HARMONIC FUNCTIONS ON MULTIPLICATIVE GRAPHS AND INTERPOLATION POLYNOMIALS

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Abstract. We construct examples of nonnegative harmonic functions on certain graded graphs: the Young lattice and its generalizations. Such functions first emerged in harmonic analysis on the infinite symmetric group. Our method relies on multivariate interpolation polynomials associated with Schur's S and P functions and with Jack symmetric functions. As a by–product, we compute certain Selberg–type integrals.

Contents

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§0. Introduction

Let $\mathbb{Y}$ denote the lattice of Young diagrams ordered by inclusion. For $\mu, \lambda \in \mathbb{Y}$, we write $\lambda \searrow \mu$ if λ covers μ, i.e., λ differs from μ by adding a box. We consider $\mathbb{Y}$ as a graph whose vertices are arbitrary Young diagrams μ and the edges are couples (μ, λ) such that $\lambda \searrow \mu$. We shall call $\mathbb{Y}$ the *Young graph.* A function $\varphi(\mu)$ is called a *harmonic function* on the Young graph [VK] if it satisfies the condition

$$\varphi(\mu) = \sum_{\lambda:\, \lambda \searrow \mu} \varphi(\lambda), \qquad \forall \mu \in \mathbb{Y}. \tag{0.1}$$

We are interested in nonnegative harmonic functions φ normalized at the empty diagram: $\varphi(\varnothing) = 1$. Such functions form a convex set denoted as $\mathcal{H}_1^+(\mathbb{Y})$.

The functions $\varphi \in \mathcal{H}_1^+(\mathbb{Y})$ have an important representation–theoretic meaning: they are in a natural bijective correspondence with central, positive definite, normalized functions on the infinite symmetric group $S(\infty)$, see [VK], [KV2]. Thoma's description of characters on $S(\infty)$ means that the extreme points of $\mathcal{H}_1^+(\mathbb{Y})$ form an infinite–dimensional simplex Ω (called the Thoma simplex), see [T], [VK], [KV2], [W]. For general elements $\varphi \in \mathcal{H}_1^+(\mathbb{Y})$, there is a (unique) Poisson–type integral representation,

$$\varphi(\lambda) = \int_\Omega K(\lambda, \omega)\, P(d\omega), \qquad \forall \lambda \in \mathbb{Y}, \tag{0.2}$$

where P is a probability measure on Ω (the 'boundary measure' for φ) and $K(\lambda, \omega)$ is a positive function on $\mathbb{Y} \times \Omega$ (the 'Poisson kernel' or 'Martin kernel' for $\mathbb{Y}$), see [KOO]. Note that any probability measure P on Ω gives rise to an element of $\mathcal{H}_1^+(\mathbb{Y})$; in particular, the extreme φ's are exactly the functions $K(\cdot, \omega)$ corresponding to Dirac measures on Ω.

This abstract result shows how large $\mathcal{H}_1^+(\mathbb{Y})$ is but it does not explain how to construct explicitly nonextreme functions φ or what nonextreme φ's could be interesting for applications.

Concrete examples of nonextreme functions φ first emerged in [KOV] in connection with a problem of harmonic analysis on the infinite symmetric group $S(\infty)$. These functions, denoted as $\varphi_{zz'}$, depend on two parameters, and the corresponding 'boundary measures' $P_{zz'}$ govern the spectral decomposition of certain natural unitary representations.[1]

[1] The measures $P_{zz'}$ are very interesting objects. They are studied in detail in our papers [P.I] – [P.V], [BO1], [BO2].

The explicit expression of the functions $\varphi_{zz'}$ (see formula (2.4) below) has an interesting combinatorial structure which raises a number of questions. For instance, one can ask whether there exist similar families of harmonic functions for other graphs. The answer is affirmative: [B1], [Ke5].[2]

The paper [B1] concerns the graph $\mathbb{S}$ of shifted Young diagrams which is related to projective representations of the symmetric groups.

The paper [Ke5] contains a generalization in another direction: a deformation of the family $\{\varphi_{zz'}\}$, which is consistent with a deformation of the basic equation (0.1):

$$\varphi(\mu) = \sum_{\lambda:\, \lambda \searrow \mu} \varkappa_\theta(\mu, \lambda)\varphi(\lambda), \qquad \forall \mu. \tag{0.3}$$

Here $\theta > 0$ is the deformation parameter and $\varkappa_\theta(\mu, \lambda) > 0$ are the coefficients that arise in (the simplest case of) Pieri's rule for Jack symmetric functions with parameter θ. The initial situation corresponds to the particular value $\theta = 1$, when Jack symmetric functions coincide with Schur's S–functions.

Note that in the limit as $\theta \to 0$, the harmonicity condition (0.3) essentially coincides with the relation which defines partition structures in the sense of Kingman [Ki1], [Ki2], while the two-parameter family of harmonic functions constructed in [Ke5] degenerates to the famous Ewens partition structures [Ew] and its generalization due to Pitman, see [Pi], [PY], [Ke4].

In the present paper, we propose a simple combinatorial construction, which allows us to get, in a unified way, all these concrete examples of harmonic functions as well as some new ones. In the new examples, the 'boundary measures' P are supported by finite–dimensional simplices, and the Poisson integral representation leads to certain Selberg–type integrals.[3]

Our construction relies on the so–called shifted (or factorial) versions of Schur's S and P functions and of Jack symmetric functions. These new combinatorial functions arise in different topics, see, e.g., [S], [KS], [OO], [OO2], [Ok1], [Ok2]. They are also called interpolation polynomials, because they give solutions to certain multivariate interpolation problems.

The paper is organized as follows. In §1, we expose the general formalism. In §2, it is applied to the Young graph to derive the family $\{\varphi_{zz'}\}$. In §§3–5, we apply it to the Young graph with Jack edge multiplicities $\mathbb{J}(\theta)$, next to the Kingman graph $\mathbb{K}$, and then to the Schur graph $\mathbb{S}$; the arguments are quite similar. Section 6 is devoted to constructing harmonic functions of a different sort — those with finite–dimensional 'boundary measures'; here we also evaluate Selberg–type integrals. The final §7 is an appendix on the Poisson integral representation.

[2]Another question, a characterization of the functions of type $\varphi_{zz'}$, was examined in [B1], [Ro].

[3]A connection between Poisson integral representation of type (0.2) and Selberg integrals was first exploited in [Ke3].

§1. The general formalism

In this section, we deal with an abstract graph $\mathbb{G}$ satisfying certain conditions listed below. In the next sections, concrete examples of $\mathbb{G}$ will be considered.

Our assumptions and conventions concerning $\mathbb{G}$ are as follows:

• To simplify the notation, we identify the graph with its set of vertices.

• The vertices are partitioned into levels, $\mathbb{G} = \mathbb{G}_0 \sqcup \mathbb{G}_1 \sqcup \mathbb{G}_2 \sqcup \dots$, so that the endpoints of any edge lie on consecutive levels. That is, $\mathbb{G}$ is a graded graph.

• The level of a vertex μ is denoted as $|\mu|$. If two vertices μ, λ form an edge, $|\lambda| = |\mu| + 1$, then we write $\lambda \searrow \mu$ or $\mu \nearrow \lambda$.

• All the levels $\mathbb{G}_n$ are finite.

• The lowest level $\mathbb{G}_0$ consists of a single vertex denoted as $\varnothing$.

• For any vertex μ there exists at least one vertex $\lambda \searrow \mu$ and for any vertex $\lambda \neq \varnothing$ there exists at least one vertex $\mu \nearrow \lambda$. This implies that the graph is connected.

(Our main example is the Young graph, see §2.)

• Finally, assume that we are given an *edge multiplicity function* which assigns to any edge $\mu \nearrow \lambda$ a strictly positive number $\varkappa(\mu, \lambda)$ — its formal multiplicity. It should be emphasized that these numbers are not necessarily integers.

(For the Young graph, all the formal multiplicities are equal to 1; graphs with nontrivial multiplicities are considered in §3 and §4.)

A complex function $\varphi(\mu)$ on $\mathbb{G}$ is called a *harmonic function on the graph* $\mathbb{G}$ if it satisfies the relation

$$\varphi(\mu) = \sum_{\lambda:\, \lambda \searrow \mu} \varkappa(\mu, \lambda)\varphi(\lambda) \tag{1.1}$$

for any vertex μ (the sum in the right–hand side is finite, because all the levels are finite). Let $\mathcal{H}(\mathbb{G})$ denote the space of all harmonic functions endowed with the topology of pointwise convergence. Let $\mathcal{H}^+(\mathbb{G})$ be the subset of nonnegative harmonic functions and $\mathcal{H}^+_1(\mathbb{G})$ be the subset of the functions $\varphi \in \mathcal{H}^+(\mathbb{G})$ with the normalization $\varphi(\varnothing) = 1$.

Clearly, $\mathcal{H}^+_1(\mathbb{G})$ is a convex subset of $\mathcal{H}(\mathbb{G})$. Moreover, it is a compact measurable space (here we again employ the finiteness assumption). We shall use some well–known general theorems about convex compact measurable sets which can be found, e.g., in [Ph].

Let $\Omega(\mathbb{G})$ denote the set of extreme points in $\mathcal{H}^+_1(\mathbb{G})$. This is a set of type G_δ, hence, a Borel measurable set. Given $\omega \in \Omega(\mathbb{G})$, let us denote by $K(\,\cdot\,, \omega)$ the corresponding extreme harmonic function on $\mathbb{G}$. Note that $K(\mu, \,\cdot\,)$ is a Borel measurable function on $\Omega(\mathbb{G})$ for any fixed $\mu \in \mathbb{G}$.

Theorem 1.1. *For each element $\varphi \in \mathcal{H}^+_1(\mathbb{G})$ there exists a unique probability measure P on $\Omega(\mathbb{G})$ such that*

$$\varphi(\mu) = \int_{\Omega(\mathbb{G})} K(\mu, \omega) P(d\omega), \qquad \forall \mu \in \mathbb{G}. \tag{1.2}$$

Proof. See §7. □

We call (1.2) the *Poisson integral representation* of the function φ.

To any *path* τ going from a vertex μ to a vertex λ with $|\lambda| > |\mu|$,

$$\tau = (\mu = \lambda_0 \nearrow \lambda_1 \nearrow \cdots \nearrow \lambda_k = \lambda), \qquad k = |\lambda| - |\mu|,$$

we assign its weight

$$w(\tau) = \prod_{i=1}^{k} \varkappa(\lambda_{i-1}, \lambda_i)$$

and then set

$$\dim_{\mathbb{G}}(\mu, \lambda) = \sum_{\tau} w(\tau), \tag{1.3}$$

summed over all paths from μ to λ. We extend this definition to all couples (μ, λ) by agreeing that $\dim_{\mathbb{G}}(\mu, \mu) = 1$ and $\dim(\mu, \lambda) = 0$ if $\mu \neq \lambda$ are such that there is no path from μ to λ. Next, we set $\dim_{\mathbb{G}} \lambda = \dim_{\mathbb{G}}(\varnothing, \lambda)$.

In all examples of the graphs $\mathbb{G}$ considered in the present paper one can embed (the vertices of) $\mathbb{G}$ into $\Omega(\mathbb{G})$ in such a way that any point $\omega \in \Omega(\mathbb{G})$ can be approximated by a sequence of vertices $\{\lambda(n) \in \mathbb{G}_n\}_{n=1,2,\dots}$, and for any such sequence

$$K(\mu, \omega) = \lim_{n\to\infty} \frac{\dim_{\mathbb{G}}(\mu, \lambda(n))}{\dim_{\mathbb{G}} \lambda(n)}\,.$$

Given a function $\varphi \in \mathcal{H}_1^+(\mathbb{G})$, we set for each n

$$M_n(\lambda) = \dim_{\mathbb{G}} \lambda \cdot \varphi(\lambda), \qquad \lambda \in \mathbb{G}_n. \tag{1.4}$$

Using the harmonicity relation (1.1) and induction on n one readily verifies that $\sum_{\lambda\in\mathbb{G}_n} M_n(\lambda) = 1$. Thus, each M_n is a probability distribution on $\mathbb{G}_n$.

For all examples of the graphs $\mathbb{G}$ considered in this paper one can transfer the measure M_n to $\Omega(\mathbb{G})$ via the embedding $\mathbb{G} \hookrightarrow \Omega(\mathbb{G})$ mentioned above. Then the measure P appearing in the integral representation (1.2) is the weak limit of the measures M_n as $n \to \infty$.

We say that $(\mathbb{G}, \varkappa(\,\cdot\,,\,\cdot\,))$ is a *multiplicative graph* [KV1], [KV2], if the following conditions are satisfied. First, the 1st floor $\mathbb{G}_1$ consists of a single vertex denoted by the symbol "(1)". Next, there exists a graded commutative unital algebra A over $\mathbb{R}$, $A = A_0 + A_1 + \dots$, and a homogeneous basis $\{\mathcal{P}_\mu\}$ in A indexed by the vertices $\mu \in \mathbb{G}$, such that $\mathcal{P}_\varnothing = 1$ and $\deg \mathcal{P}_\mu = |\mu|$. Finally, for any μ,

$$\mathcal{P}_\mu \mathcal{P}_{(1)} = \sum_{\lambda:\, \lambda \searrow \mu} \varkappa(\mu, \lambda) \mathcal{P}_\lambda\,. \tag{1.5}$$

Note that this implies $\varkappa(\varnothing, (1)) = 1$. (All the graphs considered in the present paper are multiplicative. For instance, in the case of the Young graph, the algebra

A is the algebra of symmetric functions and the basis $\{\mathcal{P}_\mu\}$ is formed by the Schur functions.)

