Hadamard matrices and strongly regular graphs with the 3-e.c. adjacency property

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Abstract

A graph is 3-e.c. if for every 3-element subset S of the vertices, and for every subset T of S, there is a vertex not in S which is joined to every vertex in T and to no vertex in $S \setminus T$. Although almost all graphs are 3-e.c., the only known examples of strongly regular 3-e.c. graphs are Paley graphs with at least 29 vertices. We construct a new infinite family of 3-e.c. graphs, based on certain Hadamard matrices, that are strongly regular but not Paley graphs. Specifically, we show

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that Bush-type Hadamard matrices of order $16n^2$ give rise to strongly regular 3-e.c. graphs, for each odd n for which $4n$ is the order of a Hadamard matrix.

Key words: n-e.c. graphs, strongly regular graphs, adjacency property, Bush-type Hadamard matrix, design

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1 Introduction

Throughout, all graphs are finite and simple. A *strongly regular graph* $SRG(v, k, \lambda, \mu)$ is a regular graph with v vertices of degree k such that every two joined vertices have exactly λ common neighbours, and every two distinct non-joined vertices have exactly μ common neighbours.

For a fixed integer $n \geq 1$, a graph G is *n-existentially closed* or *n-e.c.* if for every n-element subset S of the vertices, and for every subset T of S, there is a vertex not in S which is joined to every vertex in T and to no vertex in $S \setminus T$. N-e.c. graphs were first studied in [8], where they were called graphs with property $P(n)$. For further background on n-e.c. graphs the reader is directed to [5].

If q is a prime power congruent to 1 (mod 4), then the *Paley graph of order* q, written P_q, is the graph with vertices the elements of $GF(q)$, the field of order q, and distinct vertices are joined iff their difference is a square in GF(q). It is well-known that P_q is self-complementary and a $SRG(q, (q-1)/2, (q-5)/4, (q-1)/4)$. In [1] and [4], it was shown that for a fixed n, sufficiently large Paley graphs are n-e.c.. Few examples of strongly regular non-Paley n-e.c. graphs are known, despite the fact that for a fixed n almost all graphs are n-e.c. (see [3] and [9]). The exception is when $n = 1$ or 2; see [5] and [6]. Even for $n = 3$ it has proved difficult to find strongly regular n-e.c. graphs that are not Paley graphs. In [1] it was shown that P_{29} is the minimal order 3-e.c. Paley graph. As reported in [5], a 3-e.c. graph has order at least 20, and a computer search has revealed two non-isomorphic 3-e.c. graphs of order 28, neither of which is strongly regular.

In this article we construct new infinite families of strongly regular 3-e.c. graphs that are not Paley graphs. The graphs we study are constructed from certain Hadamard matrices; in particular, their adjacency matrices correspond to Bush-type Hadamard matrices (see Theorem 5).

A *co-clique* in a graph is a set of pairwise non-joined vertices. The matrices I_n and J_n are the $n \times n$ identity matrix and matrix of all ones, respectively. A *normalized* Hadamard matrix is one whose first row and first column is all ones. For matrices A, B, $A \otimes B$ is the *tensor* or *Kronecker product* of A and B.

2 Bush-type Hadamard matrices

A Hadamard matrix H of order $4n^2$ is called a *Bush-type Hadamard matrix* if $H = [H_{ij}]$, where H_{ij} are blocks of order $2n$, $H_{ii} = J_{2n}$ and $H_{ij}J_{2n} = J_{2n}H_{ij}$, for $i \neq j$, $1 \leq i \leq 2n$, $1 \leq j \leq 2n$.

In the language of graphs, a symmetric Bush-type Hadamard matrix of order $4n^2$ is the $\mp$-adjacency matrix (-1 for adjacency, $+1$ for non-adjacency) of a strongly regular $(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$ graph. See Haemers and Tonchev [10] for a study of such graphs.

K. A. Bush [7] proved that if there exists a projective plane of order $2n$, then there is a Bush-type Hadamard matrix. Although it is fairly simple to construct Bush-type Hadamard matrices of order $16n^2$, very little is known about the existence or non-existence of such matrices of order $4n^2$, for n odd. See [11] for details.

For completeness we include the following result of Kharaghani [12].

