Bicoloring Steiner Triple Systems

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Abstract

A Steiner triple system has a bicoloring with m color classes if the points are partitioned into m subsets and the three points in every block are contained in exactly two of the color classes. In this paper we give necessary conditions for the existence of a bicoloring with 3 color classes and give a multiplication theorem for Steiner triple systems with 3 color classes. We also examine bicolorings with more than 3 color classes.

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1 Introduction

Throughout this paper we use notation consistent with that found in [2]. Let $\mathcal{D} = (V, \mathcal{B})$ be a (v, k, λ)-design. A *coloring* of $\mathcal{D}$ is a mapping $\varphi : V \to C$. The elements of C are *colors*; if $|C| = m$, we have an *m-coloring* of $\mathcal{D}$. For each $c \in C$, the set $\varphi^{-1}(c) = \{x : \varphi(x) = c\}$ is a *color class*. A coloring φ of $\mathcal{D}$ is *weak* (*strong*) if for all $B \in \mathcal{B}$, $|\varphi(B)| > 1$ ($\varphi(B) = k$, respectively), where $\varphi(B) = \cup_{v \in B} \varphi(v)$. Each color class in a weak or strong coloring is an independent set. In a weak coloring, no block is monochromatic (i.e., no block has all its elements the same color), while in a strong coloring, the elements of any block B get $|B|$ distinct colors. The *weak* [*strong*] *chromatic number* of $\mathcal{D}$ is the smallest m for which $\mathcal{D}$ admits a weak [strong] m-coloring. Much work has been done on weak and strong colorings; for an extensive survey of these results, the reader is referred to [9].

For triple systems, the following results concerning weak colorings are known.

Theorem 1.1 *[3]. For every admissible $v \geq 5$ and any λ there exists a weakly 3-chromatic $2-(v, 3, \lambda)$-design.*

A modification of Bose's and Skolem's constructions for Steiner triple systems was used to prove:

Theorem 1.2 *[1, 4]. A weakly 4-chromatic $STS(v)$ exists for every $v \equiv 1$ or 3 (mod 6), $v \geq 21$.*

In this paper we consider a stronger coloring condition than weak coloring, termed a bicoloring. While a bicoloring is defined for any design, we examine only bicolorings of Steiner triple systems.

A coloring φ of $\mathcal{D}$ is a *bicoloring* if for all $B \in \mathcal{B}$, $|\varphi(B)| = 2$, where $\varphi(B) = \cup_{v \in B} \varphi(v)$. This definition implies that in a triple system every triple has two elements in one color class and one in another class, i.e. there are no monochromatic triples nor are there any triples receiving three colors. So, in some sense, a bicoloring of a triple system is an anti-strong weak coloring.

An m-bicoloring is a bicoloring with m color classes, and a design admitting an m-bicoloring is m-bicolorable. A design is *m-bichromatic* if it is m-bicolorable but not $(m-1)$-bicolorable.

Example 1.3 *A 3-bicolorable STS(13). First, construct an STS(13) by developing the base blocks $\{1, 3, 9\}, \{2, 5, 6\}$ mod 13. The color classes are $\{0, 1\}, \{2, 6, 8, 10, 11\}$, $\{3, 4, 5, 7, 9, 12\}$*

In the context of strict colorings of hypergraphs defined recently by Voloshin [12], a bicoloring of an STS is a strict coloring of an STS in which all triples are both edges and also co-edges. In [5, 6], Milazzo and Tuza discuss several properties of strict colorings of Steiner triple systems. An easy counting argument [8] establishes that there exist no nontrivial 2-colorable STS (or triple systems of any index λ for $v > 4$), and hence no 2-bichromatic triple systems. In the next section we consider 3-bicolorable (and hence 3-bichromatic) STSs.

While for a weak m-chromatic triple system there are some general bounds on the sizes of the color classes (see [9]), in an $m-$bichromatic triple system there is a *divisibility condition* that the sizes of the color classes must satisfy.

Proposition 1.4 *Let $(X, \mathcal{A})$ be an $m-$bicolorable triple system TS(v, λ) and assume that the m color classes have sizes $c_1, c_2, \ldots, c_m$. Then*

$$\sum_{i=1}^{m} \binom{c_i}{2} = \binom{v}{2}/3. \tag{1}$$

Proof. (See also [6].) Since exactly one of the three pairs of elements covered by any triple of $\mathcal{A}$ has both its elements within one color class, the number of monochromatic pairs must be one third of the total number of pairs. ▯

The divisibility condition is not sufficient. For example, if $c_i = c_j = 2$, and $i \neq j$, there cannot exist an m-bicoloring no matter what the size of the other color classes. This is called the *duplicity condition*.

Since every triple must have two 2-colored pairs, the total number of 2-colored pairs must be even. Hence, we obtain the *oddity condition*:

Proposition 1.5 *Let $(X, \mathcal{A})$ be an $m-$bicolorable triple system TS(v, λ) and assume that the m color classes have sizes $c_1, c_2, \ldots, c_m$, then at most one of numbers $c_1, c_2, \ldots, c_m$ can be odd.*

If v is an admissible order for STS, every m-tuple $(c_1, c_2, \ldots, c_m)$ (with the c_is in nonincreasing order by convention) satisfying the divisibility condition is an *m-split* for v.

