On the Theory Of Pfaffian Orientations.
II. T-joins, k-Cuts, and Duality of Enumeration.

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Abstract

This is a continuation of our paper "A Theory of Pfaffian Orientations I: Perfect Matchings and Permanents". We present a new combinatorial way to compute the generating functions of T-joins and k-cuts of graphs. As a consequence, we show that the computational problem to find the maximum weight of an edge-cut is polynomially solvable for the instances (G, w) where G is a graph embedded on an arbitrary fixed orientable surface and the weight function w has only a bounded number of different values. We also survey the related results concerning a duality of the Tutte polynomial, and present an application for the weight enumerator of a binary code. In a continuation of this paper which is in preparation we present an application to the Ising problem of three-dimensional crystal structures.

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1 Introduction

This is a continuation of our paper "A Theory of Pfaffian Orientations I: Perfect Matchings and Permanents". We present a new combinatorial way to compute the generating functions of T-joins and k-cuts of graphs.

A graph is a pair $G = (V, E)$ where V is a set and E is a set of unordered pairs of elements of V. The elements of V are called *vertices* and those of E are called *edges*. If $e = xy$ is an edge then x, y are called endvertices of e.

In this paper V will always be finite, $G = (V, E)$ will always be a graph and x_e will be a variable associated with each edge e of G. We let $x = (x_e : e \in E)$ denote the vector whose components are indexed by the edges of G and, for $M \subset E$, we let $x(M)$ denote the product of the variables of the edges of M.

A graph (V', E') is called a *subgraph* of graph $G = (V, E)$ if $V' \subset V$ and $E' \subset E$. A *perfect matching* of a graph is a set of pairwise disjoint edges, whose union equals the set of the vertices.

Let $P = v_1, e_1, v_2, e_2, ..., v_i, e_i, v_{i+1}, ..., e_n, v_{n+1}$ be a sequence such that each v_j is a vertex of a graph G, each $e_j = v_j v_{j+1}$ is an edge of G, and $v_i \neq v_j$ for $i < j$ except if $i = 1$ and $j = n + 1$. If also $v_1 \neq v_{n+1}$ then P is called *a path* of G. If $v_1 = v_{n+1}$ then P is called a *cycle* of G. In both cases the *length* of P equals n.

The graph $G = (V, E)$ is *connected* if any pair of vertices is joined by a path, and it is called 2-*connected* if the graph $G_v = (V - \{v\}, \{e \in E; v \notin e\})$ is connected for each vertex v of G. Each maximal 2-connected subgraph of G is called a 2-*connected component* of G.

Definition 1.1 *The generating function of the perfect matchings of G is the polynomial $\mathcal{P}(G, x)$ which equals the sum of $x(P)$ over all perfect matchings P of G.*

A subgraph $G' = (V, E')$ of a graph $G = (V, E)$ is called *eulerian* if the degree of each vertex of G' is even.

Definition 1.2 *The generating function of the eulerian subgraphs of G is the polynomial $\mathcal{E}(G, x)$ which equals the sum of $x(U)$ over all eulerian subgraphs U of G.*

Let $G = (V, E)$ be a graph and $T \subset V$. *A T-join* is a subgraph $G' = (V, E')$ such that the degree of a vertex v of G' is odd if and only if $v \in T$. Eulerian subgraphs are T-joins when $T = \emptyset$.

Definition 1.3 *Let $G = (V, E)$ be a graph and $T \subset V$. The generating function of the T-joins of G is the polynomial $\mathcal{T}_T(G, x)$ which equals the sum of $x(W)$ over all T-joins (V, W) of G.*

Next we consider k-cuts.

Definition 1.4 *Let $k \geq 1$ and let $G = (V, E)$ be a graph. A pair $(\{V_1, ..., V_k\}, E')$ is called k-cut if $\{V_1, ..., V_k\}$ is a partition of V into k non-empty disjoint subsets and E' is the set of all edges with the endvertices in different parts V_i, $i = 1, ..., k$.*

Definition 1.5 *The generating function of the k-cuts is the polynomial $\mathcal{C}_k(G,x)$ which equals the sum of $x(C)$ over all k-cuts $(\{V_1,...,V_k\},C)$ of G.*

An *embedding* of a graph on a surface is defined in a natural way: the vertices are embedded as points, and each edge is embedded as a continuous non-self-intersecting curve connecting the embeddings of its endvertices. The interiors of the embeddings of the edges are pairwise disjoint and the interiors of the curves embedding edges do not contain points embedding vertices.

