Improved LP Lower Bounds for Difference Triangle Sets

James B. Shearer
IBM Watson Research
P.O. Box 218
Yorktown Heights, NY 10598 U.S.A.
email: jbs@watson.ibm.com

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Abstract. In 1991 Lorentzen and Nilsen showed how to use linear programming to prove lower bounds on the size of difference triangle sets. In this note we show how to improve these bounds by including additional valid linear inequalities in the LP formulation. We also give some new optimal difference triangle sets found by computer search.

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Following Kløve [2] we define an (I, J) difference triangle set, Δ, as a set of integers $\{a_{ij} \mid 1 \leq i \leq I,\, 0 \leq j \leq J\}$ such that all the differences $a_{ij} - a_{ik}$, $1 \leq i \leq I$, $0 \leq k < j \leq J$ are positive and distinct. Let $m = m(\Delta)$ be the maximum difference. The difference triangle set problem is to minimize m given I and J. Kløve defined $M(I, J)$ as this minimum. Lorentzen and Nilsen [3] showed how to use linear programming (LP) methods to prove lower bounds on $M(I, J)$. This was a generalization and improvement of earlier lower bounds. Here we show how to improve the LP bound (for $I > 1$) by adding inequalities to the LP formulation. We also announce some new values of $M(I, J)$ found by computer search.

Given a difference triangle set $\{a_{ij} \mid 1 \leq i \leq I,\, 0 \leq j \leq J\}$ we have associated difference triangles $\{X_{ijk} \mid 1 \leq i \leq I,\, 1 \leq j \leq J,\, 1 \leq k \leq J+1-j\}$ where $X_{ijk} = a_{i,j+k-1} - a_{i,k-1}$. The ith difference triangle is $\{X_{ijk} \mid 1 \leq j \leq J,\, 1 \leq k \leq J+1-j\}$. Its top row is $\{X_{i1k} \mid 1 \leq k \leq J\}$. Clearly each difference triangle is determined by its top row. The maximum difference in each difference triangle is its bottom element X_{iJ1} which is the sum of the top row.

We can now formulate a linear program which gives a lower bound for $m = M(I, J)$. The LP variables will be m and the top elements $\{X_{i1k} \mid 1 \leq i \leq I, 1 \leq k \leq J\}$ of the difference triangles. Minimizing m will be the objective. Clearly since m is the maximum difference we have:

$$m \geq X_{iJ1} = \sum_{k=1}^{J} X_{i1k} \qquad 1 \leq i \leq I \tag{1}$$

Also since the differences are all distinct positive integers we know that the sum of any n of them will be at least as large as the sum of the first n positive integers or $n(n+1)/2$. So if S is a subset of the differences, we have:

$$\sum_{X_{ijk} \in S} X_{ijk} \geq n(n+1)/2 \qquad \text{where } \ |S| = n \tag{2}$$

(since $X_{ijk} = \sum_{h=k}^{j+k-1} X_{i1h}$ the inequalities (2) can be expressed in terms of our LP variables).

This is essentially the LP formulation used by Lorentzen and Nilsen [3] to bound $M(I,J)$. In practice when solving the LP, we use symmetry to reduce the IJ variables $\{X_{i1k} \mid 1 \leq i \leq I, 1 \leq k \leq J\}$ to $\lfloor (J+1)/2 \rfloor$ variables (since the difference triangles can be permuted and reflected). Also we do not use all the inequalities (2). Instead we start with a small subset and solve the LP. Then we check if any of the unused inequalities are violated by the solution. If we find a violated inequality, we add it to the subset and resolve the LP repeating as necessary. If the number of inequalities becomes too large, we delete some of the non-binding ones. In practice this procedure rapidly produces a solution without having to solve large LP's.

Given a solution to a subset problem we can quickly check the inequalities (2) as follows. We use the solution to compute the remaining elements of the difference triangles. We next sort all $IJ(J+1)/2$ elements from smallest to largest. We then compute the partial sums and compare to the corresponding partial sums of the smallest $IJ(J+1)/2$ positive integers. If, for example, we find the sum of the n smallest differences is less than $n(n+1)/2$, then we have found a violated inequality (2) which can be added to our subset LP. Otherwise we have shown the solution to the subset LP is also a solution to the entire LP. In practice the above procedure is modified to reflect the reduced number of LP variables due to symmetry.

The above produces the bounds on $M(I,J)$ in Lorentzen and Nilsen [3]. Using the following lemma we can improve these bounds when $I > 1$.

Lemma 1: Suppose we have a set, S, containing n distinct integers with average r. Let m be the maximum element of S. Then we have $m \geq r + (n-1)/2$.

Proof: Let s be the sum of the elements of S. Clearly $s = nr$. Also since the elements of S are distinct with maximum m we must have $s \leq m+(m-1)+\cdots+(m-n+1) = mn - \frac{(n-1)n}{2}$. So $nr \leq mn - \frac{(n-1)n}{2}$ or dividing by n and rearranging $m \geq r + (n-1)/2$ as claimed. ■

Now we can use Lemma 1 to strengthen the inequalities (1) when $I > 1$ as follows. By symmetry the LP solution will have $m = X_{iJ1}$ $1 \leq i \leq I$. However the X_{iJ1} are actually distinct positive integers and m is their maximum. This means using Lemma 1 we may replace (1) with

$$m \geq \frac{1}{J} \sum_{i=1}^{J} X_{iJ1} + (I-1)/2 \tag{3}$$

Clearly this will improve the lower bound on m by $(I-1)/2$. However we can do still better. (3) is just one of a family of inequalities for m each corresponding to a subset T of the $IJ(J+1)/2$ differences and derived by use of Lemma 1. When T is the set of the I bottom elements of the difference triangles, we obtain (3). In general, we have

$$m \geq \frac{1}{n} \sum_{X_{ijk} \in T} X_{ijk} + (n-1)/2 \qquad \text{where } |T| = n \tag{4}$$