Every 3-split satisfying the divisibility condition automatically satisfies the oddity condition as well. Indeed, if (a, b, c) is a 3-split satisfying the divisibility condition for an admissible order v, and a, b, c are all odd, then of the four numbers $\binom{v}{2}, \binom{a}{2}, \binom{b}{2}, \binom{c}{2}$ either three are even and one is odd, or three are odd and one is even; in either case we have a contradiction.

A similar statement can be shown to be true for 4-splits. However, for $m \geq 5$ there are many m-splits satisfying the divisibility condition which do not satisfy the oddity condition. Still, these conditions together are not sufficient. In fact, we have the following *density condition*:

Proposition 1.6 *Let* $v \equiv 1,3 \pmod 6$. *If there exists an* m*-bicolorable STS(*v*) with* m*-split* $(c_1,\ldots,c_k,d_1,\ldots,d_{m-k})$ *(with* $0<k<m$*), then the inequality*

$$0 \leq \left[\sum_{i=1}^{k}\binom{c_i}{2}\right] - \frac{1}{2}\left[\sum_{i=1}^{k-1}\sum_{j=i+1}^{k} c_ic_j\right] \leq \frac{1}{2}\left[\sum_{i=1}^{k} c_i\right]\cdot\left[\sum_{i=1}^{m-k} d_i\right] - \ell(\sum_{i=1}^{m-k} d_i)$$

holds, where

$$\ell(x) = \begin{cases} x/2 & \text{if } x \equiv 0,2 \pmod 6 \\ 0 & \text{if } x \equiv 1,3 \pmod 6 \\ (x+2)/2 & \text{if } x \equiv 4 \pmod 6 \\ 4 & \text{if } x \equiv 5 \pmod 6 \end{cases}$$

Proof. Let $(V,\mathcal{B})$ be a putative m-bicolorable STS(v) with the specified m-split. Partition the m color classes into two groups, the first with color class sizes $\{c_1,\ldots,c_k\}$ and the second with sizes $\{d_1,\ldots d_{m-k}\}$. We concentrate on triples with one point in the first group and one point in the second (and the location of the third point unknown as yet). A pair with one endpoint in a class in the first group and the other endpoint in a *different* class of the first group appears in a triple that lies wholly in the first group. Hence we can count triples that lie wholly on the first group, and from this determine the number of triples with two points in a class of the first group and one point in a class of the second group. This count is the left hand side of the inequality given. Each such triple requires two of the pairs between the first and second group. However, some of these pairs may already be required in triples containing two points from a class in the first group and one in a class of the second. The 'correction' $\ell(\ldots)$ is then the minimum number of such triples. If we examine a packing on the points of the second group, say with x points in the second group, we see that $\ell(x)$ is the smallest number of pairs left uncovered by any packing (i.e. the smallest number of edges in the leave graph of any packing). Thus the right hand side of the inequality reflects (one half of) the maximum number of pairs available to form triples with two points in a class of the first group and one point in a class of the second group. ∎

We believe that the necessary conditions given in this section taken together are sufficient for the existence of m-bicolorable STSs, at least for $m \leq 5$.

2 3-bichromatic STS

2.1 Necessary conditions

In order to establish necessary conditions for the existence of a 3-bicolorable STS we first need a number theoretic lemma.

Lemma 2.1 *Let n be an integer and $p \geq 5$ a prime factor of n^2+n+1, then $p \equiv 1 \pmod 6$.*

Proof. Since $4(n^2+n+1) = (2n+1)^2+3$, if p divides n^2+n+1, then -3 is a square modulo p. Then $1 = \left(\frac{-3}{p}\right)$, where $\left(\frac{a}{b}\right)$ is the Legendre symbol. Now $\left(\frac{-3}{p}\right) = \left(\frac{-1}{p}\right)\left(\frac{3}{p}\right) = (-1)^{(p-1)/2}\left(\frac{p}{3}\right)(-1)^{(p-1)(3-1)/4}$ by quadratic reciprocity, and hence $1 = (-1)^{p-1}\left(\frac{p}{3}\right)$. Since p is odd, $1 = \left(\frac{p}{3}\right)$, or equivalently, $p \equiv 1 \pmod 3$. Again, since p is odd, $p \equiv 1 \pmod 6$, the desired conclusion. ∎

We are now in a position to prove the following.

Theorem 2.2 *Given an STS(v), and a putative 3-bicoloring with classes of sizes a, b, and $v-a-b$. Let p be a prime, $p \equiv 5 \pmod 6$, and assume p^{2i-1} divides v for some $i \geq 1$. Then*

(a) *p^i divides a, b, and $v-a-b$, and*

(b) *p^{2i} divides v.*

Consequently, if there exists a 3-bicolorable STS(v), then any prime p dividing v with $p \equiv 5 \pmod 6$ must have an even power in the prime factorization of v.