Theorem 1. *If the order of an Hadamard matrix is $4n$, then there is a Bush-type Hadamard matrix of order $16n^2$.*

Proof. Let K be a normalized Hadamard matrix of order $4n$. Let $c_1, c_2, \ldots, c_{4n}$ be the column vectors of K. Let $C_i = c_i c_i^t$, for $i = 1, 2, \ldots, 4n$. Then it is easy to see that:

1. $C_i^t = C_i$, for $i = 1, 2, \ldots, 4n$;
2. $C_1 = J_{4n}$, $C_i J_{4n} = J_{4n} C_i = 0$, for $i = 2, 3, \ldots, 4n$;
3. $C_i C_j^t = 0$, for $i \neq j$, $1 \leq i, j \leq n$;
4. $$\sum_{i=1}^{4n} C_i C_i^t = 16n^2 I_{4n}.$$

Now consider a symmetric Latin square with entries $1, 2, \ldots, 4n$ with constant diagonal 1. Replace each i by C_i. We then obtain a Bush-type Hadamard matrix of order $16n^2$. $\square$

Example 2. We give an example of a Bush-type Hadamard matrix of order 64. For ease of notation, we use $-$ instead of -1. Let K be the following Hadamard matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & - & 1 & - & 1 & - & 1 & - \\ 1 & 1 & - & - & 1 & 1 & - & - \\ 1 & - & - & 1 & 1 & - & - & 1 \\ 1 & 1 & 1 & 1 & - & - & - & - \\ 1 & - & 1 & - & - & 1 & - & 1 \\ 1 & 1 & - & - & - & - & 1 & 1 \\ 1 & - & - & 1 & - & 1 & 1 & - \end{pmatrix}$$
$$= \begin{pmatrix} c_1 & c_2 & c_3 & c_4 & c_5 & c_6 & c_7 & c_8 \end{pmatrix}$$

Then for $i = 1, \ldots, 8$, let

$$C_i = c_i c_i^t,$$

and let

$$H = \begin{pmatrix} C_1 & C_2 & C_3 & C_4 & C_5 & C_6 & C_7 & C_8 \\ C_2 & C_1 & C_4 & C_3 & C_6 & C_5 & C_8 & C_7 \\ C_3 & C_4 & C_1 & C_2 & C_7 & C_8 & C_5 & C_6 \\ C_4 & C_3 & C_2 & C_1 & C_8 & C_7 & C_6 & C_5 \\ C_5 & C_6 & C_7 & C_8 & C_1 & C_2 & C_3 & C_4 \\ C_6 & C_5 & C_8 & C_7 & C_2 & C_1 & C_4 & C_3 \\ C_7 & C_8 & C_5 & C_6 & C_3 & C_4 & C_1 & C_2 \\ C_8 & C_7 & C_6 & C_5 & C_4 & C_3 & C_2 & C_1 \end{pmatrix}$$

By Theorem 1, H is a Bush-type Hadamard matrix of order 64.

Lemma 3. *Let $H = [H_{ij}]$ be a Bush-type Hadamard matrix of order $4n^2$. Let $M = H - I_{2n} \otimes J_{2n}$. Then M contains two $SRG(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$.*

Proof. The row sums of H are all $2n$. Thus the negative entries in H can be viewed as the incidence matrix of a $SRG(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$-graph. Since negating all the off diagonal blocks of H leaves a Bush-type Hadamard, the positive entries of M also form a $SRG(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$-graph. ☐

Note that the two graphs may not be isomorphic in general. We call the matrix M a *twin graph*.

3 Bush-type Hadamard matrices and 3-e.c. graphs

A graph G is 3-e.c. if for each triple x, y, z of distinct vertices from G, there are 8 vertices from $V(G) \setminus \{x, y, z\}$, one joined to each of x, y, z; 3 joined to exactly two of x, y, z; 3 joined to exactly one of x, y, z; and one joined to none of x, y, z. From the perspective of the $(1, -1)$-adjacency matrix A of G this is equivalent to the following condition: for each 3 distinct rows r_1, r_2, r_3 from A, representing vertices x, y, z in G, there are 8 columns in the submatrix formed by r_1, r_2, r_3, distinct from the columns representing x, y, z, which contain all the 8 possible patterns of 1's and -1's. Since our graphs are constructed as $(1, -1)$-adjacency matrices, we use the latter condition when checking whether our graphs are 3-e.c.. We first prove the following lemma.