The *genus* g of a graph G is that of the orientable surface $\mathcal{S} \subset \mathbb{R}^3$ of minimal genus on which G may be embedded.

The following theorem is the main result of [2].

Theorem 1.6 *Let G be a graph of genus g. Then $\mathcal{P}(G,x)$ may be expressed as a linear combination of 4^g square roots of determinants.*

2 T-joins.

There are several ways of relating the problem of counting eulerian subgraphs and the problem of counting perfect matchings. Each method involves replacing each vertex of a given graph by a cluster of vertices in a new "counting" graph. One of these methods turns out to be more appropriate for our purposes since it preserves the genus of the original graph. It was originally proposed by Fisher (see [1]) for counting the number of eulerian subgraphs of a graph. Here we extend his construction in order to solve the more general problem of counting the number of T-joins of a graph.

The construction may be performed in polynomial time and hence, together with Theorem 1.6, yields an algorithm to compute $\mathcal{T}_T(G,x)$.

Other constructions leading to the same result are useful in the study of crystal structures. We will discuss them in greater detail in a forthcoming paper.

Definition 2.1 *Let $G=(V,E)$ be a graph and let $v \in V$. Let $e_1,...,e_k$ be an ordering of the edges of G incident with v. The even splitting of v is a graph $G'=(V',E')$ such that $V' = V - \{v\} \cup \{v_1,...,v_{6k}\}$, and $E' = E - \{e_1,...,e_k\} \cup \{e'_1,...,e'_k\} \cup \{v_t v_{t+1}; 0 < t < 6k\} \cup \{v_{3j+1}v_{3j+3}; j=0,...,2k-1\}$ where e'_i is obtained from e_i by replacing v by $v_{6(i-1)+2}$, $i=1,\ldots,k$. We say that e'_i is the image of e_i in G'.*

The odd splitting of v is obtained from the even splitting of v by deleting vertices v_{6k}, v_{6k-1}, v_{6k-2}.

Definition 2.2 *Let $G=(V,E)$ be a graph and $T \subset V$. We denote by $G_s=(V_s,E_s)$ the graph obtained from G by odd splitting of all vertices of T and even splitting of all vertices of $V-T$. If the edge f' of G_s is the image of the edge f of G then we let $x^s_{f'} = x_f$. We let $x^s_e = 1$ for the remaining edges e of G_s.*

Theorem 2.3 *Let G be embeddable on an orientable surface $\mathcal{S}$ and let each even and odd splitting of a vertex be performed in the clockwise order of the embeddings of its incident edges. Then G_s is also embeddable on $\mathcal{S}$. Moreover $\mathcal{P}(G_s,x^s) = \mathcal{T}_T(G,x)$.*

Proof. The first statement follows from the definition of even and odd splitting. Next, observe that the T-joins W of G are in one-to-one correspondence with the perfect matchings P_W of G_s. Note that P_W contains the set of the images of the edges of W. This together with the choice of x^s implies that $x(W) = x^s(P_W)$, for each T-join W, and the theorem follows. □

3 k-cuts.

In this section we consider the generating function of the multicuts of a graph and we derive an important relation between it and the generating function of the eulerian subgraphs of the same graph. It is well known that for a planar graph G, the cuts of G are in one-to-one correspondence with the eulerian subgraphs of its geometric dual G^*. This correspondence does not hold anymore for graphs embeddable on surfaces of genus greater than zero; in these cases we need a more general duality result, due to van der Waerden (see [8], [10], [4]).

We use the following notation:

$$sinh(z,x) = \frac{z^x - z^{-x}}{2}, \quad cosh(z,x) = \frac{z^x + z^{-x}}{2}, \quad th(z,x) = \frac{sinh(z,x)}{cosh(z,x)}.$$

Note that $sinh(x) = sinh(e,x)$ and $cosh(x) = cosh(e,x)$.

Given a graph $G = (V,E)$, we denote by $\sigma \in \{1,\ldots,k\}^V$ a $|V|$-dimensional vector whose components σ_i, $i = 1,\ldots,|V|$, take values in the set $\{1,\ldots,k\}$. Clearly, any such vector identifies a partition of V into $i \leq k$ disjoint sets and, consequently, an i-cut of G.

Let us denote by δ the vector indexed by the edges of G whose component $\delta_{ij} = \delta(\sigma_i\sigma_j)$, $ij \in E$, equals 1 if $\sigma_i = \sigma_j$ and -1 otherwise.

