The characteristic polynomial of a graph is reconstructible from the characteristic polynomials of its vertex-deleted subgraphs and their complements

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Abstract

The question of whether the characteristic polynomial of a simple graph is uniquely determined by the characteristic polynomials of its vertex-deleted subgraphs is one of the many unresolved problems in graph reconstruction. In this paper we prove that the characteristic polynomial of a graph is reconstructible from the characteristic polynomials of the vertex-deleted subgraphs of the graph and its complement.

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1 Introduction

Let $G = (V, E)$ be a simple graph with a vertex set of at least three elements $V = \{1, \ldots, n\}$. We denote by $E(G)$ the set of its edges. A subgraph of G obtained by deleting vertex i and all its incident edges is called a vertex-deleted subgraph of G and is denoted by G_i. The collection of vertex-deleted subgraphs of G is known as its *deck*. The characteristic polynomial of G is the characteristic polynomial of its adjacency matrix $\mathbf{A}$ and is defined by $P_G(x) = \det(x\mathbf{I} - \mathbf{A})$. We call the collection of the characteristic polynomials of the vertex-deleted subgraphs the *polynomial deck* of G and denote it by $\mathcal{P}(G) = \{P_{G_1}, P_{G_2}, \ldots, P_{G_n}\}$. In general, a property of a graph is said to be *reconstructible* if it is uniquely determined by its deck. Tutte [11] proved that the characteristic polynomial of a graph is reconstructible from its deck. But is the full knowledge of the vertex-deleted subgraphs necessary to reconstruct the characteristic polynomial? Gutman and Cvetković [6] first raised the still unresolved question of whether the polynomial deck of a simple graph on at least three vertices contains enough information to reconstruct its characteristic polynomial. Some results are reported in [2, 8, 10]. In this paper, we prove that $P_G(x)$ is uniquely determined by the collection $\{\big(P_{G_u}(x),\, P_{\overline{G}_u}(x)\big) \,\big|\, u \in V(G)\}$.

2 Preliminaries

We begin by listing some known facts and derive lemmas that are used to prove the main theorem. The degree of vertex i is denoted by $d_{G,i}$. Let $W_G(x)$, $W_{G,i}(x)$ and $C_{G,i}(x)$ respectively be the generating functions for the total number of walks, number of walks that originate at vertex i and number of closed walks that start and end at vertex i. Then,

$$W_G(x) = \frac{1}{x}\left(\frac{(-1)^n P_{\overline{G}}(-1-1/x)}{P_G(1/x)} - 1\right), \tag{1}$$

$$W_G(x) = W_{G_i}(x) + \frac{W_{G,i}^2(x)}{C_{G,i}(x)}, \qquad i = 1, \ldots, n, \tag{2}$$

$$C_{G,i}(x) = \frac{1}{x}\frac{P_{G_i}(1/x)}{P_G(1/x)}, \qquad i = 1, \ldots, n. \tag{3}$$

Eqn. (1) is derived in [1], p.45 and (2, 3) in [5].

Lemma 2.1 *Let G and H be two graphs of the same order with $|E(G)| > |E(H)|$. Then there exists an $\varepsilon > 0$ such that $x \in (0, \varepsilon) \Rightarrow W_G(x) > W_H(x)$.*

Proof. Let $\mathbf{W}(x) = \big(w_{ij}(x)\big)$ where $w_{ij}(x)$ is the generating function for the number of walks from vertex i to j. Since $\mathbf{A}_{ij}^k$ is the number of walks of length k from i to j, we have

$$\mathbf{W}(x) = \sum_{k=0}^{\infty} x^k \mathbf{A}^k. \tag{4}$$

This may be written as $\mathbf{W}(x) = \mathbf{I} + x\mathbf{A} + x^2\mathbf{A}^2 \sum_{k=0}^{\infty} x^k \mathbf{A}^k$. Let $\mathbf{e}$ be the vector all ones. Then $W_G(x) = \mathbf{e}^t\, \mathbf{W}(x)\, \mathbf{e} = n + 2\,|E(G)|\, x + o(x^2)$, from which the claim immediately follows. ■

Lemma 2.2 *Given a graph G, there exists an $\varepsilon > 0$ such that for any pair i, j of its vertices $\big(d_{G,i} - d_{G,j}\big)\big(W_{G_i}(x) - W_{G_j}(x)\big) \le 0$ for $x \in (0, \varepsilon)$.*

