A note on the ranks of set-inclusion matrices

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Abstract

A recurrence relation is derived for the rank (over most fields) of the set-inclusion matrices on a finite ground set.

Given a finite set X of say v elements, let $W = W_{t,k}(v)$ be the (0,1)-matrix of inclusions for t-subsets versus k-subsets of X : $W_{T,K} = 1$ if T is contained in K, and 0 otherwise. These matrices play a significant part in several combinatorial investigations, see e.g. ([2], Thm. 2.4).

Let F be any field, and let $r_F(M)$ denote the rank of M over F.

Theorem. If $(k-t) \neq 0$ in the field F, then

$$r_F(W_{t,k}(v+1)) = r_F(W_{t,k-1}(v)) + r_F((k-t+1)W_{t-1,k}(v)). \tag{1}$$

Proof. The block-matrix identity

$$\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} \begin{bmatrix} AB & 0 \\ B & BC \end{bmatrix} \begin{bmatrix} I & -C \\ 0 & I \end{bmatrix} = \begin{bmatrix} 0 & -ABC \\ B & 0 \end{bmatrix}$$

implies that, over any field F,

$$r_F\left(\begin{bmatrix} AB & 0 \\ B & BC \end{bmatrix}\right) = r_F(B) + r_F(ABC). \tag{2}$$

The set-inclusion matrix has the block-triangular decomposition

$$W_{t,k}(v+1) = \begin{bmatrix} W_{t-1,k-1}(v) & 0 \\ W_{t,k-1}(v) & W_{t,k}(v) \end{bmatrix}, \tag{3}$$

as may be seen by fixing x in X and classifying t-sets and k-sets according to whether x belongs to them or not. Further, there is the elementary product formula

$$W_{t,k}(v)W_{k,l}(v) = \left(\begin{array}{c} l-t \\ k-t \end{array} \right) W_{t,l}(v) \tag{4}$$

whose proof is left as a straightforward exercise. Using (4), one may re-write (3) as

$$W_{tk}(v+1) = \left[\begin{array}{cc} \frac{1}{(k-t)} W_{t-1,t}(v) W_{t,k-1}(v) & 0 \\ W_{t,k-1}(v) & W_{t,k-1}(v) W_{k-1,k}(v) \frac{1}{(k-t)} \end{array} \right]$$

and so (2) is applicable:

$$\begin{aligned} r_F(W_{t,k}(v+1)) &= r_F(W_{t,k-1}(v)) + r_F(W_{t-1,t}(v)W_{t,k-1}(v)W_{k-1,k}(v)) \\ &= r_F(W_{t,k-1}(v)) + r_F((k-t+1)W_{t-1,k}(v)), \end{aligned}$$

which completes the proof of (1). □

Corollary Over the rational field Q, $r_Q(W_{t,k}(v)) = \left(\begin{array}{c} v \\ t \end{array} \right)$, provided $k+t \leq v$.

Proof. This is very easy using (1): note that the condition "$k+t \leq v$" is inherited by the triples $(t, k-1, v-1)$ and $(t-1, k, v-1)$; so the result follows by induction. □

The corollary is a well known result, first proved by Gottlieb [3]. Wilson [4] has worked out the modular ranks of $W_{t,k}(v)$. Unfortunately, the condition $(k-t) \neq 0$ in the hypothesis of our theorem precludes a new proof of Wilson's theorem via our recursive formula. In the special case when the characteristic p of F is larger than k, our recursion does apply, with the same conclusion and proof as the above corollary.

In conclusion, we raise the question as to whether there is a q-analogue of formula (1), i.e., for the (0,1)-inclusion matrix $W^{(q)}_{t,k}(v)$ of t-dimensional subspaces versus k-dimensional subspaces of a v-dimensional space over $GF(q)$; see [1], where the F-rank of $W^{(q)}_{t,k}(v)$ is computed when $char(F)$ does not divide q.

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References

[1] A. Frumkin and A. Yakir, "Rank of inclusion matrices and modular representation theory", Israel J. Math. 71 (1990), 309-320.

[2] C. D. Godsil, "Tools from linear algebra", in Handbook of Combinatorics (eds., Graham, Grötschel, Lovász), MIT press 1995, pp. 1705-1748.

[3] D. H. Gottlieb, "A class of incidence matrices", Proc. Amer. Math. Soc. 17 (1966), 1233-1237.

[4] R. M. Wilson, "A diagonal form for the incidence matrix of t-subsets vs. k-subsets", European J. Combin. 11 (1990), 609-615.