Moreover, for any $A \subset E$ we let

$$U_k((V,A)) = \sum_{\sigma\in\{1,\ldots,k\}^V} \prod_{ij\in A} \delta(\sigma_i\sigma_j).$$

Theorem 3.1 *Let $G = (V,E)$ be a graph, z a variable and $k > 1$. Then*

$$z^{\sum_{f\in E} x_f}[k + \sum_{i=2}^{k} i!\binom{k}{i}\mathcal{C}_i(G,(z^{-2x_f} : f \in E))] =$$

$$(\prod_{f\in E} cosh(z,x_f)) \sum_{A\subseteq E} U_k((V,A)) \prod_{f\in A} th(z,x_f).$$

Proof. Using the identity

$$z^{x\delta(\sigma_i\sigma_j)} = cosh(z,x) + \delta(\sigma_i\sigma_j)sinh(z,x)$$

the result follows after some algebraic manipulations. In fact,

$$z^{\sum_{f\in E} x_f}[k+\sum_{i=2}^{k} i!\binom{k}{i}\mathcal{C}_i(G,(z^{-2x_f}:f\in E))] = \sum_{\sigma\in\{1,\dots,k\}^V}(\prod_{ij\in E} z^{\delta(\sigma_i\sigma_j)x_{ij}}) =$$

$$\sum_{\sigma\in\{1,\dots,k\}^V}(\prod_{ij\in E}(cosh(z,x_{ij})+\delta(\sigma_i\sigma_j)sinh(z,x_{ij}))) =$$

$$(\prod_{f\in E} cosh(z,x_f))\sum_{\sigma\in\{1,\dots,k\}^V}(\prod_{ij\in E}(1+\delta(\sigma_i\sigma_j)th(z,x_{ij}))) =$$

$$(\prod_{f\in E} cosh(z,x_f))\sum_{\sigma\in\{1,\dots,k\}^V}\sum_{A\subseteq E}(\prod_{ij\in A}\delta(\sigma_i\sigma_j)th(z,x_{ij})) =$$

$$(\prod_{f\in E} cosh(z,x_f))\sum_{A\subseteq E} U_k((V,A))\prod_{f\in A} th(z,x_f)$$

where $\prod_{ij\in A}\delta(\sigma_i\sigma_j)th(z,x_{ij}) = 1$ if $A=\emptyset$. Hence, the theorem follows. □

Specializing the above result to the case of edge-cuts, i.e. $k=2$, we obtain the following result:

Theorem 3.2 *Let G be a graph, z a variable and let $\mathcal{C}'_2(G,x)=\mathcal{C}_2(G,x)+1$. Then*

$$2z^{\sum_{f\in E} x_f}\mathcal{C}'_2(G,(z^{-2x_f}:f\in E)) = (\prod_{f\in E} cosh(z,x_f))2^{|V|}\mathcal{E}(G,(th(z,x_f):f\in E)).$$

Proof. We have, from Theorem 3.1, that

$$2z^{\sum_{f\in E} x_f}\mathcal{C}'_2(G,(z^{-2x_f}:f\in E)) = (\prod_{f\in E} cosh(z,x_f))\sum_{A\subseteq E} U_2((V,A))\prod_{f\in A} th(z,x_f).$$

Now observe that if $A\subset E$ is a cycle and $\sigma\in\{1,2\}^V$ arbitrary then $\prod_{ij\in A}\delta(\sigma_i\sigma_j) = 1$. Hence $U_2((V,A)) = 2^{|V|}$ when (V,A) is an eulerian subgraph. Moreover, if (V,A) is not an eulerian subgraph, then observe that $U_2((V,A)) = 0$.

□

Theorem 3.3 *Let k be a positive integer. Let $\mathcal{G}$ be the class of graphs $G=(V,E)$ such that the edges are partitioned into at most k classes and the variables x_e are equal in each class.*

Then there is a polynomial algorithm which, given $G\in\mathcal{G}$ and $\mathcal{E}(G,x)$ as input, produces $\mathcal{C}_2(G,x)$.

