{|x_1|, |x_2|, \ldots, |x_n|\} = \{1, 2, \ldots, n\}$. Let ER_n be the set of ER_n-snakes, so e.g., ER_0 consists of the empty snake, $ER_1 = \{(1), (\omega)\}$, and

$$ER_2 = \{(2, 1), (2, \omega), (2, \omega^2), (2\omega, \omega), (2\omega, \omega^2), (1, 2\omega), (1, 2\omega^2), (\omega, 2\omega^2)\}.$$

Theorem 5.2. *For $n \geq 0$, $c_{3,n} = \operatorname{card} ER_{2n}$; for $n \geq 1$, $d_{3,n} = \operatorname{card} ER_{2n-1}$.*

Proof. In the terminology of [8], ER_n-snakes are Λ-alternating augmented 3-signed permutations of $\{1, 2, \ldots, n\}$ corresponding to $p = 2$. By Proposition 7.2 of [8], we have

$$\sum_{n=0}^{\infty} \frac{\operatorname{card} ER_n}{n!} x^n = \frac{\sin 2x + \cos x}{\cos 3x}$$

and the conclusion follows by comparison with equations (8) and (9) above.

6. Euler Polynomials. In this section we give explicit formulas for the values of the Euler polynomials at rational numbers in terms of the P_n and Q_n. The Euler polynomials $E_n(x)$ are defined by equation (2) above; the Euler numbers are $E_n = 2^n E_n(\frac{1}{2})$. In view of the translation formula for Euler polynomials (23.1.7 of [1]), it suffices to give formulas for rational arguments between 0 and 1.

Theorem 6.1. *If n, p and q are nonnegative integers with $0 \leq p \leq q$ ($0 < p < q$ if $n = 0$) and $q \geq 2$ even, then*

$$E_n(\frac{p}{q}) = \frac{2}{q^{n+1}} \sum_{k=0}^{q/2-1} \sin(\tfrac{(2k+1)\pi p}{q} - \tfrac{n\pi}{2}) \csc \tfrac{(2k+1)\pi}{q} Q_n(\cot \tfrac{(2k+1)\pi}{q})$$

if p is odd, and

$$E_n(\frac{p}{q}) = \frac{2}{q^{n+1}} \sum_{k=0}^{q/2-1} \sin(\tfrac{(2k+1)\pi p}{q} - \tfrac{n\pi}{2}) P_n(\cot \tfrac{(2k+1)\pi}{q})$$

if p is even.

Proof. We start with the Fourier series

$$E_n(\frac{p}{q}) = \frac{4 \cdot n!}{\pi^{n+1}} \sum_{k=0}^{\infty} \frac{\sin(\frac{(2k+1)\pi p}{q} - \frac{n\pi}{2})}{(2k+1)^{n+1}}$$

([1], 23.1.16). Consider

$$\psi(j)=\begin{cases}\sin(\frac{j\pi p}{q}-\frac{n\pi}{2}), & j \text{ odd},\\ 0, & j \text{ even}.\end{cases}$$

Then $\psi(j+q)=(-1)^p\psi(j)$, so ψ is alternating mod q if p is odd and periodic mod q if p is even. Further, $\psi(q-j)=(-1)^{p+n+1}\psi(j)$, so ψ has parity (as an alternating or periodic function mod q) opposite that of $p+n$. Thus, since

$$E_n(\frac{p}{q})=\frac{4\cdot n!}{\pi^{n+1}}\sum_{j=1}^{\infty}\frac{\psi(j)}{j^{n+1}}$$

the conclusion follows from Theorem 2.3.

Remark. By a proof similar to that above, one can obtain a formula for rational values $B_n(p/q)$ of the nth Bernoulli polynomial in terms of P_{n-1}. This is essentially Theorem C of [20].