Proof. The claim is obvious when $d_{G,i} = d_{G,j}$. If $d_{G,i} > d_{G,j}$ then $|E(G_i)| < |E(G_j)|$ and by Lemma 2.1, there is an interval $(0, \varepsilon_{ij})$ such that $W_{G_i}(x) < W_{G_j}(x)$. If, on the other hand $d_{G,i} < d_{G,j}$, then $|E(G_i)| > |E(G_j)|$ and there is an interval $(0, \varepsilon_{ij})$ over which $W_{G_i}(x) > W_{G_j}(x)$. Choosing $\varepsilon = \min\limits_{1 \le i,j \le n} \varepsilon_{ij}$ proves the lemma. ■

Lemma 2.3

$$P_G(1/x) \;=\; \frac{1}{x\big(xW_G(x)\big)'} \sum_{i=1}^{n} \Big(W_G(x) - W_{G_i}(x)\Big)\, P_{G_i}(1/x).$$

Proof. From (4) we have $\mathbf{W}(x) = \mathbf{I} + x\,\mathbf{A}\,\mathbf{W}(x)$ so that $\mathbf{W}(x) = (\mathbf{I} - x\mathbf{A})^{-1}$. Let $\mathbf{w}(x) = \mathbf{W}(x)\,\mathbf{e} = (W_{G,1}(x), \ldots, W_{G,n}(x))^t$. By differentiating $\mathbf{e}^t\,(\mathbf{I} - x\mathbf{A})^{-1}\,\mathbf{e}$, we first obtain

$$W_G'(x) \;=\; \mathbf{e}^t\,(\mathbf{I} - x\mathbf{A})^{-2}\,\mathbf{Ae} \;=\; \mathbf{w}(x)^t\mathbf{A}\mathbf{w}(x).$$

Next,

$$\begin{aligned} W_G(x) \;&=\; \mathbf{e}^t\mathbf{W}(x)\mathbf{W}(x)^{-1}\mathbf{W}(x)\mathbf{e} \;=\; \mathbf{w}(x)^t\,(\mathbf{I} - x\mathbf{A})\,\mathbf{w}(x) \\ &=\; \mathbf{w}(x)^t\mathbf{w}(x) - xW_G'(x). \end{aligned}$$

Therefore

$$\big(xW_G(x)\big)' \;=\; \mathbf{w}(x)^t\mathbf{w}(x). \tag{5}$$

From (2) and (3) we have

$$P_G(1/x)\,W_{G,i}^2(x) \;=\; \frac{1}{x}\Big(W_G(x) - W_{G_i}(x)\Big)\,P_{G_i}(1/x).$$

Summing this over i and using (5) proves the lemma. ∎

Lemma 2.4 *Graphs with identical characteristic polynomial decks have identical degree sequences* [6].

Proof. Let G, H be graphs such that $P_{G_i} = P_{H_i}$, $i = 1, \ldots, n$. The number of edges of a graph is determined by its characteristic polynomial. Thus, $|E(G_i)| = |E(H_i)|$, $i = 1, \ldots, n$. Now $\sum_i |E(G)| - |E(G_i)| = n|E(G)| - \sum_i |E(G_i)| = 2|E(G)$. Then, $|E(G)| = \sum_i |E(G_i)|/(n-2) = |E(H)|$. Thus, $d_{G,i} = d_{H,i}$, $i = 1, \ldots, n$. ∎

3 Main theorem

Theorem 3.1 *The characteristic polynomial of a graph G is reconstructible from the collection $\{\big(P_{G_u}(x), P_{\overline{G}_u}(x)\big) \,\big|\, u \in V(G)\}$.*

Proof. Let H be any graph such that $\big(P_{H_i}(x), P_{\overline{H}_i}(x)\big) = \big(P_{G_i}(x), P_{\overline{G}_i}(x)\big)$, $i = 1, \ldots, n$. By (1), $W_{G_i}(x) = W_{H_i}(x)$, $i = 1, \ldots, n$. This result is used to show that there is an interval $(0,\varepsilon)$ over which $W_G(x) = W_H(x)$. We then conclude by Lemma 2.3 that $P_H(x) = P_G(x)$.