Proof. By the divisibility condition,

$$\binom{a}{2} + \binom{b}{2} + \binom{v-a-b}{2} = \binom{v}{2}/3.$$

This simplifies to

$$3a(a-1)+3b(b-1)+3v(v-1)-3(2v-1)(a+b)+3(a+b)^2=v(v-1).$$

Hence

$$3a^2+3b^2+3ab+v(v-1)-3v(a+b)=0. \qquad (2)$$

Now suppose that v is a multiple of p^{2i-1}, and treat this equation modulo p^{2i-1} to get that $3a^2+3b^2+3ab\equiv 0 \pmod{p^{2i-1}}$. Since 3 and p are relatively prime,

$$a^2+b^2+ab\equiv 0 \pmod{p^{2i-1}}. \qquad (3)$$

Now one obvious solution has $a\equiv b\equiv 0 \pmod{p^i}$. If this is *not* the solution, then without loss of generality say a is not a multiple of p^i. We will obtain a contradiction in this case.

Let k be such that p^k divides a, but p^{k+1} does not divide a, with $0\le k\le i-1$. From (3), p^{2k} divides $b^2+ab=b(b+a)$. Since p^k divides a, it follows that p^k also must divide b. Let $\hat{a}=a/p^k$ and $\hat{b}=b/p^k$. Dividing (3) by p^{2k} yields the equation $\hat{a}^2+\hat{b}^2+\hat{a}\hat{b}\equiv 0 \pmod{p^{2i-2k-1}}$. Since p does not divide $\hat{a}$ we can multiply by $\hat{a}^{-1}$ modulo $p^{2i-2k-1}$ to obtain

$$1+n^2+n\equiv 0 \pmod{p^{2i-2k-1}}$$

where n is $\hat{b}\hat{a}^{-1} \pmod p$. But now by Lemma 2.1, this does not happen when $p\equiv 5 \pmod 6$. Thus, p^i divides a,b and hence also $v-a-b$.

Now look at (2) again. Since p^{2i} is a divisor of the terms $3a^2, 3b^2, 3ab$ and $3v(a+b)$, we find that p^{2i} must divide $v(v-1)$. Since p divides v, then p and $v-1$ are relatively prime, and hence p^{2i} must divide v. This completes the proof. $\square$

We conjecture that the necessary condition given in Theorem 2.2 is also sufficient for the existence of 3-bichromatic STS, and in the next section we find 3-bicolorable STS(v)s for many orders of v.

2.2 Existence

We begin this section with a multiplication theorem. One key ingredient of the construction is a special type of latin square. We first prove a theorem about these latin squares.

Write $n = a + b + c$. Let A, B and C be disjoint sets of sizes a, b and c, respectively. A latin square with rows, columns, and symbols indexed by $A \cup B \cup C$ is called $(a, b, c)-$*forbidden* if in cell (r, g) we find symbol s satisfying:

r in A and g in A implies s not in C
r in A and g in B implies s not in B
r in A and g in C implies s not in A
r in B and g in A implies s not in B
r in B and g in B implies s not in A
r in B and g in C implies s not in C
r in C and g in A implies s not in A
r in C and g in B implies s not in C
r in C and g in C implies s not in B

Now $(a, b, c)-, (b, c, a)-$ and $(c, a, b)-$forbidden latin squares are the same. Similarly, $(b, a, c)-, (a, c, b)-$ and $(c, b, a)-$forbidden latin squares are the same.

Lemma 2.3 *An (a, b, c)-forbidden latin square of order n exists if and only if $max(a, b, c) \leq n/2$.*

Proof. Necessity is obvious. To prove sufficiency, we assume without loss of generality that $c = max(a, b, c)$, so that we need only treat cases when $a \leq b \leq c$, and when $b < a \leq c$. Let X be a set of $(a + b - c)$ symbols disjoint from A, B and C. We are going to form a partial latin square P. The construction differs slightly in the two cases.

If $a \leq b$ form an $(a + b) \times (a + b)$ latin square R with rows and columns indexed by $A \cup B$, and symbols indexed by $C \cup X$, so that R contains an $a \times a$ subsquare on the rows and columns indexed by A, and the symbols in the subsquare containing all symbols in X (and possibly some of the symbols in C). From R, form a partial latin square P as follows. Remove the $a \times a$ subsquare and place an $a \times a$ subsquare on the symbols in A in its place.

When $a > b$, instead first place an $a \times a$ square on symbols in A on the subsquare with rows and columns indexed by A; a $b \times a$ latin rectangle on a subset C_a of a symbols in C on the rectangle with rows indexed by B and columns by A; an $a \times b$ latin rectangle on the subset C_a of symbols on the rectangle with rows indexed by A and columns by B. Then fill the $b \times b$

subsquare with rows and columns indexed by B using a latin square on the symbols $(C - C_a) \cup X$ (this is indeed b symbols as required).