Lemma 4. *Let K be a normalized Hadamard matrix of order $4n$ with n odd and $n > 1$. Consider any $3 \times n$ submatrix of K not including the first row of K. Each of the eight possible sign patterns appears as a column at least once among the columns of the submatrix.*

Proof. Let K be a normalized Hadamard matrix of order $4n$. Consider any three rows of K which do not include the first row. Without loss of generality, we may assume that the rows have the form

$$\begin{array}{cccccccc} \overbrace{\quad\quad}^{a} & \overbrace{\quad\quad}^{b} & \overbrace{\quad\quad}^{c} & \overbrace{\quad\quad}^{d} & \overbrace{\quad\quad}^{e} & \overbrace{\quad\quad}^{f} & \overbrace{\quad\quad}^{g} & \overbrace{\quad\quad}^{h} \\ +\cdots & +\cdots & +\cdots & +\cdots & -\cdots & -\cdots & -\cdots & -\cdots \\ +\cdots & +\cdots & -\cdots & -\cdots & +\cdots & +\cdots & -\cdots & -\cdots \\ +\cdots & -\cdots & +\cdots & -\cdots & +\cdots & -\cdots & +\cdots & -\cdots \end{array},$$

where each letter is a nonnegative integer. This leads to the linear system

$$\begin{aligned} a+b+c+d+e+f+g+h &= 4n \\ a+b+c+d &= 2n \\ a+b+e+f &= 2n \\ a+c+e+g &= 2n \\ a+b-c-d-e-f+g+h &= 0 \\ a-b+c-d-e+f-g+h &= 0 \\ a-b-c+d+e-f-g+h &= 0 \end{aligned}$$

It can be seen that the solution for this system is $b=c=e=h$ and $a=d=f=g=n-h$.

We need to find a positive solution to the system. Since K is normalized, $a>0$ so $h \neq n$. It is enough to show that $h=0$ is not possible. If $h=0$, then the three selected rows have the following form:

$$\begin{array}{cccc} \overbrace{\quad\quad}^{a=n} & \overbrace{\quad\quad}^{d=n} & \overbrace{\quad\quad}^{f=n} & \overbrace{\quad\quad}^{g=n} \\ +\cdots & +\cdots & -\cdots & -\cdots \\ +\cdots & -\cdots & +\cdots & -\cdots \\ +\cdots & -\cdots & -\cdots & +\cdots \end{array} \tag{1}$$

Now consider a fourth row, and without loss of generality, we can rearrange the columns so that the rows have the form:

$$\begin{array}{cccccccc} \overbrace{\quad\quad}^{a'} & \overbrace{\quad\quad}^{b'} & \overbrace{\quad\quad}^{c'} & \overbrace{\quad\quad}^{d'} & \overbrace{\quad\quad}^{e'} & \overbrace{\quad\quad}^{f'} & \overbrace{\quad\quad}^{g'} & \overbrace{\quad\quad}^{h'} \\ +\cdots & +\cdots & +\cdots & +\cdots & -\cdots & -\cdots & -\cdots & -\cdots \\ +\cdots & +\cdots & -\cdots & -\cdots & +\cdots & +\cdots & -\cdots & -\cdots \\ +\cdots & +\cdots & -\cdots & -\cdots & -\cdots & -\cdots & +\cdots & +\cdots \\ +\cdots & -\cdots & +\cdots & -\cdots & +\cdots & -\cdots & +\cdots & -\cdots \end{array},$$

where each primed letter is a nonnegative integer.

This leads to the system

$$
\begin{aligned}
a' + b' &= n \\
c' + d' &= n \\
e' + f' &= n \\
g' + h' &= n \\
a' - b' + c' - d' - e' + f' - g' + h' &= 0 \\
a' - b' - c' + d' - e' + f' + g' - h' &= 0 \\
a' - b' - c' + d' + e' - f' - g' + h' &= 0 \\
a' + c' + e' + g' &= 2n
\end{aligned}
$$

whose solution is $a' = b' = c' = d' = e' = f' = g' = h' = n/2$. Thus n must be even, a contradiction. □

Theorem 5. *Let $4n$ be the order of a Hadamard matrix, n odd, $n > 1$. There is a Bush-type Hadamard matrix of order $16n^2$ which is the adjacency matrix of a twin $SRG(16n^2, 8n^2-2n, 4n^2-2n, 4n^2-2n)$, whose vertices can be partitioned into $4n$ disjoint co-cliques of order $4n$. Furthermore, the graph is 3-e.c..*

Proof. Consider the Bush-type Hadamard matrix $H = [H_{ij}]$, where H_{ij} is the ij block of H of size $4n \times 4n$, constructed in Theorem 1 from a Hadamard matrix of order $4n$. The fact that there is a twin $SRG(16n^2, 8n^2 - 2n, 4n^2 - 2n, 4n^2 - 2n)$ whose vertices can be partitioned into $4n$ disjoint co-cliques of order $4n$, follows from Lemma 3.

