Queens on Non-square Tori

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Submitted: January 29, 2001; Accepted: June 11, 2001.
MR Subject Classifications: 05C69, 05B99

Abstract

We prove that for $m < n$, the maximum number of nonattacking queens that can be placed on the $n \times m$ rectangular toroidal chessboard is $\gcd(m, n)$, except in the case $m = 3, n = 6$.

The classical n-queens problem is to place n queens on the $n \times n$ chessboard such that no pair is attacking each other. Solutions for this problem exist for all for $n \neq 2, 3$ [1]. The queens problem on a rectangular board is of little interest; on the $n \times m$ board for $m < n$, one can obviously place at most m nonattacking queens and for $4 \leq m < n$, one can just take a solution on the $m \times m$ board and extend the board. Moreover, the reader will easily find solutions on the 3×2 and 4×3 boards and so these give solutions on the $n \times 2$ and $n \times 3$ boards for all $3 \leq n$ and $4 \leq n$ respectively.

In chess on a torus, one identifies the left and right edges and the top and bottom edges of the board. On the $n \times n$ toroidal board, the n-queens problem has solutions when n is not divisible by 2 or 3 [3], and the problem of placing the maximum number of queens when n is divisible by 2 or 3 is completely solved in [2]. The traditional n-queens problem and the toroidal n-queens problem are closely related, both logically and historically (see [4]). However, unlike the rectangular traditional board, the queens problem on the rectangular toroidal board is interesting and non-trivial and yet it seems that it has not been studied.

In order to work on the toroidal board we use the ring $\mathbb{Z}_i = \mathbb{Z}/(i)$, which we identify with $\{0, \ldots, i-1\}$, and the natural ring epimorphism $\mathbb{Z} \to \mathbb{Z}_i; x \mapsto [x]_i$, where $[x]_i$ is to be interpreted as the remainder of x on division by i. We give the squares of the $n \times m$ toroidal board coordinate labels (x, y), $x \in \mathbb{Z}_m, y \in \mathbb{Z}_n$, in the obvious way. The positive (resp. negative) diagonal is the subgroup $P = \{([x]_m, [x]_n) \ ; \ x \in \mathbb{Z}\}$ (resp. $N = \{([x]_m, [-x]_n) \ ; \ x \in \mathbb{Z}\}$). Notice that the diagonals are both subgroups of $\mathbb{Z}_m \times \mathbb{Z}_n$ of index $\gcd(m, n)$. In addition, there is the vertical subgroup $V = \{(0, [x]_n) \ ; \ x \in \mathbb{Z}\}$ which has index m, and the horizontal subgroup $H = \{([x]_m, 0) \ ; \ x \in \mathbb{Z}\}$ which has index n. Queens at distinct positions $(x_1, y_1), (x_2, y_2)$ are nonattacking if and only if (x_1, y_1)

and (x_2, y_2) belong to distinct cosets of V, H, P and N. In particular, the $n \times m$ toroidal board can support no more than $\gcd(m, n)$ nonattacking queens.

The aim of this paper is to prove the

Theorem. *For $m < n$, the maximum number of nonattacking queens that can be placed on the $n \times m$ rectangular toroidal chessboard is $\gcd(m, n)$, except in the case $m = 3, n = 6$.*

Proof. First let $d = \gcd(m, n)$ and suppose that $d \neq 3$. Notice that in order to place d nonattacking queens on the $n \times m$ toroidal board, it suffices to place d nonattacking queens on the $2d \times d$ toroidal board. Indeed, although the natural injection $\mathbb{Z}_d \times \mathbb{Z}_{2d} \hookrightarrow \mathbb{Z}_m \times \mathbb{Z}_n$ is not in general a group homomorphism, it is easy to see that if two queens are nonattacking in $\mathbb{Z}_d \times \mathbb{Z}_{2d}$, their images in $\mathbb{Z}_m \times \mathbb{Z}_n$ are also nonattacking. Thus, without loss of generality, we may assume that $n = 2m$. In this case $\gcd(m, n) = m$.

If $m \equiv 1, 2, 4, 5 \pmod 6$, a solution is easily obtained by placing a queen at each point in the set $A = \{(i, 2i)\ ;\ i \in \mathbb{Z}_m\}$. Indeed, it is clear that no two distinct elements of A belong to the same coset of H or V. If elements $(i, 2i)$ and $(j, 2j)$ belong to the same coset of P, then $i - j \equiv 2i - 2j \pmod m$ and so $i \equiv j \pmod m$ which implies $i = j$. If elements $(i, 2i)$ and $(j, 2j)$ belong to the same coset of N, then one has $3i \equiv 3j \pmod m$ which also gives $i = j$ when m is not divisible by 3.

Now suppose that m is divisible by 6, say $m = 2^k.6.l$, where l is odd. Here the situation is slightly more complicated; a solution is obtained by placing queens at positions $(i, f(i))$, for $i = 0, \ldots, m-1$, where

$$f(i) = \begin{cases} 2i + [i]_{6l} & ; \text{if } [i]_{3l} = [i]_{6l}, \\ 2i + 1 + [i]_{6l} & ; \text{otherwise.} \end{cases}$$

The case where $m \equiv 3 \pmod 6$ is a good deal more complicated; we consider two subcases. First if $m \equiv 3 \pmod{12}$, say $m = 12k + 3$, a solution is obtained by placing queens at positions $(i, g(i))$, for $i = 0, \ldots, m-1$, where

$$g(i) = \begin{cases} 3i & ; \text{if } i \leq 4k, \\ 2 & ; \text{if } i = 4k+1, \\ 2 + m & ; \text{if } i = 4k+3, \\ 3i - m + 4 & ; \text{if } 4k+2 \leq i \leq 10k \text{ and } i \text{ is even}, \\ 3i - m + 2 & ; \text{if } i = 10k+2, \\ 3i - m - 4 & ; \text{if } 4k+5 \leq i \leq 10k+3 \text{ and } i \text{ is odd}, \\ 3i - m & ; \text{if } i \geq 10k+4. \end{cases}$$

On the other hand, if $m \equiv 9 \pmod{12}$, say $m = 12k + 9$, a solution is obtained by placing queens at positions $(i, h(i))$, for $i = 0, \ldots, m-1$, where

$$h(i) = \begin{cases} 3i & ; \text{if } i \leq 4k+2, \\ 2 & ; \text{if } i = 4k+3, \\ 2+m & ; \text{if } i = 4k+5, \\ 3i-m+4 & ; \text{if } 4k+4 \leq i \leq 10k+6 \text{ and } i \text{ is even}, \\ 3i-2m-2 & ; \text{if } i = 10k+8, \\ 3i-m-4 & ; \text{if } 4k+7 \leq i \leq 10k+7 \text{ and } i \text{ is odd}, \\ 3i-m & ; \text{if } i \geq 10k+9. \end{cases}$$

The verification that the above functions f, g and h have the required properties is tedious but elementary.

It remains to deal with the case where $\gcd(m, n) = 3$. Here the reader will readily find that there is no solution on the 6×3 board, but there are solutions on the 9×3 board. It follows that there are solutions on the $n \times m$ board for all $m < n$ with $\gcd(m, n) = 3$ except in the case $m = 3, n = 6$. This completes the proof of the theorem. $\square$

References

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