Iterating the relation (1.5) we get the expansion

$$\mathcal{P}_{(1)}^n = \sum_{\lambda \in \mathbb{G}_n} \dim_{\mathbb{G}} \lambda \cdot \mathcal{P}_\lambda \,, \tag{1.6}$$

which is a useful tool for computing the dimensions $\dim_{\mathbb{G}} \lambda$. More generally, given $\mu \in \mathbb{G}_m$ and $n > m$,

$$\mathcal{P}_\mu \mathcal{P}_{(1)}^{n-m} = \sum_{\lambda \in \mathbb{G}_n} \dim_{\mathbb{G}}(\mu, \lambda) \cdot \mathcal{P}_\lambda \,. \tag{1.7}$$

Theorem 1.2 [KV1]**.** *Let $\mathbb{G}$ be a multiplicative graph and let A be the corresponding algebra. Given $\varphi \in \mathcal{H}_1^+(\mathbb{G})$, let $\pi : A \to \mathbb{C}$ be the linear functional sending each $\mathcal{P}_\mu$ to $\varphi(\mu)$. Then φ is extreme if and only if π is multiplicative.*

Note that a linear functional $\pi : A \to \mathbb{C}$ corresponds to a function $\varphi \in \mathcal{H}_1^+(\mathbb{G})$ if and only if $\pi(1) = 1$, $\pi(\mathcal{P}_\mu) \geq 0$ for any μ, and π factors through $A/(\mathcal{P}_{(1)} - 1)A$.

Now we shall explain our method of producing harmonic functions. Assume A^* is a commutative algebra[4], $\{\mathcal{P}_\mu^*\}$ is a family of elements in A^* indexed by the vertices $\mu \in \mathbb{G}$, $\mathcal{P}_\varnothing^* = 1$. We assume that these data obey the following condition which is a generalization of (1.5):

$$\mathcal{P}_\mu^* \mathcal{P}_{(1)}^* = a_n \mathcal{P}_\mu^* + \sum_{\lambda :\, \lambda \searrow \mu} \varkappa(\mu, \lambda) \mathcal{P}_\lambda^* \,, \qquad n = |\mu|, \tag{1.8}$$

for any μ, where $a_0 = 0, a_1, a_2, \dots$ is a sequence of numbers.

Proposition 1.3. *Under the above assumptions, let $\pi : A^* \to \mathbb{C}$ be a multiplicative linear functional, and let*

$$s = \pi(\mathcal{P}_{(1)}^*), \quad t = -s = -\pi(\mathcal{P}_{(1)}^*). \tag{1.9}$$

Assume that

$$s \neq 0, a_1, a_2, \dots, \quad \textit{i.e.,} \quad t \neq 0, -a_1, -a_2, \dots. \tag{1.10}$$

Then the function

$$\varphi(\mu) = \frac{\pi(\mathcal{P}_\mu^*)}{s(s-a_1)\dots(s-a_{n-1})} = \frac{(-1)^n \pi(\mathcal{P}_\mu^*)}{t(t+a_1)\dots(t+a_{n-1})}\,, \qquad n = |\mu|, \tag{1.11}$$

is harmonic on $\mathbb{G}$.

We agree that the denominator in (1.11) equals 1 for $\mu = \varnothing$, so that $\varphi(\varnothing) = 1$.

[4]The superscript $*$ does not mean the passage to a dual space.

Proof. Applying π to the relation (1.8) we get

$$\pi(\mathcal{P}_\mu^*)(s-a_n) = \sum_{\lambda:\,\lambda\searrow\mu} \varkappa(\mu,\lambda)\pi(\mathcal{P}_\lambda^*).$$

Dividing the both sides by $s(s-a_1)\dots(s-a_n)$ (which is possible thanks to (1.10)) we get exactly the harmonicity relation (1.1) for φ. $\square$

A trivial example is $A^* = A$, $\mathcal{P}_\mu^* = \mathcal{P}_\mu$, $a_n \equiv 0$. Then, by Theorem 1.2, φ is extreme provided that it is nonnegative. As we aim to construct interesting examples of nonextreme harmonic functions, we shall deal either with an algebra A^* distinct from A or, for $A^* = A$, with a family $\{\mathcal{P}_\mu^*\}$ distinct from $\{\mathcal{P}_\mu\}$.

In all the examples below, A^* is a filtered algebra such that the associated graded algebra $\operatorname{gr} A^*$ is canonically isomorphic to A. Thus, with any element of A^* of degree $\le n$ one can associate its *highest term* which is a homogeneous element of A of degree n. In our examples, the highest term of $\mathcal{P}_\mu^*$ coincides with $\mathcal{P}_\mu$. Furthermore, the algebra A^* can be interpreted, in a certain natural way, as an algebra of functions on the vertices of $\mathbb{G}$. Thus, for any $f \in A^*$ and $\lambda \in \mathbb{G}$, the value $f(\lambda)$ is well–defined. It turns out that the elements $\mathcal{P}_\mu^*$ can be characterized by the following

Interpolation Property. Given $\mu \in \mathbb{G}$, $\mu \ne \varnothing$, $\mathcal{P}_\mu^*$ is the only (up to a scalar factor) element of degree $|\mu|$ such that $\mathcal{P}_\mu^*(\lambda) = 0$ for any $\lambda \ne \mu$ with $|\lambda| \le |\mu|$.

The fact that the highest term of an element $\mathcal{P}_\mu^*$ defined in this way turns out to be proportional to $\mathcal{P}_\mu$ seems to be rather surprising. We normalize $\mathcal{P}_\mu^*$ in such a way that its highest term is exactly equal to $\mathcal{P}_\mu$.

Next, it turns out that $\mathcal{P}_{(1)}^*(\mu) = |\mu|$. Then a simple formal argument shows that (1.8) holds with $a_n = n$ for any $n = 0, 1, \dots$. Moreover,

$$\frac{\dim_{\mathbb{G}}(\mu,\lambda)}{\dim_{\mathbb{G}}\lambda} = \frac{\mathcal{P}_\mu^*(\lambda)}{N(N-1)\dots(N-n+1)}, \qquad \mu\in\mathbb{G}_n, \quad \lambda\in\mathbb{G}_N, \quad n\le N. \tag{1.12}$$

The argument is due to Okounkov [Ok1]; it is also reproduced in [OO].

From now on we shall assume that $a_n = n$. Then the denominator in the right–hand side of (1.11) will be equal to $(t)_n = t(t+1)\cdots(t+n-1)$, and (1.11) will take the form

$$\varphi(\mu) = \frac{(-1)^n\,\pi(\mathcal{P}_\mu^*)}{(t)_n}, \qquad t = -\pi(\mathcal{P}_{(1)}^*), \quad n = |\mu|. \tag{1.13}$$

Similarly, the formula (1.12) can be rewritten as follows

$$\frac{\dim_{\mathbb{G}}(\mu,\lambda)}{\dim_{\mathbb{G}}\lambda} = \frac{(-1)^n\,\mathcal{P}_\mu^*(\lambda)}{(-N)_n}, \qquad \mu\in\mathbb{G}_n, \quad \lambda\in\mathbb{G}_N, \quad n\le N. \tag{1.14}$$

Note that, for any fixed λ, the left–hand side of (1.14) satisfies the harmonicity relation (1.1) provided that $n < N$: this easily follows from the very definition of the dimension function (for $n > N$ the denominator in the right–hand side vanishes). On the other hand, the expression in the right–hand side of (1.14) is a particular case of that in the right–hand side of (1.13): here π is the evaluation functional $\pi_\lambda : f \mapsto f(\lambda)$ and $t = -N$. This makes it possible to interpret the construction of Proposition 1.3 as follows: we extrapolate the relation (1.14) from the points $\lambda \in \mathbb{G}$, which we identify with the corresponding evaluation functionals π_λ, to abstract multiplicative functionals.

A function $\varphi \in \mathcal{H}_1^+(\mathbb{G})$ will be called *nondegenerate* if $\varphi(\mu) \neq 0$ for all $\mu \in \mathbb{G}$; otherwise it will be called *degenerate*.

§2. The Young graph

The fundamental example of a graded graph $\mathbb{G}$ is the *Young graph* $\mathbb{Y}$ [VK], [KV2]. By definition, the vertices of $\mathbb{Y}$ are the Young diagrams including the empty diagram $\varnothing$, the n-th floor $\mathbb{Y}_n$ consists of the diagrams with n boxes, and $\mu \nearrow \lambda$ means that λ is obtained from μ by adding a single box. The numbers $\varkappa(\mu, \lambda)$ are all equal to 1. In this section the symbols μ, λ are used to denote Young diagrams.

The graph $\mathbb{Y}$ is multiplicative in the sense of the definition given in §1: here the algebra A is the algebra Λ of symmetric functions, the basis elements $\mathcal{P}_\mu$ are the Schur functions s_μ, and the relation (1.5) turns into a special case of the Pieri rule for the Schur functions,

$$s_\mu s_{(1)} = \sum_{\lambda:\, \lambda \searrow \mu} s_\lambda\,, \tag{2.1}$$

which is equivalent (under the characteristic map, see [M, I.7]) to the Young branching rule for irreducible characters of symmetric groups. For the Young graph, the expansion (1.6) takes the form

$$s_{(1)}^n = \sum_{\lambda:\, |\lambda|=n} \dim\lambda \cdot s_\lambda, \tag{2.2}$$

where $\dim\lambda = \dim_{\mathbb{Y}}\lambda$ is the number of standard Young tableaux of shape λ.

Let $b = (i, j)$ be a box of μ; here i, j are the row number and the column number of b. Recall the definition of the *content*, the *arm–length* and the *leg–length* of b:

$$c(b) = j - i, \quad a(b) = \mu_i - j, \quad l(b) = \mu'_j - i, \tag{2.3}$$

where μ' is the transposed diagram.

Theorem 2.1. *Let z, z' be arbitrary complex numbers and $t = zz'$. Assume that $t \neq 0, -1, -2, \dots$. Then the following expression is a harmonic function on the Young graph:*

$$\varphi_{zz'}(\mu) = \frac{1}{(t)_n} \prod_{b \in \mu} \frac{(z + c(b))(z' + c(b))}{a(b) + l(b) + 1}\,, \quad n = |\mu|. \tag{2.4}$$

The harmonic functions (2.4) *fit into the general scheme of Proposition 1.3 with the algebra A^* and the family $\{\mathcal{P}^*_\mu\}$ as specified below.*

The first claim of the theorem (harmonicity of $\varphi_{zz'}$) follows from the computation of a spherical function in [KOV]. Various direct combinatorial proofs for this claim were given by Kerov, Postnikov, and Borodin. Kerov's approach is explained in [Ke5]; actually, in that paper a more general result is obtained, see Theorem 3.1 below. Postnikov's argument was not published. Borodin's argument is, perhaps, the most direct and elementary; it was given in the appendix to [P.I]; actually, the present paper originated from our discussion of that argument.

For the proof we need some preparations. First, we specify the algebra A^*.

Denote by $\Lambda^*(n)$ the subalgebra in $\mathbb{C}[x_1,\dots,x_n]$ formed by the polynomials which are symmetric in 'shifted' variables $x'_j = x_j - j$, $j = 1,\dots,n$. Define the projection map $\Lambda^\theta(n) \to \Lambda^\theta(n-1)$ as the specialization $x_n = 0$ and note that this projection preserves the filtration defined by ordinary degree of polynomials. Now we take the projective limit of $\Lambda^*(n)$'s in the category of filtered algebras as $n \to \infty$. The result is a filtered algebra which is called the *algebra of shifted symmetric functions* and denoted by Λ^*.

The algebra Λ^* will be taken as the algebra A^*. As the elements $\mathcal{P}^*_\mu$ we shall take the *shifted Schur functions* s^*_μ as defined in [OO].

By the definition of Λ^*, each element $f \in \Lambda^*$ can be evaluated at any sequence $x = (x_1, x_2, \dots)$ with finitely many nonzero terms. In particular, we can evaluate shifted symmetric functions at any $\lambda = (\lambda_1, \lambda_2, \dots) \in \mathbb{Y}$, which allows one to interpret Λ^* as a certain algebra of functions on the Young diagrams. This point of view was developed in [KO]. The shifted Schur functions s^*_μ possess the Interpolation Property of §1, see [Ok1], [OO].

For the one–row shifted Schur functions there is a special notation: $h^*_m = s^*_{(m)}$. A useful tool is the following generating series for the h^* functions:

$$H^*(u) = 1 + \sum_{m=1}^{\infty} \frac{h^*_m}{u(u-1)\dots(u-m+1)}\,. \tag{2.5}$$

Here u is a formal indeterminate and the series is viewed as an element of $\Lambda^*[[\frac{1}{u}]]$. Since the elements h^*_m are algebraically independent generators of Λ^*, a multiplicative functional $\pi : \Lambda^* \to \mathbb{C}$ can be uniquely defined by assigning to $H^*(u)$ an arbitrary formal power series in $\frac{1}{u}$ with constant term 1. We shall use this fact below.

Note a useful formula

$$H^*(u)(x_1, x_2, \dots) = \prod_{i=1}^{\infty} \frac{u+i}{u+i-x_i}\,, \tag{2.6}$$

see [OO, Theorem 12.1]. Here, by definition,

$$H^*(u)(x_1, x_2, \dots) = 1 + \sum_{m=1}^{\infty} \frac{h_m^*(x_1, x_2, \dots)}{u(u-1)\dots(u-m+1)}\,. \tag{2.7}$$

The equality (2.6) can be understood as follows. We assume that only finitely many of x_i's are distinct from zero. Then the left–hand side, which is the series (2.7), converges in a left half–plane $\Re u < \mathrm{const} \ll 0$ and equals the right–hand side of (2.6).