Examples. Set $q=2$ to get

$$E_n(\frac{1}{2})=\frac{1}{2^n}\sin\frac{(1-n)\pi}{2}Q_n(0)=\begin{cases}0, & n \text{ odd}\\ (-1)^{\frac{n}{2}}2^{-n}Q_n(0), & n \text{ even}.\end{cases}\tag{10}$$

Hence (by Theorem 2.1) $Q_n(0)=|E_n|$. We have also

$$E_n(0)=-E_n(1)=\frac{1}{2^n}\sin\frac{(-n)\pi}{2}P_n(0)=\begin{cases}0, & n \text{ even}\\ (-1)^{\frac{n+1}{2}}2^{-n}P_n(0), & n \text{ odd},\end{cases}\tag{11}$$

for $n\geq 1$; this can be written in terms of Bernoulli numbers (see [1], 23.1.20) as $-2(2^{n+1}-1)B_{n+1}/(n+1)$.

Taking $q=4$, we obtain

$$E_n(\frac{1}{4})=(-1)^nE_n(\frac{3}{4})=\begin{cases}(-1)^{\frac{n}{2}}4^{-n}Q_n(1), & n \text{ even},\\ (-1)^{\frac{n+1}{2}}4^{-n}Q_n(1), & n \text{ odd}.\end{cases}\tag{12}$$

Finally, let $q=6$ to get

$$E_n(\frac{1}{6})=(-1)^nE_n(\frac{5}{6})=\begin{cases}2(-1)^{\frac{n}{2}}6^{-n-1}[2Q_n(\sqrt{3})+Q_n(0)], & n \text{ even},\\ 4(-1)^{\frac{n+1}{2}}6^{-n-1}\sqrt{3}Q_n(\sqrt{3}), & n \text{ odd},\end{cases}\tag{13}$$

and

$$E_n(\frac{1}{3})=(-1)^nE_n(\frac{2}{3})=\begin{cases}2(-1)^{\frac{n}{2}}6^{-n-1}\sqrt{3}P_n(\sqrt{3}), & n \text{ even},\\ 2(-1)^{\frac{n+1}{2}}6^{-n-1}[P_n(\sqrt{3})-P_n(0)], & n \text{ odd}.\end{cases}\tag{14}$$

If n is even, we can use Theorem 3.2(ii) and equation (13) to write

$$E_n(\frac{1}{6})=E_n(\frac{5}{6})=\frac{(-1)^{\frac{n}{2}}(3^n+1)}{2\cdot 6^n}Q_n(0)=\frac{3^n+1}{2\cdot 6^n}E_n$$

(cf. [15], Ch. 2, eqn. (46)). If n is odd, we have

$$E_n(\frac{1}{3}) = -E_n(\frac{2}{3}) = \frac{(-1)^{\frac{n+1}{2}}(3^n-1)}{2\cdot 6^n}P_n(0) = -\frac{(3^n-1)(2^{n+1}-1)}{3^n(n+1)}B_{n+1}$$

using Theorem 3.2(i) and equation (14) (cf. [15], Ch. 2, eqn. (45); [1], 23.1.22).

We close this section by expressing the Springer and Shanks numbers in terms of the Euler polynomials.

Theorem 6.2. *For integers $n \geq 1$,*

(i)
$$a_{n-1} = 2^n|E_n(\frac{(-1)^n+1}{4})|;$$

(ii)
$$b_n = 4^n|E_n(\frac{1}{4})|;$$

(iii)
$$d_n = 4^n|E_n(\frac{(-1)^n+1}{4}) - E_n(\frac{1}{4})|;$$

(iv)
$$c_{3,n} = 6^{2n}|E_{2n}(\frac{1}{3})|;$$

(v)
$$d_{3,n} = 6^{2n-1}|E_{2n-1}(\frac{1}{6})|.$$

Proof. For (i), note first that a_{n-1} is $Q_n(0)$ for n even, and $P_n(0)$ for n odd; then use equations (10) and (11). For (ii), use equation (12). For (iii), observe from equation (5) that d_n is $2^nQ_n(0) - Q_n(1)$ for n even, and $2^nP_n(0) - Q_n(1)$ for n odd; then use equation (12) together with equations (10) and (11) respectively. For (iv) and (v), use Theorem 5.1 together with equations (13) and (14).

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