Consider the interval $(0,\varepsilon)$ from Lemma 2.2 and let x be any point in this interval. Suppose $W_G(x) > W_H(x)$. Then, for any pair of vertices i, j

$$(d_{G,i} - d_{G,j})\Big(W_{G_i}(x) - W_{G_j}(x)\Big)\Big(W_G(x) - W_H(x)\Big) \le 0. \tag{6}$$

Because $W_{G_i}(x) = W_{H_i}(x)$, $W_{G_j}(x) = W_{H_j}(x)$ and using (3) we have

$$\begin{aligned}\Big(W_{G_i}(x) - W_{G_j}(x)\Big)&\Big(W_G(x) - W_H(x)\Big) \\ &= \Big(W_G(x) - W_{G_i}(x)\Big)\Big(W_H(x) - W_{H_j}(x)\Big) \\ &\qquad - \Big(W_H(x) - W_{H_i}(x)\Big)\Big(W_G(x) - W_{G_j}(x)\Big) \\ &= \frac{W^2_{G,i}(x)}{C_{G,i}(x)}\,\frac{W^2_{H,j}(x)}{C_{H,j}(x)} - \frac{W^2_{H,i}(x)}{C_{H,i}(x)}\,\frac{W^2_{G,j}(x)}{C_{G,j}(x)}.\end{aligned} \tag{7}$$

Next we note that

$$\begin{aligned} C_{G,i}(x)\; C_{H,j}(x) &= \frac{1}{x}\frac{P_{G_i}(1/x)}{P_G(1/x)}\;\frac{1}{x}\frac{P_{H_j}(1/x)}{P_H(1/x)} \\ &= \frac{1}{x}\frac{P_{H_i}(1/x)}{P_H(1/x)}\;\frac{1}{x}\frac{P_{G_j}(1/x)}{P_G(1/x)} = C_{H,i}(x)\; C_{G,j}(x).\end{aligned} \tag{8}$$

By using (8) and (7) in (6) and factoring we get

$$\begin{aligned}(d_{G,i} - d_{G,j})&\Big(W_{G,i}(x)W_{H,j}(x) - W_{H,i}(x)W_{G,j}(x)\Big) \\ &\left(\frac{W_{G,i}(x)W_{H,j}(x) + W_{H,i}(x)W_{G,j}(x)}{C_{H,i}(x)C_{G,j}(x)}\right) \le 0.\end{aligned}$$

The last term is positive when $x > 0$. Thus,

$$(d_{G,i} - d_{G,j})\Big(W_{G,i}(x)W_{H,j}(x) - W_{H,i}(x)W_{G,j}(x)\Big) \le 0.$$

By Lemma 2.4, $d_{G,i} = d_{H,i}$, $d_{G,j} = d_{H,j}$. Therefore, we derive

$$\begin{aligned}& d_{G,i}\, W_{G,i}(x)\, W_{H,j}(x) + d_{G,j}\, W_{G,j}(x)\, W_{H,i}(x) \\ & - d_{H,i}\, W_{H,i}(x)\, W_{G,j}(x) - d_{H,j}\, W_{H,j}(x)\, W_{G,i}(x) \le 0.\end{aligned} \tag{9}$$

We sum (9) over all vertices i, j to get,

$$\begin{aligned}&\sum_i d_{G,i}W_{G,i}(x) \sum_j W_{H,j}(x) + \sum_j d_{G,j}W_{G,j}(x) \sum_i W_{H,i}(x) \\ &- \sum_i d_{H,i}W_{H,i}(x) \sum_j W_{G,j}(x) - \sum_j d_{H,j}W_{H,j}(x) \sum_i W_{G,i}(x) \le 0,\end{aligned}$$

and simplify it to

$$W_H(x)\sum_i d_{G,i}W_{G,i}(x)\ -\ W_G(x)\sum_i d_{H,i}W_{H,i}(x)\ \le\ 0. \tag{10}$$

From (4) we have $\mathbf{W}(x) = \mathbf{I}+x\,\mathbf{A}\,\mathbf{W}(x)$. Hence, $W_G(x) = \mathbf{e}^t\mathbf{W}(x)\mathbf{e} = n+x\sum_{i=1}^n d_{G,i}W_{G,i}(x)$. Using this result in (10) and because $x > 0$ we get

$$W_H(x)\,(W_G(x)-n)\ -\ W_G(x)\,(W_H(x)-n)\ \le\ 0,$$

from which we conclude $W_G(x) \le W_H(x)$. This contradicts the assumption that $W_G(x) > W_H(x)$. Therefore, $W_G(x) = W_H(x)$. ■

After showing that $W_G(x)$ is reconstructible from $\{(P_{G_u}(x),\, P_{\overline{G}_u}(x))\ |\ u \in V(G)\}$, we used Lemma 2.3 to prove that the characteristic polynomial is also reconstructible. But there is an alternative argument to do this. Let $P'_G(x)$ be the derivative of the characteristic polynomial of G. Then (see [9]) $P'_G(x) = \sum_{i=1}^n P_{G_i}(x)$ so that if $P_G(x) = \sum_{k=0}^n a_{G,k}x^k$, then $a_{G,k} = (\sum_{i=1}^n a_{G_i,k-1})\,/k$, $k = 1,\ldots,n$. The constant term $a_{G,0}$ is thus the only coefficient possibly not determined by $\mathcal{P}(G)$. However, $a_{G,0}$ is reconstructible if in addition to $a_{G,k}$, $k = 1,\ldots,n$, a single eigenvalue of G is reconstructible [6].