For an element f of Λ or Λ^*, we shall abbreviate

$$f(x_1, \dots, x_k) = f(x_1, \dots, x_k, 0, 0, \dots).$$

Recall the combinatorial formula for the Schur functions:

$$s_\mu(x_1, \dots, x_k) = \sum_T \prod_{b\in\mu} x_{T(b)}\,, \tag{2.8}$$

where T ranges over the set of Young tableaux of shape μ with entries in $\{1, \dots, k\}$, see [M, I.5]. It will be convenient for us to employ here the reverse tableaux (i.e., the entries $T(b)$ decrease from left to right along the rows and down the columns). Since s_μ is symmetric, (2.8) also holds if the sum in the right–hand side is taken over all reverse tableaux of shape μ with entries in $\{1, \dots, k\}$.

We shall need a similar formula for the shifted Schur functions:

$$s_\mu^*(x_1, \dots, x_k) = \sum_T \prod_{b\in\mu} (x_{T(b)} - c(b))\,, \tag{2.9}$$

where T ranges over reverse tableaux of shape μ with entries in $\{1, \dots, k\}$, see [OO, Theorem 11.1].

Proposition 2.2. *Let $k = 1, 2, \dots$ and $z' \in \mathbb{C}$. The following specialization formula holds*

$$s_\mu^*(\underbrace{-z', \dots, -z'}_{k}) = (-1)^n \prod_{b\in\mu} \frac{(k + c(b))(z' + c(b))}{a(b) + l(b) + 1}\,, \quad n = |\mu|. \tag{2.10}$$

Proof. Compare the combinatorial formulas (2.8) and (2.9). If $x_1 = \dots = x_k = -z'$ then the product in (2.9) does not depend on T and is equal to

$$\prod_{b\in\mu} (-z' - c(b)) = (-1)^n \prod_{b\in\mu} (z' + c(b)).$$

It follows that

$$s^*_\mu(\underbrace{-z',\dots,-z'}_{k}) = (-1)^n \prod_{b\in\mu}(z'+c(b))\cdot s_\mu(\underbrace{1,\dots,1}_{k}).$$

Now we apply the well–known specialization formula

$$s_\mu(\underbrace{1,\dots,1}_{k}) = \prod_{b\in\mu}\frac{k+c(b)}{a(b)+l(b)+1}\,,$$

see [M, I.3, Ex.4], which implies (2.10). □

The argument used in the proof is borrowed from Okounkov's paper [Ok4], the derivation of formula (1.9); see also Proposition 3.2 below.

Corollary 2.3. *For any $z,z'\in\mathbb{C}$, the linear functional $\pi_{zz'}:\Lambda^*\to\mathbb{C}$ given by*

$$\pi_{zz'}(s^*_\mu) = (-1)^n\prod_{b\in\mu}\frac{(z+c(b))(z'+c(b))}{a(b)+l(b)+1},\qquad n=|\mu|, \tag{2.11}$$

is multiplicative.

Proof. Indeed, this expression depends polynomially on z. So, it suffices to prove the multiplicativity of $\pi_{zz'}$ in the case $z=k$, where $k=1,2,\dots$. By Proposition 2.2, in this case our functional is the evaluation at the point

$$x = (\underbrace{-z',\dots,-z'}_{k},0,0,\dots).$$

Consequently, the functional is multiplicative. □

Proof of Theorem 2.1. We apply Proposition 1.3 by taking $A^*=\Lambda^*$ and $\mathcal{P}^*_\mu = s^*_\mu$. The Pieri–type formula for s^*–functions ([OO, Theorem 9.1]) shows that the relation (1.8) holds with the sequence $a_n=n$. We take as π the multiplicative functional $\pi_{zz'}$ afforded by Corollary 2.3. It follows from (2.11) that

$$-\pi_{zz'}(s^*_{(1)}) = zz' = t,$$

so that we may substitute t into (1.11). Finally, the condition (1.10) is just the assumption on t given in Theorem 2.1. Thus, the expression (2.4) is a special case of (1.11), which concludes the proof. □

Remark 2.4. In terms of the generating series (2.5) for the h^* functions, the multiplicative functional $\pi_{zz'}$ can be described as follows:

$$\begin{aligned}\pi_{zz'}(H^*(u)) &= 1+\sum_{m=1}^{\infty}\frac{(z)_m(z')_m}{(-u)_m\, m!}\\ &= {}_2F_1(z,z';-u;1)=\frac{\Gamma(-u)\Gamma(-u-z-z')}{\Gamma(-u-z)\Gamma(-u-z')}\,,\end{aligned}\tag{2.12}$$

where we assume $\Re u \ll 0$; the last equality follows from Gauss' summation formula.

Proposition 2.5. *The function $\varphi_{zz'}$ afforded by Theorem 2.1 is a nondegenerate function from $\mathcal{H}_1^+(\mathbb{Y})$ if and only if the parameters satisfy one of the following two conditions:*

- *either $z' = \bar{z}$ where $z \in \mathbb{C} \setminus \mathbb{Z}$,*
- *or z, z' are real and there exists $m \in \mathbb{Z}$ such that $m < z, z' < m+1$.*

The proof is easy, see [P.I].

Let us explain the significance of the set $\mathcal{H}_1^+(\mathbb{G})$ for the Young graph. Let $S(\infty)$ be the infinite symmetric group, which is defined as the inductive limit $\varinjlim S(n)$ of the finite symmetric groups. In other words, $S(\infty)$ consists of the finite permutations of the set $\{1,2,\dots\}$. There is a natural bijective correspondence $\varphi \leftrightarrow \chi$ between functions $\varphi \in \mathcal{H}(\mathbb{Y})$ and central functions χ on $S(\infty)$. Specifically, given χ, we define the values of φ on $\mathbb{Y}_n$ from the expansion of the central function $\chi \downarrow S(n)$ on the group $S(n)$ into a linear combination of the irreducible characters χ^λ,

$$\chi \downarrow S(n) = \sum_{\lambda\colon |\lambda|=n} \varphi(\lambda)\chi^\lambda\,, \qquad n = 1,2,\dots. \tag{2.13}$$

The harmonicity of the function φ follows from the Young branching rule for the irreducible characters [JK], [OV]:

$$\chi^\lambda \downarrow S(n-1) = \sum_{\mu\colon \mu\nearrow\lambda} \chi^\mu\,, \quad n = |\lambda|. \tag{2.14}$$

Under the bijection $\varphi \leftrightarrow \chi$, the nonnegativity of φ means that the function χ is positive definite on the group $S(\infty)$, and the normalization $\varphi(\varnothing) = 1$ means that $\chi(e) = 1$, where $e \in S(\infty)$ is the trivial permutation. Thus, the elements of $\mathcal{H}_1^+(\mathbb{Y})$ correspond to the central, positive definite, normalized functions on $S(\infty)$. Such functions form a convex set, its extreme points are called the *characters* of the group $S(\infty)$ (in the sense of von Neumann). According to §1, we shall denote the set of the characters by $\Omega(\mathbb{Y})$.

Thoma proved [T] that the characters of $S(\infty)$ can be parametrized by the points of an infinite–dimensional simplex:

$$\Omega(\mathbb{Y}) = \{\alpha_1 \ge \alpha_2 \ge \dots \ge 0,\ \beta_1 \ge \beta_2 \ge \dots \ge 0 \mid \sum_i (\alpha_i + \beta_i) \le 1\} \tag{2.15}$$

called the *Thoma simplex.* It is equipped with the weakest topology in which the coordinates α_i's and β_i's are continuous functions.

Via Gelfand–Naimark–Segal construction, characters generate *finite factor representations* of the group $S(\infty)$. They also correspond to *irreducible unitary spherical representations* of the Gelfand pair (G, K) where G is the "bisymmetric group" $S(\infty) \times S(\infty)$, and K is the diagonal subgroup of G [Ol], [Ok3].

The Poisson kernel $K(\mu, \omega)$ (see Theorem 1.1) for the Young graph is given by the image of the Schur function s_μ under a certain specialization of Λ depending on ω. Namely, for $\omega = (\alpha, \beta) \in \Omega(\mathbb{Y})$ we specialize the power sums as follows

$$(x_1^k + x_2^k + \dots) \mapsto \begin{cases} 1, & k = 1, \\ \sum_{i=1}^\infty \alpha_i^k + (-1)^{k-1} \sum_{i=1}^\infty \beta_i^k, & k \geq 2, \end{cases} \tag{2.16}$$

see [VK], [KOO].

According to the general theory of §1, the functions $\varphi_{zz'} \in \mathcal{H}_1^+(\mathbb{Y})$ constructed above give rise to a family of probability measures on $\Omega(\mathbb{Y})$. These measures were thoroughly studied in [P.I–P.V], [BO1], [BO2].

§3 The Jack graph

Fix a positive number θ. Let P_μ be the Jack symmetric function with parameter θ and index μ (see [M, VI.10]; note that Macdonald uses $\alpha = \theta^{-1}$ as the parameter). The simplest case of Pieri's formula for the Jack functions reads as follows:

$$P_\mu P_{(1)} = \sum_{\lambda :\, \lambda \searrow \mu} \varkappa_\theta(\mu, \lambda) P_\lambda \,, \tag{3.1}$$

where $\varkappa_\theta(\mu, \lambda)$ are certain positive numbers,

$$\varkappa_\theta(\mu, \lambda) = \prod_b \frac{\big(a(b) + (l(b)+2)\theta\big)\big(a(b) + 1 + l(b)\theta\big)}{\big(a(b) + (l(b)+1)\theta\big)\big(a(b) + 1 + (l(b)+1)\theta\big)} \,. \tag{3.2}$$

Here b ranges over all boxes in the jth column of the diagram μ, provided that the new box $\lambda \setminus \mu$ belongs to the jth column of λ, see [M, VI.10, VI.6].

The *Jack graph* $\mathbb{J}(\theta)$ is the multiplicative graph associated with the algebra $A = \Lambda$ of symmetric functions and its basis formed by the Jack symmetric functions. I.e., this is the Young graph with the formal edge multiplicities $\varkappa(\mu, \lambda) = \varkappa_\theta(\mu, \lambda)$. When $\theta = 1$, the Jack functions turn into the Schur functions, all formal edge multiplicities are equal to 1, so that the Jack graph turns into the ordinary Young graph.

We take as A^* the algebra Λ^θ of shifted symmetric functions with parameter θ. It is defined as the projective limit of filtered algebras $\Lambda^\theta(n)$, where, in turn, $\Lambda^\theta(n)$ is formed by polynomials in $x_1, \dots, x_n$ which are symmetric with respect to the new variables $x'_j = x_j - \theta j$. The projection $\Lambda^\theta(n) \to \Lambda^\theta(n-1)$, as in the case of the Young graph, is given by the specialization $x_n = 0$.

For any μ there exists a unique element $P^*_\mu \in \Lambda^\theta$ of degree $|\mu|$, with highest term P_μ and with the Interpolation Property; it is called the *shifted Jack function.* Our reference about these functions is [OO2], [Ok2].

The relation (1.8) for shifted Jack functions has the form

$$P^*_\mu P^*_{(1)} = nP^*_\mu + \sum_{\lambda:\,\lambda\searrow\mu} \varkappa_\theta(\mu,\lambda) P^*_\lambda\,, \qquad n = |\mu|. \tag{3.3}$$

Given a box $b = (i,j)$ of a Young diagram, we denote its θ*-content* as $c_\theta(b) = (j-1) - \theta(i-1)$. When $\theta = 1$, this turns into the ordinary content.

Theorem 3.1. *Let z, z' be arbitrary complex numbers and $t = \theta^{-1}zz'$. Assume that $t \neq 0, -1, -2, \dots$. Then the following expression is a harmonic function on the Jack graph $\mathbb{J}(\theta)$:*

$$\varphi_{zz'}(\mu) = \frac{1}{(t)_n} \prod_{b\in\mu} \frac{(z + c_\theta(b))(z' + c_\theta(b))}{a(b) + \theta l(b) + \theta}\,, \quad n = |\mu|. \tag{3.4}$$

The functions (3.4) *fit into the general scheme of Proposition 1.3 with $A^* = \Lambda^\theta$ and $\mathcal{P}^*_\mu = P^*_\mu$.*

This result generalizes Theorem 2.1. The first claim is due to Kerov [Ke5]. Our proof of Theorem 3.1 is very similar to that of Theorem 2.1, so we shall only indicate necessary modifications.

The analogs of the combinatorial formulas (2.8) and (2.9) are as follows:

$$P_\mu(x_1, \dots, x_k) = \sum_T \psi_T(\theta) \prod_{b\in\mu} x_{T(b)}\,, \tag{3.5}$$

$$P^*_\mu(x_1, \dots, x_k) = \sum_T \psi_T(\theta) \prod_{b\in\mu} (x_{T(b)} - c_\theta(b))\,, \tag{3.6}$$

where, again, the summation is taken over the reverse Young tableaux of shape μ, and $\psi(\theta)$ are certain numeric factors. We do not need their exact values, the point is that they are the same in both formulas, see [Ok2], [Ok4].

Proposition 3.2. *Let $k = 1, 2, \dots$ and $z' \in \mathbb{C}$. The following specialization formula holds*

$$P^*_\mu(\underbrace{-z', \dots, -z'}_{k}) = (-1)^n \prod_{b\in\mu} \frac{(k + c_\theta(b))(z' + c_\theta(b))}{a(b) + \theta l(b) + \theta}\,, \quad n = |\mu|. \tag{3.7}$$

Proof. The argument is exactly similar to that for Proposition 2.2. We employ the combinatorial formulas (3.5), (3.6) and the well–known specialization formula for the Jack symmetric functions:

$$P_\mu(\underbrace{1,\dots,1}_{k}) = \prod_{b\in\mu} \frac{\theta k + c_\theta(b)}{a(b)+\theta l(b)+\theta}\,,$$

see [M, VI, (10.20)]. □

A more general formula appeared in [Ok4, (1.9)].

Proposition 3.2 immediately leads to the following claim.