Proof. By Theorem 3.2, we have that

$$\prod_{f\in E} cosh(z,x_f)2^{|V|}\mathcal{E}(G,(th(z,x_f):f\in E)) =$$

$$\prod_{f\in E}\frac{z^{x_f}+z^{-x_f}}{2}2^{|V|}\mathcal{E}(G,(\frac{z^{x_f}-z^{-x_f}}{z^{x_f}+z^{-x_f}}:f\in E))=$$

$$\prod_{f\in E}\frac{z^{2x_f}+1}{2z^{x_f}}2^{|V|}\mathcal{E}(G,(\frac{z^{2x_f}-1}{z^{2x_f}+1}:f\in E))=$$

$$2\prod_{f\in E}z^{x_f}\mathcal{C}_2'(G,(z^{-2x_f}:f\in E)).$$

Hence,

$$\mathcal{C}_2'(G,(z^{-2x_f},f\in E))=2^{|V|-|E|-1}\prod_{f\in E}z^{-2x_f}\mathcal{E}^*(z^{2x_f}:f\in E)),$$

where

$$\mathcal{E}^*(z^{2x_f}:f\in E))=\prod_{f\in E}(z^{2x_f}+1)\mathcal{E}(G,(\frac{z^{2x_f}-1}{z^{2x_f}+1}:f\in E)).$$

Observe that $\prod_{f\in E}z^{-2x_f}\mathcal{E}^*(z^{2x_f}:f\in E))$ is a polynomial $Q(z^{-2x_f}:f\in E)$ in the functions z^{-2x_f} and hence, its nonzero monomials correspond uniquely to nonzero terms of $\mathcal{C}_2'(G,z^{-2x_f}:f\in E)$.

It follows that, given $G\in\mathcal{G}$ and $\mathcal{E}(G,x)$, the polynomial $\mathcal{E}^*(G,x)$ and, consequently, $\mathcal{C}_2(G,x)$ may be expressed in polynomial time.

□

It immediately follows from Theorem 3.3, Theorem 2.3 and Theorem 1.6 that it is possible to find efficiently the maximum weight of an edge-cut for the graphs $G=(V,E)$ embeddable on an arbitrary fixed orientable surface, which have only a bounded number of different edge-weights.

Using similar arguments we can obtain a polynomial time algorithm when the weights are integers bounded in absolute value by a polynomial of the size of the graph. The details of these algorithms will appear elsewhere.

4 A Duality of Enumeration.

We will show now how the duality result proved in the previous section leads to interesting expressions for well-known polynomials studied in combinatorics.

We start by considering the *Tutte polynomial*; it has been defined by Tutte ([6]) and it may be expressed as a minor modification of the Whitney rank generating function ([11]).

Definition 4.1 *Let $G=(V,E)$ be a connected graph. For $A\subset E$ let $r(A)=|V|-c(A)$, where $c(A)$ denotes the number of connected components of (V,A). Then let*

$$T(G,x,y)=\sum_{A\subset E}(x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

$T(G,x,y)$ is called the Tutte polynomial of graph G.

More generally, the Tutte polynomial of a matroid (see [9] for basic notions of matroid theory) is defined as follows.

Definition 4.2 *Let M be a matroid on set E. For $A \subset E$ let $r(A)$ denote the rank of A in M. Then let*

$$T(M, x, y) = \sum_{A \subset E} (x-1)^{r(E)-r(A)}(y-1)^{|A|-r(A)}.$$

$T(M, x, y)$ is called the Tutte polynomial of matroid M.

For example if G is a graph and $\mathcal{N}_G$ the graphic matroid of G then $T(G, x, y) = T(\mathcal{N}_G, x, y)$.

If M is a matroid and M^* its dual then $r^*(E) - r^*(A) = |A| - r(A)$ and we immediately get that $T(M, x, y) = T(M^*, y, x)$. This relation is called *duality of the Tutte polynomial.*

4.1 Weight enumerator of a linear code.

Let $V = \mathbf{F}^n$ be a vector space over a field $\mathbf{F}$. Each subspace C of V of dimension k is called a *linear code of length n and dimension k.* The elements of a linear code are called *codewords.* The *weight* of a codeword is the number of its nonzero entries. The *weight distribution* of C is the sequence $A_0, A_1, \ldots, A_n$ where A_i equals the number of codewords of C of weight i, $0 \le i \le n$.

The *dual code* of C is denoted by C^* and consists of all those n-tuples $(d_1, \ldots, d_n)$ of $\mathbf{F}^n$ satisfying

$$c_1 d_1 + \ldots + c_n d_n = 0$$

for all codewords $(c_1, \ldots, c_n) \in C$. Hence, C^* is a code of length n and dimension $n-k$.