Now the two cases merge. Delete all occurrences of elements in X from the square; all appear in the subarray with rows *and* columns indexed by B. Form a bipartite graph, with one class being the rows indexed by B, the other being the columns indexed by B, with a row and column vertex made adjacent whenever the symbol was one of those in X. Color this bipartite graph in b colors, so that the coloring is equalized (proper and every color appearing the same number of times as every other, i.e. $a + b - c$ times). This can be done by a theorem of de Werra [13]. Now whenever an edge of this bipartite graph gets the ith color, place in the corresponding cell the ith symbol in B. At this point, P is a partial latin square with symbols from A, B and C. In fact, every symbol in A appears a times, every one in B occurs $a+b-c$ times, and every one in C is at least b times. It follows from Ryser's Theorem [10] that P can be completed to a latin square, since $n = a + b + c$ and so every symbol occurs at least $(a + b) + (a + b) - (a + b + c) = a + b - c$ times as required.

That this forces the $(a, b, c)-$forbidden requirements (taking of course all rows and columns added in the embedding to be those indexed by C) is easy counting. To help check it, here are the counts on numbers of symbols from each of A, B and C in each of the subarrays:

Symbols from A :

	A	B	C
A	a^2	0	0
B	0	0	ab
C	0	ab	$a(c-b)$

Symbols from B :

	A	B	C
A	0	0	ab
B	0	$b(a+b-c)$	$b(c-a)$
C	ab	$b(c-a)$	0

Symbols from C :

	A	B	C
A	0	ab	$a(c-b)$
B	ab	$b(c-a)$	0
C	$a(c-b)$	0	$c^2 - a(c-b)$

This finishes the existence proof for (a, b, c)-forbidden latin squares. □

Next we make a simple observation, that in an $(a, b, c)-$bicolored STS, the size c satisfies $c \leq (a + b)$ unless $(a, b, c) = (0, 1, 2)$ (and $v = 3$) or $(a, b, c) = (1, 2, 4)$ (and $v = 7$). The proof is easy: By the divisibility condition, $\binom{a}{2}$ +

$\binom{b}{2}+\binom{c}{2}$ must equal $\binom{a+b+c}{2}/3$. Simplify to get $a^2+b^2+c^2-(a+b+c)-(ab+ac+bc)=0$. Now assume that $c>a+b$. The left side of this expression is minimized when $a=b$, but then one can check that for this side to be less than or equal to 0, one must have $v \leq 16$. Now these remaining orders can be checked by hand.

This underlies the one exceptional case in the direct product to follow.

Theorem 2.4 *If there exists an $(a,b,c)-$bicolorable STS(u) with $c=max(a,b,c)$ and $c \leq a+b$, and if there exists an $(x,y,z)-$bicolorable STS(v), then there exists an $(ax+by+cz, ay+bz+cx, az+bx+cy)-$bicolorable STS(uv).*

Proof. Define sets V_{ij} of elements with $0 \leq i < v$ and $j \in \{0,1,2\}$, so that V_{ij} has a,b,c elements for $j=0,1,2$ respectively, when $0 \leq i < x$; b,c,a elements for $j=0,1,2$ respectively, when $x \leq i < x+y$; and c,a,b elements for $j=0,1,2$ respectively, when $x+y \leq i < v$. The union of V_{ij} for $0 \leq i < v$ then has $ax+by+cz$, $bx+cy+az$, and $cx+ay+bz$ elements for $j=0,1,2$, respectively.

For $i=0,\dots,v-1$, place on the union of V_{ij} for $j=0,1,2$ an $(a,b,c)-$bicolored STS(u) in which the color classes are V_{i0}, V_{i1} and V_{i2}. Now choose an $(x,y,z)-$bicolored STS(v) in which the color classes are $\{0,...,x-1\}$, $\{x,...,x+y-1\}$, $\{x+y,...,v-1\}$, and let $\mathcal{B}$ be its blocks. Call the colors in this coloring $0,1$ and 2. Whenever $\{f,g,h\} \in \mathcal{B}$, suppose without loss of generality that f and g have the same color k in the bicoloring of the STS(v). When h has color $(k+2)$ mod 3, form an $(a,b,c)-$forbidden latin square; when h has color $(k+1)$ mod 3, instead form a $(b,a,c)-$forbidden latin square. Use the latin square to construct triples in the obvious way (i.e., form the transversal design TD$(3,u)$ from the latin square and align the row, column, and symbol classes of sizes a,b,c on the corresponding $V'_{fj}s, V'_{gj}s$ and $V'_{hj}s$).