Corollary 3.3. *For any $z, z' \in \mathbb{C}$, the linear functional $\pi_{zz'} : \Lambda^\theta \to \mathbb{C}$ given by*

$$\pi_{zz'}(P^*_\mu) = (-1)^n \prod_{b\in\mu} \frac{(z+c_\theta(b))(z'+c_\theta(b))}{a(b)+\theta l(b)+\theta}, \qquad n = |\mu|, \tag{3.8}$$

is multiplicative.

Proof. The argument is just the same as for Corollary 2.3. □

Proof of Theorem 3.1. Exactly the same as for Theorem 2.1. □

Extreme points of the convex set $\mathcal{H}^+_1(\mathbb{J}(\theta))$ of nonnegative normalized harmonic functions on the Jack graph, as in the case of the Young graph ($\theta = 1$), can be parametrized by points of the Thoma simplex (2.15), see [KOO].

The Poisson kernel $K(\mu,\omega)$ is defined as the image of the Jack function P_μ under the specialization of Λ which sends power sums to the following expressions (cf. (2.16)):

$$(x_1^k + x_2^k + \dots) \mapsto \begin{cases} 1, & k=1, \\ \sum_{i=1}^\infty \alpha_i^k + (-\theta)^{k-1}\sum_{i=1}^\infty \beta_i^k, & k \ge 2, \end{cases} \tag{3.9}$$

see [KOO].

As was mentioned in §2, the set $\mathcal{H}^+_1(\mathbb{Y}) = \mathcal{H}^+_1(\mathbb{J}(1))$ has a representation theoretic meaning. There is one more value of θ, namely $\theta = 1/2$, when harmonic functions on the Jack graph can be related to representations. We shall briefly explain this connection.

Let G be the group of finite permutations of the set $\{\pm1, \pm2, \dots\}$ and K be its subgroup consisting of the permutations which commute with the involution $i \mapsto -i$. The group G is just another realization of the infinite symmetric group $S(\infty)$, and the group K can be considered as an infinite version of the hyperoctahedral groups. Note that (G,K) is a Gelfand pair [Ol].

It turns out that there exists a natural one–to–one correspondence between $\mathcal{H}^+_1(\mathbb{J}(1/2))$ and the set of positive definite, K–biinvariant functions on G normalized at the unity (this correspondence is based on classical facts explained in [M, VII.2]). In particular, extreme functions from $\mathcal{H}^+_1(\mathbb{J}(1/2))$ correspond to the spherical functions of irreducible unitary spherical representations of (G,K). For more details about these representations, see [Ol], [Ok3].

4. The Kingman graph

Let

$$\mu = (\mu_1 \geq \dots \geq \mu_l > 0) = (1^{r_1(\mu)} 2^{r_2(\mu)} \dots)$$

denote an arbitrary partition also viewed as a Young diagram.

In this section we are dealing with the monomial symmetric functions m_μ [M, I.2]. They form a basis of the algebra Λ and obey the relation:

$$m_\mu m_{(1)} = \sum_{\lambda:\, \lambda \searrow \mu} \varkappa_0(\mu, \lambda) m_\lambda \,, \tag{4.1}$$

where the positive integers $\varkappa_0(\mu, \lambda)$ are defined as follows: if k stands for the length of the row in λ containing the box $\lambda \setminus \mu$ then $\varkappa_0(\mu, \lambda) = r_k(\lambda)$.

The *Kingman graph* $\mathbb{K}$ is the multiplicative graph associated with the algebra Λ and its basis $\{m_\mu\}$ [Ke1]. I.e., this is the Young graph with the formal edge multiplicities $\varkappa_0(\mu, \lambda)$. Since the numbers $\varkappa_0(\mu, \lambda)$ are integers, one can regard $\mathbb{K}$ as a graph with multiple edges.

Next, introduce the *factorial monomial symmetric functions* m^*_μ, which are also elements of Λ. By definition [Ke1], m^*_μ is the sum of all *distinct* expressions obtained from

$$\prod_{i=1}^{l} x_i(x_i - 1) \dots (x_i - \mu_i + 1)$$

by permutations of the variables $x_1, x_2, \dots$. Thus, the definition of m^*_μ is similar to that of m_μ, the only difference is that the ordinary powers x^m are replaced by the falling factorial powers $x(x-1)\dots(x-m+1)$.

The function m^*_μ can be characterized as the only symmetric function with the highest term m_μ and such that $m^*_\mu(\lambda_1, \lambda_2, \dots) = 0$ for any diagram $\lambda \neq \mu$, $|\lambda| \leq |\mu|$. Thus, m^*_μ possesses the Interpolation Property of §1.

One can directly verify that

$$m^*_\mu m^*_{(1)} = n m^*_\mu + \sum_{\lambda:\, \lambda \searrow \mu} \varkappa_0(\mu, \lambda) m^*_\lambda, \qquad n = |\mu| \tag{4.2}$$

(this also follows from the Interpolation Property).

Theorem 4.1. *Let t, α be complex parameters, $t \neq -1, -2, \dots$. Then the function*

$$\varphi_{t,\alpha}(\mu) = \frac{(\mu_1 - 1)! \dots (\mu_l - 1)!}{r_1(\mu)! r_2(\mu)! \dots} \cdot \frac{t(t+\alpha)\dots(t+(l-1)\alpha)}{(t)_n} \cdot \prod_{\substack{b=(i,j)\in\mu \\ j \geq 2}} \left(1 - \frac{\alpha}{j-1}\right), \tag{4.3}$$

where $n = |\mu|$, is harmonic on the graph $\mathbb{K}$. Here l is the length (number of nonzero parts) of μ.

The functions (4.3) *fit into the general scheme of Proposition 1.3 with* $A^* = \Lambda$ *and* $\mathcal{P}^*_\mu = m^*_\mu$.

As is explained below, the first claim is equivalent to a result of Pitman [Pi].

Proof. According to Proposition 1.3, it suffices to check that there exists a multiplicative functional $\pi_{t,\alpha} : \Lambda \to \mathbb{C}$ such that

$$\begin{aligned} \pi_{t,\alpha}(m^*_\mu) = (-1)^n \, \frac{(\mu_1-1)!\dots(\mu_l-1)!}{r_1(\mu)!r_2(\mu)!\dots} \\ \cdot t(t+\alpha)\dots(t+(l-1)\alpha) \cdot \prod_{\substack{b=(i,j)\in\mu \\ j\geq 2}} \left(1-\frac{\alpha}{j-1}\right) . \end{aligned} \tag{4.4}$$

As the functions m^*_μ form a basis in Λ, we can define a linear functional $\pi_{t,\alpha} : \Lambda \to \mathbb{C}$ by the formula (4.4). We claim that it is multiplicative if $t = -k\alpha$, where $k = 1, 2, \dots$. To see this we shall prove that $\pi_{-k\alpha,\alpha}$ coincides with the specialization at the point $(\underbrace{\alpha,\dots,\alpha}_{k})$.

Indeed, from the definition of m^*_μ it follows that

$$m^*_\mu(\underbrace{\alpha,\dots,\alpha}_{k}) = \frac{k(k-1)\dots(k-l+1)}{r_1(\mu)!r_2(\mu)!\dots} \prod_{i=1}^{l} \alpha(\alpha-1)\dots(\alpha-\mu_i+1), \tag{4.5}$$

and a direct verification shows that this expression coincides with $\pi_{-k\alpha,\alpha}(m^*_\mu)$.

Finally, as the right–hand side of (4.4) depends on the parameters t, α polynomially, $\pi_{t,\alpha}$ is multiplicative for all values of the parameters. □

Proposition 4.2. *The function* $\varphi_{t,\alpha}$ *afforded by Theorem 4.1 is a nondegenerate function from* $\mathcal{H}^+_1(\mathbb{K})$ *if and only if the parameters* t, α *are real and satisfy the inequalities* $0 \leq \alpha < 1$, $t > -\alpha$.

The proof is straightforward.

There is a bijective correspondence between the functions $\varphi \in \mathcal{H}^+_1(\mathbb{K})$ and the *partition structures* in the sense of Kingman [Ki1], [Ki2]. According to Kingman, a partition structure is a sequence $M = (M_n)$ of probability distributions on the partitions of n, $n = 1, 2, \dots$, such that for each n, M_n and M_{n+1} are connected by a certain consistency relation. These sequences are nothing else than the sequences (M_n) as defined in §1, see (1.4). Thus, the passage from a harmonic function $\varphi \in \mathcal{H}^+_1(\mathbb{K})$ to the corresponding partition structure is given by the formula

$$M_n(\mu) = \varphi(\mu) \cdot \dim_{\mathbb{K}}(\mu), \qquad |\mu| = n,$$

where

$$\dim_{\mathbb{K}}(\mu) = \frac{n!}{\mu_1!\dots\mu_l!} \tag{4.6}$$

(the latter formula readily follows from the general relation (1.12) if we substitute $\lambda = \mu$ and $\mathcal{P}^*_\mu = m^*_\mu$).

Under this correspondence, the functions $\varphi_{t,0}$ with $t > 0$ turn into Ewens' partition structures [Ew]. More general functions $\varphi_{t,\alpha}$ with the restrictions $t > -\alpha$, $0 \le \alpha < 1$, of Proposition 4.2 correspond to Pitman's two–parameter generalization of Ewens' partition structures [Pi], [Ke4]. Note that the harmonic functions $\varphi_{-k\alpha,\alpha}$ with $k = 1, 2, \dots$, which appear in the proof of Theorem 4.1, are nonnegative and *degenerate* provided that $\alpha < 0$; the significance of the corresponding partition structures is explained in the introduction to [Pi].

Here is yet another interpretation of the harmonic functions $\varphi_{t,\alpha}$.

For $n \ge 2$ there exists a unique map $S(n) \to S(n-1)$ which commutes with the two–sided action of the smaller group $S(n-1)$. This map, called the *canonical projection,* can be defined as follows: if $s, s_1, s_2 \in S(n-1)$ and $(n-1, n)$ stands for the elementary transposition of "$n-1$" and "n", then $s \mapsto s$ and $s_1 \cdot (n-1, n) \cdot s_2 \mapsto s_1 s_2$. In other words, the canonical projection is defined by removing "n" from the cycle $\dots i \to n \to j \dots$ containing it, see [KOV].

The projective limit $\mathfrak{X} = \varprojlim S(n)$ taken with respect to the canonical projections is a compact topological space. Its elements are called *virtual permutations.* There is a natural embedding $S(\infty) \to \mathfrak{X}$ whose image is dense, so that X is a certain compactification of the discrete set $S(\infty)$. The two–sided action of $S(\infty)$ on itself can be extended to $\mathfrak{X}$, which makes $\mathfrak{X}$ a $S(\infty) \times S(\infty)$-space. This construction and its meaning for the representation theory of the group $S(\infty)$ is discussed in [KOV].

A probability measure $\mathcal{M}$ on $\mathfrak{X}$ is called *central* if it is invariant under the action of the diagonal subgroup in $S(\infty) \times S(\infty)$ (that action extends the action of $S(\infty)$ on itself by conjugations). There is a natural bijective correspondence $\varphi \leftrightarrow \mathcal{M}$ between the elements $\varphi \in \mathcal{H}^+_1(\mathbb{K})$ and the central measures $\mathcal{M}$ on $\mathfrak{X}$. It is specified as follows: for each $n = 1, 2, \dots$, the image of $\mathcal{M}$ under the composite projection $\mathfrak{X} \to S(n) \to \mathbb{K}_n$ coincides with M_n; here the second arrow $S(n) \to \mathbb{K}_n$ assigns to a permutation its cycle structure which is identified with a partition, and (M_n) is the partition structure corresponding to φ.

Let us denote by $\mathcal{M}_{t,\alpha}$ the measures corresponding to the harmonic functions $\varphi_{t,\alpha}$, where the parameters satisfy the conditions of Proposition 4.2. The measure $\mathcal{M}_{1,0}$ is invariant with respect to $S(\infty) \times S(\infty)$ and it is the only probability measure with this property. The measures $\mathcal{M}_{t,0}$ with $t > 0$ were employed in [KOV] for a geometric construction of generalized regular representations of the group $S(\infty) \times S(\infty)$ which are closely related to the harmonic functions $\varphi_{zz'}$ defined in §2. Note that all the measures $\mathcal{M}_{t,\alpha}$ are quasi–invariant under the action of this group, see [KOV], [Ke4].

Extreme points of the convex set $\mathcal{H}^+_1(\mathbb{K})$ can be parametrized by nonincreasing sequences of nonnegative numbers with sum less or equal to 1 (cf. (2.15)):

$$\Omega(\mathbb{K}) = \{\alpha_1 \ge \alpha_2 \ge \dots \mid \sum_i \alpha_i \le 1\}, \tag{4.7}$$

and the Poisson kernel $K(\mu,\omega)$ is given in this case by *extended monomial symmetric functions*:

$$K(\mu,\alpha)=K(1^{r_1}2^{r_2}\dots,\alpha)=\sum_{k=0}^{r_1}\frac{(1-\sum_i\alpha_i)^k}{k!}\,m_{(1^{r_1-k}2^{r_2}\dots)}(\alpha_1,\alpha_2,\dots), \tag{4.8}$$

see [Ki2], [Ke4].

Let us also note that the Kingman graph may be viewed as the degeneration of the Jack graph $\mathbb{J}(\theta)$ as $\theta\to 0$. Indeed, according to the definition of the Jack functions, their expansion in the monomial functions has the form

$$P_\mu=m_\mu+\text{lower terms}$$

relative to the dominance order on partitions [M, VI, (10.13)]. It is well known that in the limit $\theta\to 0$ the coefficients of all the lower terms vanish. In this sense, the Jack functions P_μ degenerate to the monomial functions m_μ as $\theta\to 0$. This implies, in particular, the limit relation

$$\varkappa_0(\mu,\lambda)=\lim_{\theta\to 0}\varkappa_\theta(\mu,\lambda),$$

which can also be checked directly from (3.2).