The *weight enumerator* of C is the polynomial

$$A_C(t) = \sum_{i=0}^{n} A_i t^i.$$

The following theorem was proved by MacWilliams ([5]) and it states a fundamental relation between the weight enumerators of C and of its dual C^*.

Theorem 4.3 *Let C be a linear code of length n and dimension k over $GF[q]$ and $1 + (q-1)t \neq 0$. Then*

$$A_{C^*}(t) = \frac{[1+(q-1)t]^n}{q^k} A_C\left(\frac{(1-t)}{1+(q-1)t}\right).$$

If the linear code of length n is given as the row space of a $k \times n$ matrix A over a field $\mathbf{F}$, i.e. $C = \{A^T x; x \in \mathbf{F}^k\}$, then we will denote it as $C(\mathbf{F}, A)$. Moreover, if a matroid M is represented by the columns of A, then we let $M = M(\mathbf{F}, C)$ and $C = C(\mathbf{F}, M)$. In this case we have $C^* = \{x \in \mathbf{F}^n; Ax = 0\}$.

The following theorem was proved by Greene ([3]).

Theorem 4.4 *Let C be a linear code of length n and dimension k over $GF[q]$ and $0 \neq t \neq 1$. Then*

$$A_C(t) = (1-t)^k t^{n-k} T(M(GF[q], C), \frac{1+(q-1)t}{(1-t)}, \frac{1}{t}).$$

Note that Theorem 4.3 also follows immediately from Theorem 4.4 and the duality of the Tutte polynomial. As an immediate corollary we get

Corollary 4.5 *Let M be a matroid represented over $GF[q]$. If $(x-1)(y-1) = q$ and $0 \neq y \neq 1$ then*

$$T(M, x, y) = y^n (y-1)^{-k} A_{C(GF[q],M)}(y^{-1}).$$

Consider now the following example. Let $G = (V, E)$ be a graph and let $\mathcal{N}_G$ be the graphic matroid of G. Let O_G be an oriented incidence matrix of G, i.e. an $|V| \times |E|$ matrix obtained from the incidence matrix of G by replacing exactly one '1' of each column by '-1'. The columns of O_G represent $\mathcal{N}_G$ over an arbitrary field $\mathbf{F}$.

The set of the characteristic vectors of 2-cuts of a graph G (including the empty cut) equals $C(GF[2], \mathcal{N}_G)$ and the set of the characteristic vectors of eulerian subgraphs of G equals $C(GF[2], \mathcal{N}_G)^*$. Using this terminology, it may be checked easily that Theorem 4.3 generalizes Theorem 3.2.

Theorem 4.6 *Let $G = (V, E)$ be a connected graph. Then*

$$A_{C(GF[q],\mathcal{N}_G)}(t) = 1 + \sum_{i=2}^{q} (q-1)\dots(q-i+1)\mathcal{C}_i(G, (t, \dots, t)).$$

Proof. Let $C = C(GF[q], \mathcal{N}_G)$. We have $C = \{O_G^T x; x \in GF[q]^V\}$ and $A_C(t)$ is the weight enumerator of C. Let us define an equivalence on $GF[q]^V$ by $x \equiv y$ if $O_G^T x = O_G^T y$. Observe that each equivalence class consists of q elements since $O_G^T x = O_G^T y$ if and only if $x - y$ is a constant vector, i.e. $(x-y)_i = (x-y)_j$ for each $i, j \in \{1, \dots, |V|\}$. Let C^+ be the system (in contrast with *a set*, some elements of a system may appear several times) defined by $C^+ = (O_G^T x; x \in GF[q]^V)$. Let $A_{C^+}(t) = \sum_{i=0}^{|E|} A_i^+ t^i$, where A_i^+ equals the number of vectors of C^+ with i non-zero components. Since each equivalence class of $\equiv$ consists of q elements we have $A_{C^+}(t) = q A_C(t)$.

If $x = (x_1, ..., x_{|V|}) \in GF[q]^V$ then let l_x be the number of different values of x_i, $i = 1, ..., |V|$ and let $V_1^x, ..., V_{l_x}^x$ be the partition of V into l_x non-empty classes such that the components of x are equal in each class.

Let $cut(x)$ be the subset of E formed by those edges having endvertices in different sets V_i^x, $i = 1, ..., l_x$ and let $Cut(x) = (\{V_1, ..., V_{l_x}\}, cut(x))$.