The result is a bicolorable STS(uv) whose color classes have the specified sizes. ∎

Corollary 2.5 *If there exists a 3-bicolorable STS(u) and a 3-bicolorable STS(v), then there exists a 3-bicolorable STS(uv).*

Proof. The corollary follows from Theorem 2.4 except possibly when u and v both are 3 or 7. Examples of 3-bicolorable STS(9), STS(21), and STS(49) are easily found. ∎

If there exists a 3-bicolorable STS(v) with a 3-split (a, b, c), then (2) must be satisfied. Using (2) for $v \equiv 1, 3 \pmod 6$, $v \leq 97$ yields the following solutions.

v	a	b	c	$\exists$?	v	a	b	c	$\exists$?	v	a	b	c	$\exists$?	v	a	b	c	$\exists$?
7	4	2	1	yes	9	4	4	1	yes	9	5	2	2	dupl	13	6	5	2	yes
15		none			19	9	6	4	yes	21	9	8	4	yes	21	10	6	5	yes
25	10	10	5	yes	27	12	9	6	yes	31	14	9	8	yes	33		none		
37	16	12	9	yes	39	16	14	9	yes	39	17	12	10	yes	43	17	16	10	yes
45		none			49	20	17	12	yes	49	21	14	14	yes	51		none		
55		none			57	22	21	14	yes	57	24	17	16	yes	61	25	20	16	yes
63	25	22	16	yes	63	26	20	17	yes	67	26	24	17	yes	69		none		
73	30	22	21	yes	75	30	25	20	yes	79	32	25	22	yes	81	30	30	21	yes
81	33	24	24	yes	85		none			87		none			91	34	33	24	yes
91	36	30	25	yes	93	36	32	25	yes	93	37	30	26	yes	97	37	34	26	yes

We have obtained solutions for all of these meeting the duplicity condition, using Stinson's hill-climbing algorithm for triple systems [11] (or see [2], p. 730) modified so that only 2-colored triples could be constructed. We have also constructed a 3-bicolorable STS(v)s for all $v \equiv 1, 3 \pmod 6$, $99 \leq v < 1000$, satisfying the condition given in Theorem 2.2 and every possible 3-split satisfying the divisibility condition.

Theorem 2.6 *For every* $v \equiv 1, 3 \pmod 6$*,* $v < 1000$ *satisfying the necessary condition given in Theorem 2.2 and for all 3-splits* (a, b, c) *satisfying the divisibility condition, there exists a 3-bicolorable STS(v) with color classes of sizes* $a, b,$ *and* c*.*

Conjecture 2.7 *For every* $v \equiv 1, 3 \pmod 6$*, satisfying the condition in Theorem 2.2 and for all 3-splits* (a, b, c) *for* v *satisfying the divisibility and duplicity conditions, there exists a 3-bicolorable STS(v) with color classes of sizes* $a, b,$ *and* c*.*

¿From Theorem 2.6 and Corollary 2.5, we have our main result on the existence of 3-bicolorable STSs:

Theorem 2.8 *Let* $v \equiv 1, 3 \pmod 6$ *and assume that in the prime factorization of* v *no prime congruent to 5 (mod 6) appears with an odd exponent. Further assume that all prime factors* p *congruent to 1 (mod 6) are less than 1000 and that all prime factors* p *congruent to 5 (mod 6) satisfy* $p^2 < 1000$*, then there exists a 3-bicolorable STS*(v)*.*

3 4- and 5-bicolorable STS

The following has been observed, in effect, in [6].

Theorem 3.1 *If there exists an m-bicolorable STS(v) with m-split $(c_1, \dots, c_m)$, then there exists an $(m+1)$-bicolorable STS($2v+1$) with an $(m+1)$-split $(v+1, c_1, \dots, c_m)$.*

Proof. Apply the well-known $2v+1$ construction (cf. [3], Chapter 3), and color all elements not in the sub-STS(v) with a new color. □

A variant of this gives a different $(m+1)$-split:

Theorem 3.2 *If there exists an m-bicolorable STS(v) with m-split $(c_1, \dots, c_m)$, then there exists an $(m+1)$-bicolorable STS($2v+1$) with an $(m+1)$-split $(2c_1, 2c_2, \dots, 2c_m, 1)$.*

Proof. Let $(V, \mathcal{B})$ be the m-bicolorable STS(v). We form an STS($2v+1$) on $(V \times \{0,1\}) \cup \{\infty\}$ as follows. Whenever $\{x, y, z\} \in \mathcal{B}$, form the blocks $\{x_i, y_j, z_k\}$ for $(i, j, k) =$ (0,0,0), (0,1,1), (1,0,1), and (1,1,0). Then form the blocks $\{\infty, x_0, x_1\}$ for $x \in X$. Color x_i for $i \in \{0, 1\}$ using the same color as was assigned to x in the STS(v). Then assign ∞ a new color. □

That there exist m-bicolorable STSs with arbitrarily large m is now immediate. However, in this section we restrict our attention to the case of 4- and 5-bicolorings as these display some marked contrasts to the case of 3-bicolorings. For example, while Theorem 2.2 ensures that there exist infinitely many admissible orders v for which no 3-bicolorable STS can exist, this does not appear to be so in the case of 4- or 5-bicolorings.

Another somewhat less obvious recursion is proved:

Theorem 3.3 *If there exists a m-bicolorable STS(v) with m-split $(c_1, \dots, c_m)$, then there exists a $(m+2)$-bicolorable STS($5v+4$) with $(m+2)$-split $(2v+2, 2v+2, c_1, \dots, c_m)$.*

Proof. Let $V = \{a_j : j = 1, \dots, v\}$, and suppose $(V, \mathcal{B})$ is the m-colorable STS(v). We construct a $(m+2)$-bicolorable STS($5v+4$) on the $(5v+4)$-set $W = Z_{2v+2} \times \{1, 2\} \cup V$ as follows.