§5. The Schur graph

Recall that a partition is said to be *strict* if its nonzero parts are pairwise distinct. In this section, the symbols μ and λ always mean strict partitions. Using the standard correspondence between partitions and Young diagrams we introduce the relation $\mu\nearrow\lambda$ as before, see §2. Then the *Schur graph* $\mathbb{S}$ is defined as follows: the vertices of the nth floor $\mathbb{S}_n$ are the strict partitions of n, and the edges are the couples $\mu\nearrow\lambda$. By definition, the empty partition $\varnothing$ is included into the set of the strict partitions. All the edge multiplicities are equal to 1. Thus, the Schur graph is a subgraph of the Young graph.

Let Γ denote the subalgebra in Λ generated by the odd power sums $p_1=\sum_i x_i$, $p_3=\sum_i x_i^3,\dots$. Equivalently, Γ consists of those symmetric functions $f(x_1,x_2,\dots)$ which satisfy the following cancellation condition: for any $i\ne j$, the result of specializing $x_i=y$, $x_j=-y$ in f does not depend on y (see [Pr]).

In this section, the symbol P_μ stands for the Schur P function indexed by a strict partition μ. The Schur P functions form a homogeneous basis of Γ, $\deg P_\mu=|\mu|$. They obey the following Pieri–type rule:

$$P_\mu P_{(1)}=\sum_{\lambda:\,\lambda\searrow\mu}P_\lambda\,, \tag{5.1}$$

see [M, III.8]. Thus, $\mathbb{S}$ is a multiplicative graph with $A=\Gamma$ and $\{\mathcal{P}_\mu\}=\{P_\mu\}$.

Note that

$$\dim_{\mathbb{S}} \mu = \frac{n!}{\prod_{i=1}^{\ell(\mu)} \mu_i!} \prod_{1 \le i < j \le \ell(\mu)} \frac{\mu_i - \mu_j}{\mu_i + \mu_j}. \tag{5.2}$$

Here and below $\ell(\mu)$ denotes the length of μ.

We shall employ the *factorial Schur P functions* P^*_μ, see [I1]. These are inhomogeneous elements of Γ with the Interpolation Property; the highest term of P^*_μ coincides with P_μ. According to the general formalism, the P^* functions satisfy the relation

$$P^*_\mu P^*_{(1)} = nP^*_\mu + \sum_{\lambda:\, \lambda \searrow \mu} P^*_\lambda, \qquad n = |\mu|, \tag{5.3}$$

which has the same form as for the shifted Schur functions (see [OO, Theorem 9.1]), except that now μ and λ are not arbitrary but strict partitions.

There is a convenient generating series for the one–row P^* functions,

$$F^*(u) = 1 + \sum_{m=1}^{\infty} \frac{P^*_{(m)}}{u(u-1)\dots(u-m+1)}, \tag{5.4}$$

whose evaluation at a point $x = (x_1, x_2, \dots)$ has the form

$$F^*(u)(x) = \prod_{i=1}^{\infty} \frac{u+1+x_i}{u+1-x_i}, \tag{5.5}$$

see [I2]. The formulas (5.4), (5.5) should be understood in the same way as the formulas (2.6), (2.7).

Theorem 5.1. *Let $t > 0$. The following expression is a positive harmonic function on the Schur graph:*

$$\varphi_t(\mu) = \frac{1}{(t)_n} \frac{\prod_{(i,j)\in\mu}(2t + (j-1)j)}{2^{\ell(\mu)} \prod_{i=1}^{\ell(\mu)} \mu_i!} \prod_{1 \le i < j \le \ell(\mu)} \frac{\mu_i - \mu_j}{\mu_i + \mu_j}, \qquad n = |\mu|. \tag{5.6}$$

The harmonic functions (5.6) *fit into the general scheme of Proposition 1.3 with $A^* = \Gamma$ and $\mathcal{P}^*_\mu = P^*_\mu$.*

The first claim was established in [B1].

Let $\widetilde{\mu}$ denote the *shifted* Young diagram corresponding to μ (see [M, III.8]). Define t_1, t_2 from the conditions

$$t_1 + t_2 = 1, \qquad t_1 t_2 = 2t. \tag{5.7}$$

Then (cf. (2.4))

$$\prod_{(i,j)\in\mu} (2t + (j-1)j) = \prod_{b\in\widetilde{\mu}} (t_1 + c(b))(t_2 + c(b)) \tag{5.8}$$

so that the parameters t_1, t_2 are to a certain extent similar to z, z'.

Sketch of proof. As usual, we shall employ Proposition 1.3.

Consider the linear functional $\pi_t : \Gamma \to \mathbb{C}$ defined by

$$\pi_t(P^*_\mu) = (-1)^n \frac{\prod_{(i,j)\in\mu}(2t+(j-1)j)}{2^{\ell(\mu)}\prod_{i=1}^{\ell(\mu)}\mu_i!} \prod_{1\le i<j\le\ell(\mu)} \frac{\mu_i-\mu_j}{\mu_i+\mu_j}, \qquad n=|\mu|. \tag{5.9}$$

Here $t\in\mathbb{C}$ is arbitrary. Note that $\pi_t(P^*_{(1)}) = -t$, as prescribed by Proposition 1.3. According to the general formalism, it suffices to prove that π_t is multiplicative.

Since the expression (5.9) depends on t polynomially, it suffices to prove the claim for a countable number of different values of t. We shall assume that

$$t = \frac{k(1-k)}{2}, \qquad k=1,2,\dots. \tag{5.10}$$

That is, in the notation of (5.7),

$$t_1 = k, \quad t_2 = 1-k, \qquad k=1,2,\dots. \tag{5.11}$$

To prove the multiplicativity property for the values (5.10) we shall show that

$$\pi_{k(1-k)/2} = \text{ evaluation at the staircase diagram } (k,k-1,\dots,1). \tag{5.12}$$

This claim is an analog of Propositions 2.2, 3.2, and formula (4.5). It does not seem to have appeared in the literature before, so we give here a sketch of the proof.

We shall consecutively check (5.12) on the one–row P^* functions, next on the two–row P^* functions, and finally on arbitrary P^* functions.

First, consider the generating series (5.4) for the one–row functions. By (5.5), its evaluation at the kth staircase diagram is as follows:

$$F^*(u)(k,k-1,\dots,1) = \prod_{i=1}^{k} \frac{u+1+i}{u+1-i}. \tag{5.13}$$

Here and below we assume $\Re u \ll 0$.

On the other hand, by (5.4) and (5.9) we have

$$\begin{aligned}\pi_t(F^*(u)) &= 1+\sum_{m=1}^{\infty}\frac{(t_1)_m(t_2)_m}{(-u)_m m!}\\ &= {}_2F_1(t_1,t_2;-u;1) = \frac{\Gamma(-u)\Gamma(-u-t_1-t_2)}{\Gamma(-u-t_1)\Gamma(-u-t_2)},\end{aligned} \tag{5.14}$$

where the last equality is Gauss' summation formula, cf. (2.12). When t_1, t_2 are as in (5.11), this coincides with (5.13). Thus, we have checked (5.12) on the one–row functions.

Next, we employ recurrence relations which make it possible to express the two–row functions through the one–row ones. It is convenient to extend the definition of two–row functions $P^*_{(p,q)}$ with $p > q \geq 1$ to a larger set of indices by adopting the convention $P^*_{(p,q)} = -P^*_{(q,p)}$. Thus, $P^*_{(p,q)}$ makes sense for any $p, q = 1, 2, \dots$ (we assume that $P^*_{(p,q)} = 0$ for $p = q$). Then we have two families of relations:

$$P^*_{(p,1)} = P^*_{(p)} P^*_{(1)} - p P^*_{(p)}\,, \tag{5.15}$$

$$P^*_{(p+1,q)} + P^*_{(p,q+1)} + (p+q) P^*_{(p,q)} = P^*_{(p)} P^*_{(q+1)} - P^*_{(p+1)} P^*_{(q)} - (p-q) P^*_{(p)} P^*_{(q)}\,. \tag{5.16}$$

Here p, q range over $\{1, 2, \dots\}$. These relations were proved in [I2]. Note that (5.15) can be formally obtained from (5.16) by substituting $q = 0$.

Using double induction on $p + q$ and q we see that these relations indeed allow to express the two–row functions through the one–row functions.

A direct computation shows that the relations (5.15), (5.16) remain valid if we apply π_t to each P^* function involved. This means that (5.12) holds on the two–row functions.

Finally, to handle arbitrary P^* functions we employ the following relation proved in [I2]:

$$P^*_\mu = \operatorname{Pf}\left[P^*_{(\mu_i,\mu_j)}\right]_{1 \leq i,j \leq \ell(\mu)+\varepsilon}\,. \tag{5.17}$$

Here the symbol Pf means Pfaffian and ε equals 1 for odd $\ell(\mu)$ and 0 for even $\ell(\mu)$, so that the order of the matrix is always even. Note that this relation has exactly the same form as in the case of the classical Schur P functions, see [M, III.8].

To conclude that (5.12) holds on any P^*_μ we must verify that (5.17) remains valid if we apply π_t to each P^* function. This readily follows from the well–known relation

$$\prod_{1 \leq i < j \leq \ell(\mu)} \frac{\mu_i - \mu_j}{\mu_i + \mu_j} = \operatorname{Pf}\left[\frac{\mu_i - \mu_j}{\mu_i + \mu_j}\right]_{1 \leq i,j \leq \ell(\mu)+\varepsilon}\,, \tag{5.18}$$

see [M, III.8]. □

Proposition 5.2. *The harmonic function φ_t afforded by Theorem 5.1 is strictly positive if and only if $t > 0$.*

The proof is straightforward.

It should be noted that the values $t = k(1-k)/2$ used in the proof of Theorem 5.1 lie outside the region $t > 0$.

Quite similarly to the case of the Young graph, extreme points of $\mathcal{H}_1^+(\mathbb{S})$ correspond to *projective characters* or *projective finite factor representations* of the group

$S(\infty)$. They can be parametrized by the points of the infinite–dimensional simplex $\Omega(\mathbb{S}) = \Omega(\mathbb{K})$ described by (4.7), see [N], [I1].

The Poisson kernel $K(\mu, \omega)$ for the Schur graph is given by the image of the Schur P function P_μ under the specialization of the algebra Γ which sends odd powers sums to the following expressions (cf. (2.16), (3.9))

$$(x_1^k + x_2^k + \dots) \mapsto \begin{cases} 1, & k = 1, \\ \sum_{i=1}^{\infty} \alpha_i^k, & k = 2m + 1 \geq 3. \end{cases} \tag{5.19}$$

§6. Finite–dimensional specializations

In the previous four sections we described four different examples of graphs which fit into the general scheme introduced in §1. For each of these graphs we produced a nontrivial family of specializations of the corresponding algebras A^* which defined, according to Proposition 1.3, a certain family of (nonnegative) harmonic functions on the graph.

Every such family, in its turn, gives rise to a family of probability measures on the space $\Omega(\mathbb{G})$ (Theorem 1.1), and this space in all our examples is infinite–dimensional, see (2.15), (4.7).

For Young and Kingman graphs such measures have been thoroughly studied, see [P.I–P.V], [BO1], [BO2], [Ki3], [Pi], [Ke4]. They lead to certain stochastic processes on the real line, for the Young graph the processes are closely related to those arising in Random Matrix Theory, while for the Kingman graph the theory is connected with Poisson processes.

Our goal in this section is to construct 'simpler' families of harmonic functions for Young, Kingman, and Schur graphs. The word 'simpler' means that the corresponding measures on $\Omega(\mathbb{G})$ will be supported by finite–dimensional subspaces. These measures will be explicitly computed.

The corresponding Poisson integrals (which arise due to Theorem 1.1) will give (possibly new) integral formulas involving products of Schur S and P functions, see (6.10), (6.15), (6.23), (6.26) below.

6.1. Truncated Young branching. Recall that for the Young graph $\mathbb{Y}$ the algebra A^* is the algebra Λ^* of functions in infinitely many variables $x_1, x_2, \dots$ symmetric in 'shifted' variables $x'_j = x_j - j$.

We shall consider the most natural specializations of this algebra obtained by fixing finitely many variables $x_1, \dots, x_l$ and sending remaining variables $x_{l+1}, x_{l+2}, \dots$ to zero.

In any such specialization all functions s^*_μ with the length (number of nonzero parts) of μ greater than l vanish, see [OO]. This implies that the harmonic function on $\mathbb{Y}$ afforded by Proposition 1.3 vanishes on all Young diagrams with more than l rows. Thus, one can consider such a function as a harmonic function on the subgraph $\mathbb{Y}(l)$ of $\mathbb{Y}$ consisting of all Young diagrams with length $\leq l$. This graph fits into the general formalism of §1: the algebra A is the algebra of symmetric polynomials in l variables, and the algebra A^* is the algebra of shifted symmetric polynomials in l variables. The elements $\mathcal{P}_\mu$ and $\mathcal{P}^*_\mu$ are conventional and shifted Schur polynomials in l variables, respectively. The graph $\mathbb{Y}(l)$ is called the *truncated Young graph.* Harmonic functions on such graphs were considered by Kerov, see [Ke3]; he used them to derive certain Selberg–type integrals. Our arguments below are similar to those of Kerov's work.

Let us fix a Young diagram λ. Denote by l the length of λ. We shall assume that $l \geq 2$. Denote by π_λ the algebra homomorphism $\pi_\lambda : \Lambda^* \to \mathbb{R}$ defined by

$$\begin{cases} x_i \mapsto -\lambda_i - 2(l-i) - 1, & 1 \leq i \leq l, \\ x_i \mapsto 0 & i > l. \end{cases} \tag{6.1}$$

Convenience of such choice of notation will be clear in a while.