An eigenvalue λ of a graph is called *main* if it has an associated eigenvector $\mathbf{x}$ such that $\mathbf{e}^t\mathbf{x} \ne 0$. Let $M(G)$ denote the set of main eigenvalues of G. Deo, Harary and Schwenk [4] have shown that $W_G(x) = W_H(x)$ iff $M(G) = M(H)$ and $M(\overline{G}) = M(\overline{H})$. They call such graphs *comain*. Thus, by Theorem 3.1 $M(G)$ is reconstructible from $\{(P_{G_u}(x),\, P_{\overline{G}_u}(x))\ |\ u \in V(G)\}$ and since a graph has at least one main eigenvalue, so is $a_{G,0}$.

4 Discussion

The original problem of whether $\mathcal{P}(G)$ uniquely determines $a_{G,0}$ is still open. It is part of a general class of reconstruction problems which ask whether a graph invariant $I(G)$ is uniquely determined by the collection $I(G_u)$, $u \in V(G)$. In [8] Schwenk expresses his suspicion that $P_G(x)$ is not reconstructible from $\mathcal{P}(G)$ but that counter-examples will be difficult to find.

While searching for a counter-example, I found many pairs of non-cospectral graphs G, H such that $P'_G(x) = P'_H(x)$. An example of two such graphs is shown in Figure 1 where $P_G(x) = x^{12} - 13x^{10} + 56x^8 - 102x^6 + 80x^4 - 22x^2$ and $P_H(x) = P_G(x) + 1$. The two graphs have identical degree sequence and $P_{G_2}(x) = P_{H_2}(x)$, $P_{G_3}(x) = P_{H_3}(x)$. Moreover, the characteristic polynomials of the pairs $\{G_7, H_7\}$, $\{G_{11}, H_{12}\}$ and $\{G_{12}, H_{11}\}$ differ only in their respective coefficients of x. The list of the characteristic polynomials of the vertex deleted subgraphs of the two graphs is shown in Table 1. This is hardly an indication that counter-examples exist and it may well turn out that $P_{G_i}(x) = P_{H_i}(x)$, $i = 1,\ldots,n$ if and only if $P_{\overline{G}_i}(x) = P_{\overline{H}_i}(x)$, $i = 1,\ldots,n$.

graph	a_{12}	a_{11}	a_{10}	a_9	a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0
G	1	0	-13	0	56	0	-102	0	80	0	-22	0	0
G_1		1	0	-12	0	47	0	-76	0	51	0	-11	0
G_2		1	0	-12	0	47	0	-76	0	51	0	-11	0
G_3		1	0	-12	0	45	0	-67	0	40	0	-8	0
G_4		1	0	-11	0	37	0	-44	0	16	0	0	0
G_5		1	0	-11	0	37	0	-47	0	22	0	-2	0
G_6		1	0	-11	0	40	0	-57	0	27	0	0	0
G_7		1	0	-11	0	35	0	-40	0	14	0	0	0
G_8		1	0	-11	0	37	0	-53	0	34	0	-8	0
G_9		1	0	-10	0	30	0	-34	0	12	0	0	0
G_{10}		1	0	-10	0	34	0	-46	0	22	0	-2	0
G_{11}		1	0	-10	0	33	0	-43	0	20	0	-2	0
G_{12}		1	0	-9	0	26	0	-29	0	11	0	0	0
H	1	0	-13	0	56	0	-102	0	80	0	-22	0	1
H_1		1	0	-12	0	46	0	-69	0	37	0	-2	0
H_2		1	0	-12	0	47	0	-76	0	51	0	-11	0
H_3		1	0	-12	0	45	0	-67	0	40	0	-8	0
H_4		1	0	-11	0	38	0	-49	0	23	0	-3	0
H_5		1	0	-11	0	37	0	-46	0	20	0	-2	0
H_6		1	0	-11	0	41	0	-63	0	37	0	-4	0
H_7		1	0	-11	0	35	0	-40	0	14	0	-1	0
H_8		1	0	-11	0	37	0	-53	0	33	0	-7	0
H_9		1	0	-10	0	30	0	-35	0	15	0	-2	0
H_{10}		1	0	-10	0	33	0	-42	0	19	0	-2	0
H_{11}		1	0	-9	0	26	0	-29	0	11	0	-1	0
H_{12}		1	0	-10	0	33	0	-43	0	20	0	-1	0

Table 1: Coefficients of the characteristic polynomials of the graphs of Figure 1 and their vertex-deleted subgraphs.