Consider the graph G with the vertex-set Z_{2v+2} and the edge-set $\{\{x, y\} : |x - y| \in \{(v+3)/2, (v+5)/2, \dots, v+1\}$; G is regular of degree v, and thus, by the Stern-Lenz Lemma (cf. [3], Chapter 1), G has a 1-factorization, say $\{F_1, \ldots, F_v\}$. Let further $S = \{(a_r, b_r) : r = 1, 2, \ldots, (v+1)/2\}, b_r - a_r = r$ be a Skolem sequence of order $(v+1)/2$ if $(v+1)/2 \equiv 0, 1 \pmod 4$, or a hooked Skolem sequence of order $(v+1)/2$ if $(v+1)/2 \equiv 2, 3 \pmod 4$ (cf. [3], Chapter 2).

Define now the following sets of triples (for the sake of brevity we write x_i for (x, i) etc.).

$\mathcal{P} = \{a_j, x, y\} : \{x, y\} \in F_j^i, i = 1, 2, j = 1, 2, \ldots, v\}$,

$\mathcal{Q} = \{\{i_2, (i + r - 1)_1, (i + a_r - 1)_1\} : r = 1, \ldots, (v+1)/2, i \in Z_{2v+2}\}$,

$\mathcal{R} = \{\{i_1, (i + 2v + 1 - r)_2, (i + 2v + 1 - a_r)_2\} : r = 1, \ldots, (v+1)/2, i \in Z_{2v+2}\}$.

Then $(W, \mathcal{B} \cup \mathcal{P} \cup \mathcal{Q} \cup \mathcal{R})$ is an STS$(5v + 4)$. Moreover, this STS is $(m + 2)$-bicolorable with color classes $Z_{2v+2} \times \{1\}$, $Z_{2v+2} \times \{2\}$, and the m color classes of the original m-bicoloring of $(V, \mathcal{B})$. ∎

A specific recursion for special 4-splits can also be established:

Theorem 3.4 *Suppose that there exists a 4-bicolorable STS(v) with 4-split $(x, y, z, 1)$ and a 4-bicolorable STS(u) with 4-split $(a, b, c, 1)$, so that $c = max(a, b, c)$ and $c \leq a + b$. Then there exists a 4-bicolorable STS($\frac{1}{2}(u-1)(v-1)+1$) with 4-split $(\frac{1}{2}(ax+by+cz), \frac{1}{2}(ay+bz+cx), \frac{1}{2}(az+bx+cy), 1)$.*

Proof. Define sets V_{ij} of elements with $0 \leq i < \frac{v-1}{2}$ and $j \in \{0, 1, 2\}$, so that V_{ij} has a, b, c elements for $j = 0, 1, 2$ respectively, when $0 \leq i < \frac{x}{2}$; b, c, a elements for $j = 0, 1, 2$ respectively, when $\frac{x}{2} \leq i < \frac{x+y}{2}$; and c, a, b elements for $j = 0, 1, 2$ respectively, when $\frac{x+y}{2} \leq i < \frac{v-1}{2}$. The union of V_{ij} for $0 \leq i < \frac{v-1}{2}$ then has $\frac{1}{2}(ax + by + cz)$, $\frac{1}{2}(ay + bz + cx)$, $\frac{1}{2}(az + bx + cy)$ elements for $j = 0, 1, 2$, respectively. Add a new point ∞.

For $i = 0, \ldots, \frac{v-1}{2} - 1$, place on the union of V_{ij} for $j = 0, 1, 2$ together with ∞, an $(a, b, c, 1)-$bicolored STS(u) in which the color classes are V_{i0}, V_{i1}, V_{i2}, and $\{\infty\}$. Next consider the 4-bicolored STS(v) with color classes $\{i, \overline{i} : 0 \leq i < \frac{x}{2}\}$, $\{i, \overline{i} : \frac{x}{2} \leq i < \frac{x+y}{2}\}$, $\{i, \overline{i} : \frac{x+y}{2} \leq i < \frac{v-2}{2}\}$, and $\{\infty\}$. Call the colors in this coloring $0, 1$, 2, and 3, where ∞ gets color 3. Partition each set V_{ij} into two sets X_{ij} and $\overline{X}_{ij}$ of equal size, for each $0 \leq i < \frac{v-1}{2}$ and $j \in \{0, 1, 2\}$. Consider a block of the STS(v) not containing ∞, say $\{f, g, h\}$. For each element $t \in \{f, g, h\}$, choose one of X_{ij} or $\overline{X}_{ij}$ depending upon whether t is of the form i or $\overline{i}$. On the three corresponding sets, latin squares of side $\frac{u-1}{2}$ are used to form triples as in Theorem 2.4. ∎

Table 1 lists all possible 4-splits for $v \leq 99$, and Table 2 all possible 5-splits for $v \leq 87$ meeting the divisibility conditions. The entry 'yes' indicates that the corresponding STS has been found by hill climbing or by the constructions given in this paper. The entries 'dupl', 'odd', and 'den' indicate that the parameters fail the duplicity, oddity, or density condition, respectively. As in the case of 3-bicoloring, we have actually generated all possible 4-splits satisfying the divisibility and duplicity conditions for all $v \leq 157$, and all possible 5-splits satisfying the divisibility, duplicity, oddity, and density conditions for $v \leq 105$. We then used the hill-climbing algorithm to verify that in each case there exists a bicolorable STS with the corresponding split.