According to the general scheme of §1 (Proposition 1.3), the corresponding harmonic function on $\mathbb{Y}$ has the form

$$\varphi_\lambda(\mu) = \frac{(-1)^{|\mu|}\pi_\lambda(s^*_\mu)}{(-\pi_\lambda(s^*_{(1)}))_{|\mu|}} = \frac{(-1)^{|\mu|} s^*_\mu(-\lambda - 2\delta - 1)}{(|\lambda| + l^2)_{|\mu|}}, \tag{6.2}$$

where s^*_μ is the shifted Schur function and

$$\begin{cases} \delta_i = l - i, & 1 \leq i \leq l, \\ \delta_i = 0, & i > l. \end{cases}$$

Proposition 6.1. *The function φ_λ defined by* (6.2) *is nonnegative.*

Proof. Let the symbol $(a \downharpoonleft k)$ denote the *kth falling factorial power of a*:

$$(a \downharpoonleft k) = \begin{cases} a(a-1)\cdots(a-k+1), & k = 1, 2, \dots, \\ 1, & k = 0. \end{cases}$$

Recall the definition of the shifted Schur polynomials in finitely many variables [OO]

$$s^*_\mu(x_1, \dots, x_l) = \frac{\det[(x_i + \delta_i \downharpoonleft \mu_j + \delta_j)]^l_{i,j=1}}{\det[(x_i + \delta_i \downharpoonleft \delta_j)]^l_{i,j=1}}, \tag{6.3}$$

where δ_j are as above.

Let us plug in our x_i from (6.1) to (6.3). We get

$$\begin{aligned} (-1)^{|\mu|}\pi_\lambda(s^*_\mu) &= (-1)^{|\mu|} \frac{\det[(-\lambda_i - \delta_i - 1 \downharpoonleft \mu_j + \delta_j)]^l_{i,j=1}}{\det[(-\lambda_i - \delta_i - 1 \downharpoonleft \delta_j)]^l_{i,j=1}} \\ &= \frac{\det[\Gamma(\lambda_i + \delta_i + 1 + \mu_j + \delta_j)]^l_{i,j=1}}{\det[\Gamma(\lambda_i + \delta_i + 1 + \delta_j)]^l_{i,j=1}}. \end{aligned}$$

As the first l members of sequences $\lambda+\delta+1$, $\mu+\delta$, and δ decrease, the claim follows from the inequality

$$\det[\Gamma(x_i+y_j)]_{i,j=1}^{l}>0, \qquad x_1>\cdots>x_l>0, \quad y_1>\cdots>y_l>0,$$

which is a special case of Problem VII.66 in [PS]. $\square$

Next, we form the probability distributions $M_n=M_n^\lambda$ on $\mathbb{Y}_n$ for each $n=0,1,2,\dots$ according to (1.4):

$$M_n^\lambda(\nu)=\dim_{\mathbb{Y}}(\nu)\,\varphi_\lambda(\nu), \qquad |\nu|=n. \tag{6.4}$$

Let us embed $\mathbb{Y}_n\hookrightarrow\Omega(\mathbb{Y})$ via

$$\nu\mapsto\Bigl(\frac{\nu_1}{n},\frac{\nu_2}{n},\dots;\,0,0,\dots\Bigr)\in\Omega(\mathbb{Y}). \tag{6.5}$$

As was mentioned in §1, the probability measure P involved in the Poisson integral (1.2) is the weak limit of the images of the measures M_n under appropriate embeddings $\mathbb{Y}_n\hookrightarrow\Omega(\mathbb{Y})$; as such embeddings one can take (6.5).

Let us denote by $V(a)$ the Vandermonde determinant

$$V(a_1,\dots,a_s)=\prod_{1\le i<j\le s}(a_i-a_j).$$

Proposition 6.2. *The images of M_n^λ under the embeddings* (6.5) *weakly converge to a probability measure P on $\Omega(\mathbb{Y})$ supported by the finite–dimensional face*

$$\Delta_l=\{(\alpha,\beta)\in\Omega(\mathbb{Y})\mid \alpha_{l+1}=\alpha_{l+2}=\cdots=\beta_1=\beta_2=\cdots=0,\ \sum_{i=1}^{l}\alpha_i=1\}$$
$$\backsimeq\{(\alpha_1,\dots,\alpha_l)\in\mathbb{R}_+^l\mid \alpha_1\ge\alpha_2\ge\cdots\ge\alpha_l,\ \sum_{i=1}^{l}\alpha_i=1\}.$$

The density of P with respect to the Lebesgue measure on Δ_l equals

$$\frac{\Gamma(|\lambda|+l^2)\,s_\lambda(\alpha_1,\dots,\alpha_l)V^2(\alpha_1,\dots,\alpha_l)}{\det[\Gamma(\lambda_i+\delta_i+\delta_j+1)]_{i,j=1}^{l}}. \tag{6.6}$$

Proof. We shall need the following lemma.

Lemma 6.3. *As* $n \to \infty$,

$$\max_{\nu \in \mathbb{Y}_n} M_n^\lambda(\nu) = O(n^{1-l}).$$

Furthermore, if $\nu_l \geq \varepsilon n$, *where* $\varepsilon > 0$ *is arbitrary, then, as* $n \to \infty$,

$$M_n^\lambda(\nu) = n^{1-l} \frac{\Gamma(|\lambda| + l^2)\, s_\lambda(\alpha_1, \dots, \alpha_l) V^2(\alpha_1, \dots, \alpha_l)}{\det[\Gamma(\lambda_i + \delta_i + \delta_j + 1)]_{i,j=1}^l} (1 + o(1)) \tag{6.7}$$

with

$$\alpha_1 = \frac{\nu_1}{n}, \dots, \alpha_l = \frac{\nu_l}{n},$$

and the estimate (6.7) *is uniform in* ν.

Let us postpone the proof of this statement and proceed with the proof of Proposition 6.2 taking Lemma 6.3 for granted.

The Thoma simplex $\Omega(\mathbb{Y})$ defined in (2.15) is a compact topological space. Let $C(\Omega(\mathbb{Y}))$ denote the algebra of continuous functions on $\Omega(\mathbb{Y})$. Take any $f \in C(\Omega(\mathbb{Y}))$. Note that f is bounded. The value on f of the image of M_n^λ under the nth embedding (6.5) have the form

$$\sum_{\nu \in \mathbb{Y}_n} M_n^\lambda(\nu) \cdot f\left(\frac{\nu_1}{n}, \frac{\nu_2}{n}, \dots ; 0, 0, \dots\right). \tag{6.8}$$

Recall that M_n^λ is supported by the Young diagrams ν with the length $\leq l$. Let us first consider the part of the sum (6.8) involving diagrams ν with $\nu_l \geq \varepsilon n$, $\nu_{l+1} = 0$. Using Lemma 6.3 and the boundedness of f, we get

$$\begin{aligned}\sum_{\substack{\nu \in \mathbb{Y}_n \\ \nu_l \geq \varepsilon n,\, \nu_{l+1}=0}} M_n^\lambda(\nu) \cdot f\left(\frac{\nu_1}{n}, \frac{\nu_2}{n}, \dots ; 0, 0, \dots\right) &= \frac{\Gamma(|\lambda| + l^2)\,(1 + o(1))}{\det[\Gamma(\lambda_i + \delta_i + \delta_j + 1)]_{i,j=1}^l} \\ \times \sum_{\substack{\nu \in \mathbb{Y}_n \\ \nu_l \geq \varepsilon n,\, \nu_{l+1}=0}} & f(\alpha_1, \dots, \alpha_l, 0, \dots ; 0, 0, \dots) \cdot s_\lambda(\alpha) V^2(\alpha) \cdot n^{1-l},\end{aligned} \tag{6.9}$$

where $\alpha = (\alpha_1, \dots, \alpha_l)$ is as in Lemma 6.3.

The sum in the right–hand side of (6.9) is a Riemannian sum for the integral

$$\int f(\alpha_1, \dots, \alpha_l, 0, \dots ; 0, 0, \dots) \cdot s_\lambda(\alpha) V^2(\alpha)\, d\alpha$$

over the part of Δ_l specified by the condition $\alpha_l \geq \varepsilon$.

Thus, it remains to prove that the part of the sum (6.8) involving diagrams ν with $\nu_l < \varepsilon n$ is ε–negligible. Since $M_n^\lambda(\,\cdot\,) = O(n^{1-l})$, this follows from the fact that the number of Young diagrams $\nu = (\nu_1, \dots, \nu_l, 0, \dots)$ with $\nu_1 + \dots + \nu_l = n$ and such that $\nu_l < \varepsilon n$ is bounded by $const \cdot \varepsilon\, n^{l-1}$. This completes the proof of Proposition 6.2 modulo Lemma 6.3.

Proof of Lemma 6.3. We shall employ formulas (6.2), (6.3), (6.4). Denote by m the length of ν. We apply a well–known dimension formula

$$\dim_{\mathbb{Y}} \nu = \frac{n!}{\prod_{i=1}^{m}(\nu_i + m - i)!} \prod_{1 \le i < j \le m} (\nu_i - i - \nu_j + j)$$

which can be derived, e.g., from [M, I.7, Ex.6]. Then

$$M_n^{\lambda}(\nu) = \dim_{\mathbb{Y}}(\nu)\varphi_\lambda(\nu) = \frac{n!}{\prod_{i=1}^{m}(\nu_i + m - i)!} \prod_{1 \le i < j \le m} (\nu_i - i - \nu_j + j)$$
$$\times \frac{1}{(|\lambda| + l^2)_n} \cdot \frac{\det[\Gamma(\lambda_i + \delta_i + 1 + \nu_j + \delta_j)]_{i,j=1}^{l}}{\det[\Gamma(\lambda_i + \delta_i + 1 + \delta_j)]_{i,j=1}^{l}}.$$

Here $\nu_j = 0$ for $j > m$.

Next, we have the following asymptotic relations as $n \to \infty$:

$$\prod_{1 \le i < j \le m} (\nu_i - i - \nu_j + j) = O(n^{\frac{m(m-1)}{2}}),$$

$$\frac{\Gamma(\lambda_i + \delta_i + 1 + \nu_j + \delta_j)}{(\nu_j + m - j)!} = \frac{(\lambda_i + \delta_i + \nu_j + \delta_j)!}{(\nu_j + m - j)!}$$
$$\le (\lambda_i + \delta_i + \nu_j + \delta_j)^{\lambda_i + \delta_i + l - m} = O(n^{\lambda_i + \delta_i + l - m}),$$
$$\frac{n!}{(|\lambda| + l^2)_n} = O(n^{-|\lambda| - l^2 + 1}),$$

where estimates are uniform in $\nu \in \mathbb{Y}_n$.

Expanding the determinant $\det[\Gamma(\lambda_i + \delta_i + 1 + \nu_j + \delta_j)]_{i,j=1}^{l}$ and using the above estimates we get that each of $l!$ terms in the expansion, after multiplication by the remaining factors, is at most of order n^{1-l}. This proves the first claim of the lemma.

Let us proceed to the second claim. Assume that $\nu_l \ge \varepsilon n$. We have

$$\prod_{1 \le i < j \le l} (\nu_i - i - \nu_j + j) = n^{\frac{l(l-1)}{2}}\, V(\alpha) \cdot (1 + o(1)),$$

$$\frac{\Gamma(\lambda_i + \delta_i + 1 + \nu_j + \delta_j)}{(\nu_j + \delta_j)!} = \frac{\Gamma(\lambda_i + \delta_i + 1 + \nu_j + \delta_j)}{\Gamma(\nu_j + \delta_j + 1)} = (\alpha_j n)^{\lambda_i + \delta_i} \cdot (1 + o(1)),$$

$$\frac{n!}{(|\lambda| + l^2)_n} = \Gamma(|\lambda| + l^2)\, \frac{\Gamma(n+1)}{\Gamma(|\lambda| + l^2 + n)} = \Gamma(|\lambda| + l^2)\, n^{-|\lambda| - l^2 + 1} \cdot (1 + o(1)).$$

All estimates are uniform in $\nu \in \mathbb{Y}_n$ provided that $\nu_l \ge \varepsilon n$. This assumption has been used in the second estimate above. This implies the desired estimate (6.7). $\square$

As a consequence, we have the Poisson integral representation for our harmonic functions (see Theorem 1.1):

Proposition 6.4.

$$\frac{(-1)^{|\mu|}s^*_\mu(-\lambda-2\delta-1)}{(|\lambda|+l^2)_{|\mu|}}=\frac{\Gamma(|\lambda|+l^2)}{\det[\Gamma(\lambda_i+\delta_i+\delta_j+1)]_{i,j=1}^l}\int_{\Delta_l} s_\mu(\alpha)\,s_\lambda(\alpha)\,V^2(\alpha)\,d\alpha. \tag{6.10}$$

Note that the Poisson kernel $K(\mu,\omega)$ on Δ_l coincides with the ordinary Schur function $s_\mu(\alpha)$, see the end of §2.

It should be noted that the integration in (6.10) can be carried out directly by making use of the formulas

$$s_\mu(\alpha)=\frac{\det[\alpha_i^{\mu_j+\delta_j}]_{i,j=1}^l}{V(\alpha)},\quad s_\lambda(\alpha)=\frac{\det[\alpha_i^{\lambda_j+\delta_j}]_{i,j=1}^l}{V(\alpha)},$$

and the well–known Dirichlet integrals

$$\int\limits_{\substack{\alpha_1\ge\dots\ge\alpha_l\ge0\\ \alpha_1+\dots+\alpha_l=1}} \alpha_1^{\kappa_1-1}\cdots\alpha_l^{\kappa_l-1}d\alpha=\frac{1}{l!}\int\limits_{\substack{\alpha_1\ge0,\dots,\alpha_l\ge0\\ \alpha_1+\dots+\alpha_l=1}} \alpha_1^{\kappa_1-1}\cdots\alpha_l^{\kappa_l-1}d\alpha$$
$$=\frac{1}{l!}\,\frac{\Gamma(\kappa_1)\cdots\Gamma(\kappa_l)}{\Gamma(\kappa_1+\dots+\kappa_l)}.$$

However, in more complicated cases, see below, a direct evaluation of integrals like (6.10) seems to be difficult.