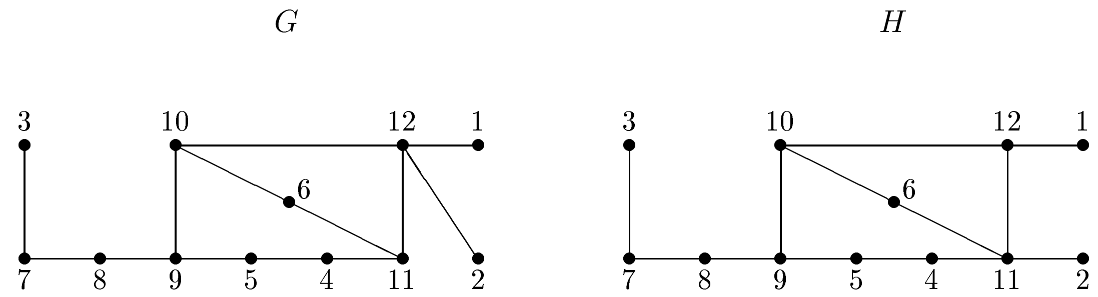

Figure 1: Two non-cospectral graphs with $P'_G(x) = P'_H(x)$.

Theorem 3.1 suggests that perhaps the correct problem to pose in general is: Is $I(G)$ uniquely determined by the collection $\{\big(I(G_u),\, I(\overline{G}_u)\big) \,\big|\, u \in V(G)\}$? The vertex-deleted subgraphs are not essential to reconstruct the elementary invariants of a graph. For example, the number of edges of a graph is determined from that of the subgraphs as $|E(G)| = \sum_i |E(G_i)|\big/(n-2)$. The degree sequence of G, denoted by D_G, is uniquely determined by $\big\{D_{G_i} \,\big|\, i \in V(G)\big\}$. The degree sequence of a vertex i, denoted by $D_{G,i}$, is the list of the degrees of its neigbours in ascending order. It is easy to show that $\big\{D_{G,i} \,\big|\, i \in V(G)\big\}$ is reconstructible from $\big\{D_{G_i,j} \,\big|\, i \in V(G),\ j \in V(G_i)\big\}$. In each of these cases, we note that the invariants are equal for two graphs iff they are also equal for their complements. We rely on these observations and Theorem 3.1 to suggest the following problem.

Problem. *Find examples of non-trivial invariants $I(G)$ of a graph G with at least three vertices which are reconstructible from their collection $\{\big(I(G_u),\, I(\overline{G}_u)\big) \,\big|\, u \in V(G)\}$.*

The celebrated reconstruction conjecture which asserts that the isomorphism class of a graph on at least three vertices is uniquely determined by the isomorphism classes of its vertex-deleted subgraphs is the most general case of problems of this type. There are counter-examples to the question of whether a graph invariant $I(G)$ is reconstructible from the collection $\{\big(I(G_u),\, I(\overline{G}_u)\big) \,\big|\, u \in V(G)\}$. This was pointed out to the author (who originally hazarded it as a conjecture) by Brendan Mckay [7] who observed that Hamiltonicity is not reconstructible in this sense. His observation was: "Let $I(G) =$ 'G is Hamiltonian'. Choose a large even number n. Let G be a cubic hypohamiltonian graph (which exist for all large even orders). All the vertex-deleted subgraphs of both G and $\overline{G}$ are Hamiltonian (for G, by definition; for $\overline{G}$, because the degree is high enough to imply it), yet G is not Hamiltonian. For the second graph, take $H = \overline{G}$. Again all the vertex-deleted subgraphs of both H and $\overline{H}$ are Hamiltonian, yet this time H is also Hamiltonian."

A different problem than that proved by Theorem 3.1 is the question of whether $P_G(x)$ is reconstructible from the two decks $\mathcal{P}(G)$, $\mathcal{P}(\overline{G})$. Unlike the condition of

Theorem 3.1, here it is not known *a priori* which characteristic polynomials from the two decks belong to a vertex-deleted subgraph and its complement. This is crucial to the proof of Theorem 3.1.

Finally, the referee noted that the result from the title can be reformulated as follows: The eigenvalues and main angles of a graph can be uniquely reconstructed from the eigenvalues and main angles of its vertex deleted subgraphs. This follows from a formula connecting characteristic polynomials of a graph and its complement and main angles (see, for example, [3] p. 99).

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