Diagonal Sums of Boxed Plane Partitions

David B. Wilson
Microsoft Research
One Microsoft Way
Redmond, WA 98052
dbwilson@microsoft.com

Submitted: December 8, 2000; Accepted: December 22, 2000

2000 AMS subject classifications: primary 60C05, secondary 05A15.

Abstract: We give a simple proof of a nice formula for the means and covariances of the diagonal sums of a uniformly random boxed plane parition.

An $a \times b \times c$ boxed plane partition is an $a \times b$ grid of integers between 0 and c inclusive, such that the numbers decrease weakly in each row and column. At the right is a $4 \times 5 \times 6$ boxed plane parition, which for convenience we have drawn rotated 45°. We have added up these numbers in the direction along the main diagonal of the lattice to obtain the diagonal sums $S_{-a+1}, \ldots, S_{b-1}$. If we pick the boxed plane partition uniformly at random, these form a sequence of random variables, and we show that their means and covariances are given by

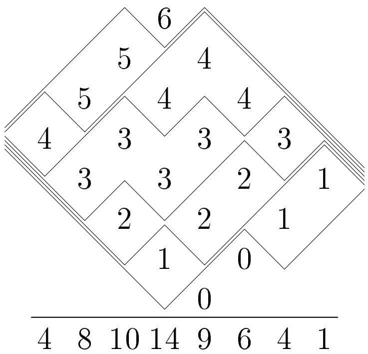

Figure 1: A $4 \times 5 \times 6$ boxed plane partition with its contours and diagonal sums.

$$E[S_i] = \begin{cases} (a+i)bc/(a+b) & i \leq 0 \\ (b-i)ac/(a+b) & i \geq 0 \end{cases}$$

$$\operatorname{Cov}(S_i, S_j) = (a+i)(b-j) \times \frac{abc(a+b+c)}{(a+b)^2((a+b)^2-1)} \qquad (i \leq j).$$

Notice that while the expected values "see" the corner at the origin, the other corner does not enter the formula, and neither corner enters the formula for the covariances. For given values of a, b, and c, the covariances are just proportional to the product of the distances from i and j to the endpoints. As Kenyon points out, a similar covariance property holds for Brownian bridges, and indeed can be deduced from this formula by taking $c = 1$ and $a, b \to \infty$. We are unaware of similarly nice formulas for e.g. the row sums.

As Stembridge points out, diagonal sums appear in some generating functions such as

$$\sum_{a \times b \times \infty \text{ bpp's}} \prod_i x_i^{S_i} = \prod_{i=-a+1}^{0} \prod_{j=0}^{b-1} \frac{1}{1 - x_i x_{i+1} \cdots x_j},$$

which is due to Stanley (see [3, Chapter 7]). The corresponding generating function for $a \times b \times c$ bpp's is not so nice, but Krattenthaler [2, pp 192] expresses it in terms of a determinant. We do not know a derivation of the covariance formula using this approach.

To prove these formulas we look at the contours associated with a boxed plane partition. The contours come from viewing the numbers in the plane partition as heights, so that between adjacent cells with heights z_1 and z_2 there will be $|z_1 - z_2|$ contours (see figure). The contours are noncrossing, but may share vertices and edges.

Consider the ith diagonal, and condition on the locations where the contours cross the two adjacent diagonals. Each contour will either make two down moves or two up moves, or it will be flexible and make an up and down move in some random order. The flexible lattice paths do not interact at all unless they intersect diagonals $i-1$ and $i+1$ in the same locations. Since the boxed plane partition is uniformly random, if a group of k flexible contours start and end at the same locations, the expected number that go down and then up is $k/2$. If we let Y_i be the sum over contours of the height of the contour above the line connecting the left and right corners ($Y_i + S_i$ is a deterministic function of i), then we have

$$E[Y_i|\text{diagonals } i-1 \text{ and } i+1] = (Y_{i-1} + Y_{i+1})/2.$$

Since $Y_{-a} = 0 = Y_b$, we find $E[Y_i] = 0$, which leads to the formula for $E[S_i]$.

Next observe that for $i \le j$

$$\begin{aligned} E[Y_iY_j] &= \sum_y \Pr[Y_i = y] y E[Y_j|Y_i = y] \\ &= \sum_y \Pr[Y_i = y] y \frac{b-j}{b-i} y = \frac{b-j}{b-i} E[Y_i^2]. \end{aligned}$$

Similarly

$$\begin{aligned} E[Y_iY_j] &= \sum_y \Pr[Y_j = y] y E[Y_i|Y_j = y] \\ &= \sum_y \Pr[Y_j = y] y \frac{a+i}{a+j} y = \frac{a+i}{a+j} E[Y_j^2]. \end{aligned}$$

Equating these formulas gives $E[Y_j^2] = (a+j)(b-j)/(a+b-1)E[Y_{-a+1}^2]$, and hence

$$\operatorname{Cov}(S_i, S_j) = \operatorname{Cov}(Y_i, Y_j) = E[Y_iY_j] = \frac{(a+i)(b-j)}{(a+b-1)} \operatorname{Var}[S_{-a+1}]. \tag{1}$$

To compute $\operatorname{Var}[S_{-a+1}]$ we let $S = \sum_i S_i$ and write

$$\begin{aligned} \operatorname{Var}[S] &= 2\sum_{i<j} \operatorname{Cov}(S_i, S_j) + \sum_i \operatorname{Cov}(S_i, S_i) \\ &= \left[2 \sum_{-a<i<j<b} (a+i)(b-j) + \sum_{-a<i=j<b} (a+i)(b-j) \right] \frac{\operatorname{Var}[S_{-a+1}]}{a+b-1} \\ &= \frac{(a+b)^2((a+b)^2-1)}{12} \frac{\operatorname{Var}[S_{-a+1}]}{a+b-1} \end{aligned} \tag{2}$$

It is straightforward to calculate $\mathrm{Var}[S]$ using the q-analogue of MacMahon's formula

$$\sum_{a \times b \times c \text{ bpp's}} q^S = \prod_{i=1}^{a} \prod_{j=1}^{b} \prod_{k=1}^{c} \frac{(i+j+k-1)_q}{(i+j+k-2)_q},$$

where $n_q = q^{n-1} + \cdots + q + 1$. This calculation, perhaps first carried out by Blum [1], yields

$$\mathrm{Var}[S] = abc(a+b+c)/12. \tag{3}$$

Combining (1), (2), and (3) yields the formula for the covariance.

References

[1] M. D. Blum, unpublished notes (1996).

[2] C. Krattenthaler, Generating functions for plane partitions of a given shape, *Manuscripta Mathematica* 69:173–201 (1990).

[3] R. P. Stanley, *Enumerative Combinatorics*, volume 2 (1999).