Again as with (2) these inequalities can be rewritten in terms of our LP variables. Similarly it is easy to check given a solution of a LP containing a subset of the inequalities (4) whether any of the remaining inequalities (4) are violated. Simply compute and sort the differences X_{ijk} and verify (4) for sets T consisting of the n largest differences $1 \leq n \leq IJ(J+1)/2$. Again in practice this procedure is modified to reflect the reduced number of variables due to symmetry.

Remark 1: Let S and T be the entire set of differences in (2) and (4) respectively. Then we have $m \geq \frac{n+1}{2} + \frac{n-1}{2} = n$ where $n = IJ(J+1)/2$. So the bound given by our LP formulation is always at least as good as the trivial bound $M(I,J) \geq IJ(J+1)/2$ from the fact that all $IJ(J+1)/2$ differences must be distinct. This is not true for Lorentzen and Nilsen's formulation.

We have solved the LP with inequalities (2) and (4) using the above described iterative procedure for $1 \leq I \leq 15, 5 \leq J \leq 20$ (for $J \leq 4$ the solution gives the trivial bound $M(I,J) \geq IJ(J+1)/2$ mentioned in Remark 1 above). The resulting lower bounds on $M(I,J)$ are listed in Table 1. These bounds are generally improvements (compare [1], [3]). In most cases however there is still a considerable gap between these bounds and the best upper bounds known. For example the lower bound for $M(15,10)$ in [1] is 958, the lower bound using Lorentzen and Nilsen's formulation is 962 and the lower bound in this paper is 974. This remains far from the best upper bound known for $M(15,10)$, 1415 (from [1]). The fault may be more with the upper bound than with the lower bound however as the known ways of constructing (I,J) difference triangle sets do not seem to be very good for large I. In a few cases the lower bound in this paper is known to be quite good. For example exhaustive search (see below) has shown $M(5,5) = 79$ (the claim that $M(5,5) = 83$ in [2] and propagated elsewhere is incorrect). The Lorentzen and Nilsen [3] lower bound for this case is 75, the lower bound in this paper is 77.

Remark 2: For small values of J we can solve the LP for all values of I obtaining:

$$M(I,5) \geq (91I+6)/6 \tag{5}$$
$$M(I,6) \geq (179I+9)/8 \tag{6}$$

We have also found some new exact values of $M(I,J)$ by computer search using a program similar to that described in [4]. We have $M(2,9) = 121$, $M(3,7) = 100$, $M(5,5) = 79$ and $M(9,4) = 91$. Examples obtaining these values are given in Table 2. The first two are unique, the third is probably not unique and the last is far from unique. The web page `<http://www.research.ibm.com/people/s/shearer/dtsub.html>` lists these values as well as numerous additional improvements on the results in [4].

Table 1 - Lower Bounds for M(I,J)

J/I	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
5	17	32	47	62	77	92	108	123	138	153	168	183	199	214	229
6	24	46	69	91	113	136	158	181	203	225	248	270	292	315	337
7	32	63	94	124	155	186	217	247	278	309	339	370	401	431	462
8	43	83	124	164	205	245	286	327	367	408	448	489	529	570	610
9	54	106	158	209	261	313	365	417	469	521	572	624	676	728	780
10	67	132	197	261	326	391	456	520	585	650	715	780	844	909	974
11	82	161	240	320	399	478	558	637	716	796	875	954	1034	1113	1193
12	98	194	289	385	480	576	671	767	862	958	1054	1149	1245	1340	1436
13	116	229	343	456	569	683	796	909	1023	1136	1249	1363	1476	1589	1703
14	136	268	401	534	667	800	933	1066	1198	1331	1464	1597	1730	1863	1996
15	157	311	464	618	772	926	1080	1234	1388	1542	1696	1850	2003	2157	2311
16	180	356	533	709	886	1062	1239	1415	1592	1768	1945	2121	2298	2475	2651
17	204	405	606	807	1007	1208	1409	1610	1811	2012	2213	2413	2614	2815	3016
18	230	457	684	910	1137	1364	1591	1818	2044	2271	2498	2725	2952	3178	3405
19	258	512	767	1021	1276	1530	1784	2039	2293	2548	2802	3057	3311	3565	3820
20	287	571	855	1139	1422	1706	1990	2274	2557	2841	3125	3409	3692	3976	4260

Table 2 - New Values for M(I,J)

M(2,9) = 121

0	4	13	45	46	69	94	109	116	121
0	3	19	21	29	57	87	101	107	118

M(3,7) = 100

0	12	15	31	55	87	88	93
0	7	30	41	51	77	90	99
0	2	29	37	54	82	96	100

M(5,5) = 79

0	12	20	41	72	75
0	6	25	49	62	79
0	4	15	51	65	74
0	2	18	44	66	71
0	1	33	40	68	78

M(9,4) = 91

0	26	42	79	90
0	20	25	69	76
0	17	21	82	84
0	13	35	75	87
0	10	33	80	83
0	9	38	66	81
0	8	27	68	86
0	6	45	77	91
0	1	31	55	89

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