The identity (6.10) is similar to a result of Hua, see [H, p. 104], [Ri, (3.3)].

Remark 6.5. All claims of the present subsection remain true if we replace the integers $(\lambda_1\ge\lambda_2\ge\dots\ge\lambda_l>0)$ by any positive real numbers satisfying the same system of inequalities. Then the Schur function $s_\lambda(\alpha)$ should be understood just as the ratio $\det[\alpha_i^{\lambda_j+l-j}]/\det[\alpha_i^{l-j}]$. We restricted ourselves to integral λ_i's in order to emphasize the symmetry $\lambda\leftrightarrow\mu$ in the integral (6.10). By analytic continuation, the formula (6.10) can be extended to arbitrary complex λ_i's. When λ has the form $((l-1)\theta+a,(l-2)\theta+a,\dots,a)$, where $\theta>0$ and $a>-1$, the measure (6.6) is related to the so–called Laguerre biorthogonal ensemble, see [B2].

6.2. Γ–shaped Young branching. As in the previous subsection, we work with the Young graph $\mathbb{Y}$. This time we shall use another, so–called super realization of the algebra $A^*=\Lambda^*$ of shifted symmetric functions. Since a detailed exposition of the material of this subsection would be rather tedious, we shall only state the results and outline the ideas used in the proofs.

Let $\widetilde{\Lambda}$ be the algebra of supersymmetric functions in $x=(x_1,x_2,\dots)$ and $y=(y_1,y_2,\dots)$, see [BR], [M, Ex.I.3.23-24, Ex.I.5.23]. It can be identified with the algebra Λ of symmetric functions; under this identification power sums $p_m \in \Lambda$ correspond to their super analogs

$$p_m(x;y) = \sum_i x_i^m + (-1)^{m-1} \sum_i y_i^m.$$

Note that our notation slightly differs from that of Macdonald's book: his supersymmetric functions in x and y coincide with ours in x and $-y$.

Below we shall use Frobenius notation for Young diagrams, its description can be found in [M, I.1].

Theorem 6.6 [KO]. *There exists an algebra isomorphism $\rho : \Lambda^* \to \widetilde{\Lambda}$ such that for any $f \in \Lambda^*$ and any Young diagram $\lambda = (\lambda_1, \lambda_2, \dots)$ with Frobenius coordinates $(p_1, \dots p_d |\, q_1, \dots, q_d)$ the following equality holds:*

$$f(\lambda_1, \lambda_2, \dots) = \rho(f) \left(p_1 + \frac{1}{2}, \dots, p_d + \frac{1}{2};\, q_1 + \frac{1}{2}, \dots, q_d + \frac{1}{2} \right).$$

Now we shall identify $\widetilde{\Lambda}$ and Λ^* using the isomorphism ρ. We shall denote the elements $\rho(s^*_\mu) \in \widetilde{\Lambda}$ as FS_μ and call them *Frobenius–Schur functions*, see [ORV].

Let us consider specializations of the algebra $\Lambda^* \simeq \Lambda$ obtained by fixing finitely many variables $x_1, \dots, x_d$; $y_1, \dots, y_d$ and sending remaining variables $x_i, y_i,\ i = d, d+1, \dots,$ to zero. In any such specialization all functions s^*_μ with the depth (number of diagonal boxes) greater than d vanish, this follows from results of [ORV]. Hence, the corresponding harmonic functions are concentrated on a subgraph of the Young graph $\mathbb{Y}$ consisting of diagrams with depth $\le d$. These are exactly the Young diagrams which fit into the Γ–shaped figure with d rows and d columns. We denote the subgraph of such Young diagrams by $\mathbb{Y}(d,d)$ and call it the *Γ–shaped Young graph*. Like the truncated Young graphs considered in 6.1, the Γ–shaped Young graphs also fit into the general formalism of §1. The algebras A and A^* are both identified with the algebra of supersymmetric polynomials in $d+d$ variables, the elements $\mathcal{P}_\mu$ are supersymmetric Schur polynomials, and the elements $\mathcal{P}^*_\mu$ are supersymmetric Frobenius–Schur polynomials.

Fix a Young diagram λ with Frobenius coordinates $(p_1, \dots p_d \,|\, q_1, \dots, q_d)$. We shall denote by π_λ the algebra homomorphism $\pi_\lambda : \Lambda^* \to \mathbb{R}$ defined by

$$\begin{cases} x_i \mapsto -p_i - \frac{1}{2}, & 1 \le i \le d, \\ y_i \mapsto -q_i - \frac{1}{2}, & 1 \le i \le d, \\ x_i,\, y_i \mapsto 0, & i > d. \end{cases} \tag{6.11}$$

According to §1, the harmonic function on $\mathbb{Y}$ corresponding to π_λ has the form

$$\varphi_\lambda(\mu) = \frac{(-1)^{|\mu|}\pi_\lambda(s^*_\mu)}{(-\pi_\lambda(s^*_{(1)}))_{|\mu|}} = \frac{(-1)^{|\mu|} FS_\mu(-p - \frac{1}{2}; -q - \frac{1}{2})}{(|\lambda|)_{|\mu|}}. \tag{6.12}$$

Unfortunately, it is not clear how to prove directly that φ_λ is nonnegative. However, we can go around this obstacle.

Set

$$M_n^\lambda(\nu) = \dim_{\mathbb{Y}}(\nu)\,\varphi_\lambda(\nu), \qquad |\nu| = n.$$

Consider the embeddings $\mathbb{Y}_n \hookrightarrow \Omega(\mathbb{Y})$ defined as follows. For a Young diagram $\nu \in \mathbb{Y}_n$ with Frobenius coordinates $(P_1, \dots, P_D \,|\, Q_1, \dots, Q_D)$

$$\nu \mapsto \left(\frac{P_1 + 1/2}{n}, \dots, \frac{P_D + 1/2}{n}, 0, \dots; \; \frac{Q_1 + 1/2}{n}, \dots, \frac{Q_D + 1/2}{n}, 0, \dots \right) \in \Omega(\mathbb{Y}). \tag{6.13}$$

Proposition 6.7. *The images of (possibly signed) measures M_n^λ under the embeddings* (6.13) *weakly converge, as $n \to \infty$, to a (positive) probability measure P on $\Omega(\mathbb{Y})$. This measure is supported by the finite–dimensional face*

$$\Delta_{d,d} = \{(\alpha, \beta) \in \Omega(\mathbb{Y}) \,|\, \alpha_{d+1} = \beta_{d+1} = \alpha_{d+2} = \beta_{d+2} = \dots = 0, \; \sum_{i=1}^d (\alpha_i + \beta_i) = 1\}$$

$$\backsimeq \{(\alpha_1, \dots, \alpha_d; \beta_1, \dots, \beta_d) \in \mathbb{R}_+^{2d} | \alpha_1 \ge \dots \ge \alpha_d;\; \beta_1 \ge \dots \ge \beta_d, \; \sum_{i=1}^d (\alpha_i + \beta_i) = 1\},$$

and its density with respect to the Lebesgue measure $d(\alpha;\beta)$ on $\Delta_{d,d}$ equals

$$\frac{\Gamma(|\lambda|)}{\prod_{i=1}^d p_i! q_i!} \left[\det\left(\frac{1}{p_i + q_j + 1} \right) \right]^{-1} s_\lambda(\alpha;\beta) \det{}^2 \left(\frac{1}{\alpha_i + \beta_j} \right), \tag{6.14}$$

where $s_\lambda(\alpha;\beta)$ is the supersymmetric Schur polynomial in $d + d$ variables.

The proof of this proposition is quite similar to that of Proposition 6.2. An analog of Lemma 6.3 is proved using the Sergeev–Pragacz formula for $s_\lambda(\alpha;\beta)$ (see [PT], [M, I.3, Ex.24]) and its analog for the Frobenius–Schur polynomials (see [ORV]).

It turns out that Proposition 6.7 implies the existence of the Poisson integral representation (1.2) for the harmonic function (6.12). The proof of this claim is quite similar to the proof of Theorem B in [KOO]. Since the Poisson kernel is always nonnegative, the existence of the Poisson integral representation implies that our harmonic function is nonnegative.

Explicitly, the Poisson integral representation has the following form, cf. Proposition 6.4.

Proposition 6.8.

$$\begin{aligned} \frac{(-1)^{|\mu|} FS_\mu(-p - \frac{1}{2}; -q - \frac{1}{2})}{(|\lambda|)_{|\mu|}} &= \frac{\Gamma(|\lambda|)}{\prod_{i=1}^d p_i! q_i!} \left[\det\left(\frac{1}{p_i + q_j + 1} \right) \right]^{-1} \\ &\times \int_{\Delta_{d,d}} s_\mu(\alpha;\beta)\, s_\lambda(\alpha;\beta) \det{}^2 \left(\frac{1}{\alpha_i + \beta_j} \right) d(\alpha;\beta). \end{aligned} \tag{6.15}$$

Remark 6.9. The formula (6.15) gives an expression for the integral

$$\int_{\Delta_{d,d}} s_\mu(\alpha;\beta)\, s_\lambda(\alpha;\beta) \det{}^2\left(\frac{1}{\alpha_i+\beta_j}\right) d(\alpha;\beta). \tag{6.16}$$

The integrand is symmetric in λ and μ. However, our assumptions on λ and μ are different: the depth of λ must be equal to d while μ is arbitrary. Actually, if the depth of μ is $> d$ then both sides of (6.15) vanish. If the depth of μ is equal to d, the integration can be carried out directly in a rather simple way using the Berele–Regev formula

$$s_\nu(\alpha_1,\dots,\alpha_d;\,\beta_1,\dots,\beta_d) = \left[\det\left(\frac{1}{\alpha_i+\beta_j}\right)_{i,j=1}^{d}\right]^{-1} \det[\alpha_i^{P_j}]_{i,j=1}^d \det[\beta_i^{Q_j}]_{i,j=1}^d;$$

here $(P|\,Q)$ are the Frobenius coordinates of ν, see [BR], [M, I.3, Ex.23]. But if the depth of μ is strictly less than d, the Berele–Regev formula must be replaced by the more complicated Sergeev–Pragacz formula, and a direct integration seems to be more difficult.

Remark 6.10. All claims of this subsection remain true if we replace integral Frobenius coordinates $p_1 > \dots > p_d \ge 0$, $q_1 > \dots > q_d \ge 0$ of a fixed Young diagram λ with any ordered sequences of real numbers $> -\frac{1}{2}$. Then the Schur function $s_\lambda(\alpha;\beta)$ should be understood as

$$\left[\det\left(\frac{1}{\alpha_i+\beta_j}\right)_{i,j=1}^{d}\right]^{-1} \det[\alpha_i^{p_j}]_{i,j=1}^d \det[\beta_i^{q_j}]_{i,j=1}^d,$$

cf. Remark 6.5. As in 6.1, we restricted ourselves to integral p's and q's in order to demonstrate the symmetry $\lambda \leftrightarrow \mu$ in (6.15). By analytic continuation, the formula (6.15) can be extrapolated to any pairwise distinct complex p_i's and q_i's.

6.3. Truncated Kingman branching. For the Kingman graph $\mathbb{K}$ (see §4) the algebra A^* coincides with the algebra Λ of symmetric functions. We consider specializations of Λ obtained by fixing finitely many indeterminates, say, $x_1,\dots,x_l$, and sending remaining indeterminates to zero.

Under such a specialization all functions $\mathcal{P}^*_\mu = m^*_\mu$ with $\ell(\mu) > l$ vanish (this easily follows from the definition of the factorial monomial functions, see §4). This means that the corresponding harmonic function lives on the subgraph $\mathbb{K}(l)$ of the Kingman graph K consisting of all Young diagrams with the length $\le l$. We call this subgraph the *truncated Kingman graph.* It fits into the general formalism of §1 with algebras A and A^* both equal to the algebra of symmetric polynomials in l variables. The elements $\mathcal{P}_\mu$ and $\mathcal{P}^*_\mu$ are monomial symmetric polynomials and factorial monomial symmetric polynomials, respectively.

Certain harmonic functions on truncated Kingman graphs and their applications to Selberg–type integrals were previously considered by Kerov [Ke3].

Let us fix a Young diagram

$$\lambda = (\lambda_1 \geq \cdots \geq \lambda_l > 0) = (1^{r_1(\lambda)} 2^{r_2(\lambda)} \ldots).$$

We define an algebra homomorphism $\pi_\lambda : \Lambda \to \mathbb{R}$ as follows, cf. (6.1), (6.11),

$$\begin{cases} x_i \mapsto -\lambda_i - 1, & 1 \leq i \leq l, \\ x_i \mapsto 0, & i > l. \end{cases} \tag{6.17}$$

The corresponding harmonic function on $\mathbb{K}$ has the form

$$\varphi_\lambda(\mu) = \frac{(-1)^{|\mu|}\pi_\lambda(m^*_\mu)}{(-\pi_\lambda(m^*_{(1)}))_{|\mu|}} = \frac{(-1)^{|\mu|} m^*_\mu(-\lambda_1 - 1, \ldots, -\lambda_l - 1)}{(|\lambda| + l)_{|\mu|}}. \tag{6.18}$$

Proposition 6.11. *The harmonic function φ_λ defined by* (6.18) *is nonnegative.*

Proof. It suffices to note that $m^*_\mu(-\lambda_1 - 1, \ldots, -\lambda_l - 1)$, by definition, is a sum of expressions of the form (σ is a permutation here)

$$\prod_{i=1}^{l(\mu)} (-\lambda_{\sigma(i)} - 1)(-\lambda_{\sigma(i)} - 2) \cdots (-\lambda_{\sigma(i)} - \mu_i),$$

each of which has sign $(-1)^{|\mu|}$. □

According to §1, we have probability distributions M_n^λ on $\mathbb{K}_n$ for all $n = 1, 2, \ldots$ given by

$$M_n^\lambda(\nu) = \dim_{\mathbb{K}}(\nu)\, \varphi_\lambda(\nu), \quad |\nu| = n. \tag{6.19}$$

Embeddings $\mathbb{K}_n \hookrightarrow \Omega(\mathbb{K})$ are defined as follows ($\Omega(\mathbb{K})$ was defined in (4.7)):

$$\nu \in \mathbb{K}_n \mapsto \left(\frac{\nu_1}{n}, \frac{\nu_2}{n}, \ldots\right) \in \Omega(\mathbb{K}). \tag{6.20}$$

The images of the probability distributions M_n^λ via these embeddings, according to the general theory, weakly converge, as $n \to \infty$, to a certain probability measure P on $\Omega(\mathbb{K})$.