In view of the above, the following conjectures appear reasonable.

Conjecture 3.5 *A 4-bicolorable STS(v) exists if and only if* $v \equiv 1, 3 \pmod 6$ *,* $v \geq 15, v \neq 21, 31$.

Conjecture 3.6 *A 5-bicolorable STS(v) exists if and only if* $v \equiv 1, 3 \pmod 6$*,* $v \geq 31, v \neq 33, 37, 43, 45, 61$.

4 Conclusion

If an STS(v) is m-bicolorable then $m \leq \lceil log_2(v + 1)\rceil$ [5, 6]. Moreover, for all $v = 2^n - 1$ there exists an STS(v) for which this bound is attained. The corresponding STS is the projective STS(v) (see, e.g., [3]).

If we define the spectrum $\mathcal{C}(v)$ for bicolorings by $\mathcal{C}(v) = \{m$: there exists an m-bicolorable STS(v) $\}$ then we have $\mathcal{C}(7) = \mathcal{C}(9) = \mathcal{C}(13) = \{3\}$, $\mathcal{C}(15) = \{4\}$, $\mathcal{C}(19) = \{3, 4\}$, $\mathcal{C}(21) = \{3\}$, $\mathcal{C}(25) = \mathcal{C}(27) = \{3, 4\}$, $\mathcal{C}(31) = \{3, 5\}$, $\mathcal{C}(33) = \{4\}$.

Thus $\mathcal{C}(v)$ is not necessarily an interval. Can one characterize those orders v for which $\mathcal{C}(v)$ is an interval? It is conceivable that 31 is the only admissible order v for which $\mathcal{C}(v)$ is *not* an interval.

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v	a	b	c	d	$\exists$?	v	a	b	c	d	$\exists$?	v	a	b	c	d	$\exists$?
15	8	4	2	1	yes	19	10	5	2	2	dupl	19	10	4	4	1	yes
19	8	8	2	1	yes	21			none			25	12	8	4	1	yes
27	14	6	5	2	yes	27	12	10	4	1	yes	31			none		
33	16	10	5	2	yes	33	16	8	8	1	yes	33	14	13	4	2	yes
37	16	14	5	2	yes	39	20	9	6	4	yes	39	18	13	6	2	yes
39	18	12	8	1	yes	43	22	10	6	5	yes	43	22	9	8	4	yes
43	21	10	10	2	yes	43	20	12	10	1	yes	43	18	17	4	4	yes
43	18	16	8	1	yes	45	22	13	6	4	yes	45	21	14	8	2	yes
49	24	14	6	5	yes	49	21	18	8	2	yes	51	26	10	10	5	yes
51	25	14	8	4	yes	51	20	20	10	1	yes	55	28	12	9	6	yes
55	26	17	8	4	yes	55	24	18	12	1	yes	57	24	22	6	5	yes
57	24	21	10	2	yes	57	24	20	12	1	yes	61	30	16	10	5	yes
61	30	14	13	4	yes	61	29	18	10	4	yes	61	28	20	9	4	yes
61	28	18	13	2	yes	61	25	24	8	4	yes	63	32	14	9	8	yes
63	29	18	14	2	yes	63	28	22	9	4	yes	63	28	18	16	1	yes
67	33	18	8	8	yes	67	32	20	9	6	yes	67	30	21	14	2	yes
69	34	16	14	5	yes	69	32	22	10	5	yes	69	29	26	10	4	yes
69	28	26	13	2	yes	73	36	18	13	6	yes	73	34	22	13	4	yes
73	33	24	12	4	yes	73	32	26	10	5	yes	73	32	20	20	1	yes
73	29	28	14	2	yes	73	28	28	16	1	yes	75	38	16	12	9	yes
75	37	18	14	6	yes	75	36	22	9	8	yes	75	36	18	17	4	yes
75	34	21	18	2	yes	75	33	26	12	4	yes	75	33	24	16	2	yes
75	32	28	9	6	yes	75	32	24	18	1	yes	75	30	30	10	5	yes
79	40	17	12	10	yes	79	40	16	14	9	yes	79	38	22	13	6	yes
79	36	25	14	4	yes	79	34	29	10	6	yes	79	34	24	20	1	yes
79	33	28	16	2	yes	79	32	28	18	1	yes	81	40	21	10	10	yes
81	38	24	14	5	yes	81	34	30	13	4	yes	81	33	32	8	8	yes
85	42	21	14	8	yes	85	41	24	12	8	yes	85	38	29	12	6	yes
85	38	24	21	2	yes	85	36	32	9	8	yes	85	36	29	18	2	yes
87	44	17	16	10	yes	87	41	22	20	4	yes	87	40	26	17	4	yes
87	34	32	20	1	yes	91	44	25	14	8	yes	91	38	34	14	5	yes
93	46	21	18	8	yes	93	45	26	12	10	yes	93	45	22	20	6	yes
93	44	28	12	9	yes	93	42	30	16	5	yes	93	42	29	18	4	yes
93	40	30	21	2	yes	93	38	36	13	6	yes	97	46	29	12	10	yes
97	46	24	22	5	yes	97	45	26	22	4	yes	97	42	34	16	5	yes
97	40	37	14	6	yes	97	40	36	17	4	yes	97	40	34	21	2	yes
97	40	32	24	1	yes	99	50	21	14	14	yes	99	50	20	17	12	yes
99	49	24	16	10	yes	99	48	26	17	8	yes	99	46	25	24	4	yes
99	42	37	10	10	yes	99	42	28	28	1	yes	99	41	38	12	8	yes
99	40	34	24	1	yes	99	38	38	21	2	yes						