It remains to show that the graph is 3-e.c.. Given three rows of H, consider the submatrix L consisting of these three rows. We need to show that each of the eight possible sign patterns appears as a column among the columns of the submatrix.

The rows of H can be partitioned into $4n$ "zones", corresponding to the rows of the $4n$ $4n \times 4n$ subblocks of H. We consider three cases, based on where the three rows of L are located relative to the zones.

Case 1: The rows of L are selected from the same zone, say the j-th zone. Referring to the proof of Theorem 1 we see that the leading columns of C_i's form a rearrangement of the columns of the original Hadamard matrix. The only case when not all 8 patterns appear among the leading columns is if the leading columns appear in form like in matrix (1). However, all C_i's are of rank 1, so the negatives of each of the patterns in the columns of matrix (1) appear among the columns of H, off the block H_{jj}. Thus all eight patterns appear off the block H_{jj}.

Case 2: Exactly two rows of L belong to the same zone. Suppose two rows r_1 and r_2 are from the j-th zone and the other row, r_3, is from the k-th zone, where $k \neq j$. Without loss of generality we can assume that the entries of the blocks in the r_1 and r_2

rows have the form:

$$\begin{array}{cccc} \overbrace{+\cdots}^{n} & \overbrace{+\cdots}^{n} & \overbrace{-\cdots}^{n} & \overbrace{-\cdots}^{n} \\ +\cdots & -\cdots & +\cdots & -\cdots \end{array} \qquad (2)$$

We now look at the possible arrangement of row r_1 relative to row r_3. We observe that in each block a similar arrangement as in (2) occurs. Since in each block, row r_2 is a multiple of row r_1, we see that all eight patterns appear off the blocks H_{jj} and H_{kk}.

Case 3: The three rows of L belong to three different zones, zones i, j, and k, with i, j, and k distinct. Select l distinct from i, j, and k. Consider the three rows restricted to the blocks H_{il}, H_{jl}, and H_{kl}. The rows are multiples of three distinct rows of the original Hadamard matrix, so by Lemma 4 all eight patterns appear off the blocks H_{ii}, H_{jj}, and H_{kk}. (Note that the assumption that n is odd is only used in this part of the proof.) □

Of course, none of the graphs in Theorem 5 are Paley graphs. We think that the assumption that n is odd can be dropped from Theorem 5, in view of the following example and remark, and the proof above.

Example 6. A Bush-type Hadamard matrix of order 64 is not included in the previous theorem. However, Example 2 leads to two (isomorphic) graphs which were verified to be 3-e.c. by a computer calculation. We have verified that this graph is not 4-e.c.. We do not know an example of a 4-e.c. graph that is not a Paley graph.

Remark 7. The only known Bush-type Hadamard matrix of order $4n^2$, n odd, $n > 1$ is of order 324 and is constructed in [11]. We tested this Bush-type Hadamard matrix of order 324 by computer and have established that its graph is 3-e.c..

These observations lead us to the following conjecture.

Conjecture 8. *Every Bush-type Hadamard matrix of order $4n^2$ with $n > 1$ contains a twin 3-e.c. $SRG(4n^2, 2n^2 - n, n^2 - n, n^2 - n)$.*

We are grateful to the referee for pointing out a few minor errors and the following. There are some strongly regular 3-e.c. graphs that are not Paley graphs. Some of them, however, do have the same parameters. For example, there are 3-e.c. graphs that are not Paley graphs, but have the same parameters as P_{37}, P_{41}, and P_{49}. Furthermore, although there does not exist a symmetric Bush-type Hadamard matrix of order 36 (see for example [2]), there is a unique 3-e.c. $(36, 15, 6, 6)$ graph which is reproduced in Figure 1.