Proposition 6.12. *The probability measure P on $\Omega(\mathbb{K})$ is supported by the finite–dimensional face*

$$\Delta_l = \{\alpha \in \Omega(\mathbb{K}) \mid \alpha_{l+1} = \alpha_{l+2} = \cdots = 0,\ \sum_{i=1}^{l} \alpha_i = 1\}$$

$$\backsimeq \{(\alpha_1, \ldots, \alpha_l) \in \mathbb{R}_+^l \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l,\ \sum_{i=1}^{l} \alpha_i = 1\}.$$

Its density with respect to the Lebesgue measure $d\alpha$ on Δ_l equals

$$\frac{\Gamma(|\lambda|+l)\cdot r_1(\lambda)!r_2(\lambda)!\cdots}{\prod_{i=1}^{l}\lambda_i!}\,m_\lambda(\alpha_1,\dots,\alpha_l). \tag{6.21}$$

The proof is very similar to that of Proposition 6.2, so we shall just state the analog of Lemma 6.3 in this case.

Lemma 6.13. *As $n\to\infty$,*

$$\max_{\nu\in\mathbb{K}_n} M_n^\lambda(\nu)=O(n^{1-l}).$$

Furthermore, if $\nu_{l+1}=0$ and $\nu_l\ge\varepsilon n$, where $\varepsilon>0$ is arbitrary, then, as $n\to\infty$,

$$M_n^\lambda(\nu)=n^{1-l}\,\frac{\Gamma(|\lambda|+l)\cdot r_1(\lambda)!r_2(\lambda)!\cdots}{\prod_{i=1}^{l}\lambda_i!}\,m_\lambda(\alpha_1,\dots,\alpha_l)\,(1+o(1)), \tag{6.22}$$

with $\alpha_1=\nu_1/n,\dots,\alpha_l=\nu_l/n$. The estimate (6.22) *is uniform in ν.*

Proof. Using the formula (4.6) for $\dim_{\mathbb{K}}(\nu)$ we get

$$M_n^\lambda(\nu)=\dim_{\mathbb{K}}(\nu)\,\varphi_\lambda(\nu)=\frac{n!}{(|\lambda|+l)_n}\,\frac{\sum_\sigma\prod_{i=1}^{\ell(\nu)}(\lambda_{\sigma(i)}+1)_{\nu_i}}{\nu_1!\cdots\nu_{\ell(\nu)}!},$$

where the summation is taken over all permutations $\sigma\in S_l$ which produce different products $\prod_{i=1}^{\ell(\nu)}(\lambda_{\sigma(i)}+1)_{\nu_i}$.

Next, we have asymptotic relations

$$\frac{n!}{(|\lambda|+l)_n}=O(n^{1-l-|\lambda|}),$$

$$\frac{(\lambda_j+1)_{\nu_i}}{\nu_i!}\le(\lambda_j+\nu_i)^{\lambda_j}=O(n^{\lambda_j}).$$

They imply that each term of the sum above, after multiplication by remaining factors, is at most of order n^{1-l}. This proves the first part of the lemma.

For the second part of the lemma, assume $\nu_l\ge\varepsilon n$. Then we have

$$\frac{n!}{(|\lambda|+l)_n}=\Gamma(|\lambda|+l)\,n^{1-l-|\lambda|}(1+o(1)),$$

$$\frac{(\lambda_j+1)_{\nu_i}}{\nu_i!}=\frac{(\alpha_i n)^{\lambda_j}}{\lambda_j!}\,(1+o(1))$$

as $n\to\infty$, all estimates are uniform in $\nu\in\mathbb{K}_n$ provided that $\nu_l\ge\varepsilon n$. This yields the estimate (6.22). $\square$

As a corollary, we get the Poisson integral representation, cf. Propositions 6.4 and 6.8.

Proposition 6.14.

$$\begin{aligned} &\frac{(-1)^{|\mu|} m^*_\mu(-\lambda_1-1,\dots,-\lambda_l-1)}{(|\lambda|+l)_{|\mu|}} \\ &= \frac{\Gamma(|\lambda|+l)\cdot r_1(\lambda)! r_2(\lambda)!\cdots}{\prod_{i=1}^{l}\lambda_i!}\int_{\Delta_l} m_\mu(\alpha) m_\lambda(\alpha)d\alpha. \end{aligned} \tag{6.23}$$

This claim, similarly to Proposition 6.4, can be proved directly by making use of Dirichlet integrals.

Remark 6.15. All claims above remain valid for any positive ordered sequence $\lambda = (\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_l \ge 0)$ with the obvious modification of the definition of monomial symmetric function $m_\lambda(\alpha)$, cf. Remarks 6.5, 6.10. By analytic continuation, the formula (6.23) can be extended to arbitrary complex λ_i's.

6.4. Truncated Schur branching. In this subsection we shall deal with the Schur graph, see §5. For this graph the algebras A and A^* coincide with a subalgebra Γ of the algebra of symmetric functions; Γ is generated by the odd power sums $\sum_i x_i^{2k+1}$, $k=0,1,\dots$. Again, we consider specializations of Γ obtained by fixing variables $x_1,\dots,x_l$ and sending remaining variables to zero. As in 6.2, we shall state the results and sketch the ideas of the proofs.

In such a specialization, the elements $\mathcal{P}^*_\mu = P^*_\mu$ vanish if $\ell(\mu) > l$. This means that the corresponding harmonic function can be viewed as a harmonic function on the *truncated Schur graph* $\mathbb{S}(l)$ — the subgraph of the Schur graph $\mathbb{S}$ consisting of diagrams with length $\le l$. The truncated Schur graphs also fit into the formalism of §1 with algebras A and A^* coinciding with the subalgebra of the algebra of symmetric polynomials in l variables generated by odd power sums. The elements $\mathcal{P}_\mu$ and $\mathcal{P}^*_\mu$ are the Schur P polynomials and the factorial Schur P polynomials, respectively.

Let us fix a strict partition λ and denote its length by l. We define a multiplicative linear functional $\pi_\lambda : \Gamma \to \mathbb{R}$ as follows, cf. (6.1), (6.11), (6.17):

$$\begin{cases} x_i \mapsto -\lambda_i - 1, & 1 \le i \le l, \\ x_i \mapsto 0, & i > l. \end{cases}$$

The corresponding harmonic function on $\mathbb{S}$ has the form

$$\varphi_\lambda(\mu) = \frac{(-1)^{|\mu|}\pi_\lambda(P^*_\mu)}{(-\pi_\lambda(P^*_{(1)}))_{|\mu|}} = \frac{(-1)^{|\mu|}P^*_\mu(-\lambda_1-1,\dots,-\lambda_l-1)}{(|\lambda|+l)_{|\mu|}}. \tag{6.24}$$

As in 6.2, it is not evident that this function is nonnegative.

For all strict partitions $\nu \in \mathbb{S}_n$, $n = 1,2,\dots$, we define, as usual,

$$M_n^\lambda(\nu) = \dim_{\mathbb{S}}\nu\cdot\varphi_\lambda(\nu).$$

Embeddings $\mathbb{S}_n \hookrightarrow \Omega(\mathbb{S}) = \Omega(\mathbb{K})$ are defined exactly as in the case of the Kingman graph, see (6.20), the only difference is that now all partitions are strict.

Proposition 6.16. *The sequence of images of (possibly signed) measures M_n^λ under the embeddings defined above weakly converges, as $n \to \infty$, to a (positive) probability measure P on $\Omega(\mathbb{S})$. This measure is supported by the finite–dimensional face*

$$\begin{aligned}\Delta_l &= \{\alpha \in \Omega(\mathbb{S}) \mid \alpha_{l+1} = \alpha_{l+2} = \cdots = 0,\ \sum_{i=1}^{l} \alpha_i = 1\} \\ &\backsimeq \{(\alpha_1, \dots, \alpha_l) \in \mathbb{R}_+^l \mid \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_l,\ \sum_{i=1}^{l} \alpha_i = 1\},\end{aligned}$$

and its density with respect to the Lebesgue measure $d\alpha$ on Δ_l equals

$$\frac{\Gamma(|\lambda| + l)}{\prod_{i=1}^{l} \lambda_i!} \left[\mathrm{Pf} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j + 2} \right) \right]^{-1} P_\lambda(\alpha_1, \dots, \alpha_l)\, \mathrm{Pf}^2 \left(\frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right). \tag{6.25}$$

This result is parallel to Propositions 6.2, 6.7, 6.12. Its proof is based on an appropriate analog of the approximation Lemmas 6.3, 6.13. The proof of such a lemma in this case follows from explicit formulas for Schur P–functions and factorial Schur functions [M, III.8], [I1].

Similarly to Proposition 6.7, Proposition 6.16 implies the existence of the Poisson integral representation for φ_λ. Thanks to the positivity of (6.25), this implies that φ_λ is nonnegative.

The explicit Poisson integral representation (1.2) in this case takes the following form.

Proposition 6.17.

$$\begin{aligned}\frac{(-1)^{|\mu|} P_\mu^*(-\lambda_1 - 1, \dots, -\lambda_l - 1)}{(|\lambda| + l)_{|\mu|}} &= \frac{\Gamma(|\lambda| + l)}{\prod_{i=1}^{l} \lambda_i!} \left[\mathrm{Pf} \left(\frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j + 2} \right) \right]^{-1} \\ &\quad \times \int_{\Delta_l} P_\mu(\alpha) P_\lambda(\alpha)\, \mathrm{Pf}^2 \left(\frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right) d\alpha.\end{aligned} \tag{6.26}$$

Remark 6.18. If $\ell(\mu) = l$ $(= \ell(\lambda))$, then the integration in (6.26) can be carried out directly using Dirichlet integrals and the formula [M, III.8, Ex.12]

$$P_\nu(\alpha_1, \dots, \alpha_l) = \left[\mathrm{Pf} \left(\frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j} \right) \right]^{-1} \det[\alpha_i^{\nu_j}]_{i,j=1}^{l}\,, \tag{6.27}$$

which holds for all ν of length l. If $\ell(\mu) < l$, the integration seems to be more complicated, cf. Remark 6.9.

Remark 6.19. All claims of the present subsection remain true for any nonnegative strictly ordered sequence $\lambda = (\lambda_1 > \lambda_2 > \cdots > \lambda_l \geq 0)$. Then the Schur P–function $P_\lambda(\alpha)$ should be understood as

$$\left[\mathrm{Pf}\left(\frac{\alpha_i - \alpha_j}{\alpha_i + \alpha_j}\right)\right]^{-1} \det[\alpha_i^{\lambda_j}],$$

cf. Remarks 6.5, 6.10, 6.15. By analytic continuation, the formula (6.26) can be extended to arbitrary complex mutually distinct λ_i's.

§7. Appendix

Proof of Theorem 1.1. Existence of the integral representation (1.2) follows from Choquet's theorem, see, e.g., [Ph]. To prove its uniqueness one can apply another theorem, due to Choquet and Meyer, [DM]. Then we have to verify that the cone $\mathcal{H}^+(\mathbb{G})$ is a lattice, i.e., for any $\varphi, \psi \in \mathcal{H}^+(\mathbb{G})$, there exist their lowest upper bound $\varphi \vee \psi$ and greatest lower bound $\varphi \wedge \psi$. Let us prove that

$$(\varphi \vee \psi) = \lim_{n\to\infty} \sum_{\lambda\in\mathbb{G}_n} \dim_{\mathbb{G}}(\mu, \lambda) \max(\varphi(\lambda), \psi(\lambda)), \tag{7.1}$$

$$(\varphi \wedge \psi) = \lim_{n\to\infty} \sum_{\lambda\in\mathbb{G}_n} \dim_{\mathbb{G}}(\mu, \lambda) \min(\varphi(\lambda), \psi(\lambda)). \tag{7.2}$$

Indeed, take (7.1) and (7.2) as the definition of the functions $\varphi \vee \psi$ and $\varphi \wedge \psi$. For any fixed μ, the sum in the right–hand side of (7.1) increases as $n \to \infty$ and remains bounded from above by $\varphi(\mu) + \psi(\mu)$. Similarly, the sum in the right–hand side of (7.2) decreases and remains bounded from below by 0. Hence, the limits exist.

Next, remark that for any fixed vertex ν, the function $\mu \mapsto \dim_{\mathbb{G}}(\mu, \nu)$ satisfies the harmonicity relation up to level $|\nu| - 1$. This implies that $\varphi \vee \psi$ and $\varphi \wedge \psi$ are harmonic functions. Clearly, they are nonnegative and are upper and lower bounds, respectively.

Finally, it is readily verified that they are the lowest upper bound and the greatest lower bound, respectively. □

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