Table 1: 4-splits

v	a	b	c	d	e	∃?	v	a	b	c	d	e	∃?	v	a	b	c	d	e	∃?
31	16	8	4	2	1	yes	33			none				37			none			
39	20	10	5	2	2	dupl	39	20	10	4	4	1	yes	39	20	8	8	2	1	yes
39	19	11	7	1	1	odd	39	17	15	3	3	1	odd	39	16	16	4	2	1	yes
43	21	13	5	3	1	odd	43	21	11	9	1	1	odd	45			none			
49	20	20	5	2	2	dupl	49	20	20	4	4	1	yes	51	26	13	6	4	2	den
51	26	12	8	4	1	yes	51	25	15	5	5	1	odd	51	24	17	4	4	2	den
51	24	16	8	2	1	yes	51	23	17	9	1	1	odd	55	28	14	6	5	2	yes
55	28	12	10	4	1	yes	55	27	15	9	3	1	odd	55	26	18	5	4	2	den
55	25	19	7	3	1	odd	55	24	20	8	2	1	yes	57	28	17	4	4	4	den
57	28	16	8	4	1	yes	57	26	20	5	4	2	den	61	30	18	5	4	4	den
61	28	21	6	4	2	den	63	31	15	13	3	1	odd	63	30	20	6	5	2	yes
67	34	17	8	4	4	den	67	34	16	10	5	2	yes	67	34	16	8	8	1	yes
67	34	14	13	4	2	yes	67	32	21	6	6	2	den	67	32	20	10	4	1	yes
67	32	16	16	2	1	yes	67	31	17	17	1	1	odd	67	30	24	6	5	2	yes
67	29	25	7	5	1	odd	67	29	23	13	1	1	odd	67	28	26	8	4	1	yes
69	34	20	6	5	4	den	69	30	26	5	4	4	den	69	30	25	10	2	2	dupl
69	28	28	6	5	2	yes	73	36	21	6	6	4	den	73	36	20	10	5	2	yes
73	36	20	8	8	1	yes	73	36	16	16	4	1	yes	73	34	24	8	5	2	yes
73	30	29	8	4	2	yes	75	38	16	14	5	2	yes	75	37	21	9	5	3	odd
75	37	19	13	5	1	odd	75	35	25	5	5	5	odd	75	34	26	8	5	2	yes
75	33	27	9	5	1	odd	75	32	28	10	4	1	yes	79	40	20	9	6	4	yes
79	40	18	13	6	2	yes	79	40	18	12	8	1	yes	79	39	21	11	7	1	odd
79	39	19	15	5	1	odd	79	38	24	9	4	4	yes	79	37	26	6	6	4	den
79	37	25	11	3	3	odd	79	37	23	15	3	1	odd	79	36	26	12	4	1	yes
79	36	24	16	2	1	yes	79	35	27	13	3	1	odd	79	34	30	6	5	4	den
79	33	31	7	5	3	odd	81	38	26	9	6	2	yes	81	37	28	6	6	4	den
81	36	29	8	6	2	den	81	36	28	12	4	1	yes	85	42	24	8	6	5	den
85	41	26	6	6	6	den	85	40	26	13	4	2	yes	85	38	30	9	6	2	yes
87	44	22	10	6	5	yes	87	44	22	9	8	4	yes	87	44	21	10	10	2	yes
87	44	20	12	10	1	yes	87	44	18	17	4	4	yes	87	44	18	16	8	1	yes
87	43	23	13	5	3	odd	87	43	23	11	9	1	odd	87	42	26	10	5	4	yes
87	42	20	20	4	1	yes	87	41	28	8	6	4	den	87	41	27	11	7	1	odd
87	41	23	19	3	1	odd	87	40	29	10	6	2	yes	87	40	24	20	2	1	yes
87	39	31	7	5	5	odd	87	39	23	23	1	1	odd	87	38	32	9	4	4	yes
87	36	34	10	5	2	yes	87	36	34	8	8	1	yes	87	36	32	16	2	1	yes
87	35	35	9	7	1	odd														

Table 2: 5-splits

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