$$
\begin{bmatrix}
0 1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 \\
1 0 1 1 1 1 1 1 0 0 0 0 0 0 0 0 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 \\
1 1 0 1 1 1 1 0 1 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0 1 1 1 1 1 1 1 0 0 0 0 0 \\
1 1 1 0 1 0 0 1 0 1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 1 1 1 0 0 0 0 1 1 1 0 0 \\
1 1 1 1 0 0 0 0 0 1 0 1 1 0 0 0 0 1 0 1 1 0 0 0 1 0 0 1 1 0 0 0 0 0 1 1 \\
1 1 1 0 0 0 0 1 1 0 1 1 0 0 0 0 0 0 0 0 0 1 1 1 0 0 0 1 0 1 1 1 0 0 1 0 \\
1 1 1 0 0 0 0 1 0 0 0 0 0 1 1 1 1 0 1 1 0 0 0 0 0 0 0 0 1 1 1 0 1 0 0 1 \\
1 1 0 1 0 1 1 0 0 0 0 0 0 1 1 0 0 0 0 0 0 1 1 0 1 0 0 0 0 0 0 1 1 1 1 1 \\
1 0 1 0 0 1 0 0 0 0 0 1 0 1 1 1 1 1 0 0 0 0 0 1 0 1 1 1 0 0 0 0 0 1 1 0 \\
1 0 0 1 1 0 0 0 0 0 1 1 1 0 0 1 1 0 0 0 0 1 0 1 0 1 0 0 1 0 0 1 1 0 0 1 \\
1 0 0 1 0 1 0 0 0 1 0 1 1 0 0 1 0 0 1 1 0 0 1 0 0 0 1 0 0 1 1 1 0 1 0 0 \\
1 0 0 0 1 1 0 0 1 1 1 0 0 1 0 0 0 0 1 0 1 0 0 1 0 0 0 0 1 1 0 0 0 1 1 1 \\
1 0 0 0 1 0 0 0 0 1 1 0 0 1 1 1 0 0 0 1 1 1 1 0 1 1 0 1 0 0 1 0 0 0 0 0 \\
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1 0 0 0 0 0 1 0 1 1 1 0 1 0 1 0 1 1 0 1 0 1 0 0 0 0 1 0 0 1 0 0 1 0 1 0 \\
0 1 1 0 0 0 1 0 1 1 0 0 0 0 0 1 0 0 0 1 0 1 1 1 0 1 1 0 1 0 0 0 0 1 0 1 \\
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0 1 0 1 0 0 1 0 0 0 1 1 0 1 0 0 0 1 0 1 1 0 0 1 0 1 0 0 0 1 1 0 1 1 0 0 \\
0 1 0 0 1 0 1 0 0 0 1 0 1 0 0 1 1 1 1 0 0 0 1 0 0 0 0 1 0 0 1 0 0 1 1 1 \\
0 1 0 0 1 0 0 0 0 0 0 1 1 1 1 0 0 1 1 0 0 1 1 1 1 0 1 0 1 1 0 0 0 0 0 0 \\
0 1 0 0 0 1 0 1 0 1 0 0 1 0 0 1 1 0 0 0 1 0 1 1 1 1 0 0 0 1 0 0 1 0 1 0 \\
0 1 0 0 0 1 0 1 0 0 1 0 1 1 0 0 1 0 0 1 1 1 0 0 0 0 1 1 1 0 0 1 0 1 0 0 \\
0 1 0 0 0 1 0 0 1 1 0 1 0 0 1 0 1 1 1 0 1 1 0 0 0 1 0 0 0 0 1 1 0 0 0 1 \\
0 0 1 1 1 0 0 1 0 0 0 0 1 0 1 0 0 0 0 0 1 1 0 0 0 1 1 0 0 1 1 0 0 1 1 1 \\
0 0 1 1 0 0 0 0 1 1 0 0 1 1 0 0 1 0 1 0 0 1 0 1 1 0 0 1 0 0 1 0 1 1 0 0 \\
0 0 1 1 0 0 0 0 1 0 1 0 0 0 1 1 1 1 0 0 1 0 1 0 1 0 0 0 1 1 0 1 0 1 0 0 \\
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0 0 1 0 1 0 1 0 0 1 0 1 0 1 0 0 1 0 0 0 1 0 1 0 0 0 1 1 0 1 0 1 1 0 0 1 \\
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0 0 0 1 0 0 0 1 1 0 1 1 0 1 0 0 1 0 1 1 0 0 1 0 1 1 1 0 0 0 0 0 0 0 1 1 \\
0 0 0 0 1 1 0 1 1 0 0 1 0 0 0 1 0 1 0 1 0 1 0 0 1 0 0 1 0 1 0 0 1 1 0 1 \\
0 0 0 0 1 0 1 1 0 1 0 1 0 0 1 0 1 0 0 1 0 0 0 1 1 0 0 0 1 0 1 1 0 1 1 0
\end{bmatrix}
$$

Figure 1: A 3-e.c. $(36, 15, 6, 6)$ graph

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