Each $Cut(x)$ is an l_x-cut of G and the weight of the codeword $O_G^T x$ equals $|cut(x)|$. Let C^{++} be the system defined by $C^{++} = (cut(x); x \in GF[q]^V)$. Let $A_{C^{++}}(t) = \sum_{W \in C^{++}} t^{|W|}$. We have $A_{C^+}(t) = A_{C^{++}}(t)$.

For $i = 1, ..., q$ let $X_i = \{x \in GF[q]^V; l_x = i\}$. Define an equivalence on X_i by $x \equiv^* y$ if $Cut(x) = Cut(y)$. Observe that each equivalence class of $\equiv^*$ consists of $q(q-1)...(q-i+1)$ elements. Hence

$$qA_C(t) = A_{C^+}(t) = A_{C^{++}}(t) = q + \sum_{i=2}^{q} q(q-1)...(q-i+1)\mathcal{C}_i(G, (t, ..., t)).$$

This proves the Theorem.

□

By Corollary 4.5 and Theorem 4.6 we have

Corollary 4.7 *Let $G = (V, E)$ be a connected graph and let $\mathcal{N}_G$ be the graphic matroid of G. If $(x-1)(y-1) = 2$ and $0 \neq y \neq 1$ then*

$$T(G, x, y) = T(\mathcal{N}_G, x, y) = y^{|E|}(y-1)^{1-|V|}[1 + \mathcal{C}_2(G, (y^{-1}, ..., y^{-1})].$$

It follows that the Tutte polynomial of a graph of genus g may be expressed along the hyperbola $(x-1)(y-1) = 2$ as a linear combination of 4^g Pfaffians, and hence it may be determined efficiently for the graphs embeddable on an arbitrary fixed orientable surface.

It is natural to ask whether there is an analogy of this statement for binary matroids.

4.2 Flow polynomial of graphs.

We have $C(GF[q], \mathcal{N}_G)^* = \{z \in GF[q]^E; O_G z = 0\}$. The elements of $C(GF[q], \mathcal{N}_G)^*$ are *flows* on G with values in $GF[q]$. An element of $C(GF[q], \mathcal{N}_G)^*$ is called a *nowhere-zero flow* if its weight equals $|E|$. Let $F'(G, q)$ be the subset of $C(GF[q], \mathcal{N}_G)^*$ consisting of nowhere-zero flows. $F(G, q) = |F'(G, q)|$ is called the *flow polynomial* of G.

Theorems 4.3, 4.6 express a duality between flows and cuts of a graph. It is a duality of the Tutte polynomial.

Nowhere-zero flows are studied extensively. The following theorem was proved by Tutte ([7]).

Theorem 4.8 *Let $G=(V,E)$ be a graph and let q be a power of a prime. Then*

$$F(G,q)=(-1)^{|V|-|E|-c(G)}T(G,0,1-q).$$

We give an interesting expression for $F(G,q)$ which is new as far as we know.

Theorem 4.9 *Let $G=(V,E)$ be a graph. Then*

$$F(G,q)=q^{-|V|}2^{-|E|}\sum_{A\subset E}U_q((V,A))q^{|A|}(q-2)^{|E|-|A|}.$$

Proof. Let $C=C(GF[q],\mathcal{N}_G)$ and let $D=C(GF[q],\mathcal{N}_G)^*$. By Theorem 4.3 we have

$$A_D(t)=q^{1-|V|}[1+(q-1)t]^{|E|}A_C((1-t)(1+(q-1)t)^{-1}).$$

From Theorem 4.6 and Theorem 3.1 we get for $z>0$

$$qA_C(z^{-1})=q+\sum_{i=2}^{q}q(q-1)...(q-i+1)\mathcal{C}_i(G,(z^{-1},...,z^{-1}))=$$

$$2^{-|E|}z^{-|E|}\sum_{A\subset E}U_q((V,A))(z-1)^{|A|}(z+1)^{|E|-|A|}.$$

If we let $z=(1+(q-1)t)(1-t)^{-1}$ we get for all $t>0$, $t\neq 1$

$$A_D(t)=q^{-|V|}2^{-|E|}\sum_{A\subset E}U_q((V,A))(qt)^{|A|}(2+(q-2)t)^{|E|-|A|}.$$

It follows that the leading coefficient of $A_D(t)$, which equals $F(G,q)$, is equal to

$$q^{-|V|}2^{-|E|}\sum_{A\subset E}U_q((V,A))q^{|A|}(q-2)^{|E|-|A|}.$$

□

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