# Meet and Join within the Lattice of Set Partitions

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Submitted: January 21, 2001; Accepted: March 1, 2001.
MR Subject Classifications: 05A18, 05A15, 05A16, 06C10

**Abstract**

We build on work of Boris Pittel [5] concerning the number of $t$-tuples of partitions whose meet (join) is the minimal (maximal) element in the lattice of set partitions.

## 1 Introduction

Recall that a *partition* of the set $[n] = \{1, 2, \ldots n\}$ is a collection of nonempty, pairwise disjoint subsets of $[n]$ whose union is $[n]$. The subsets are called *blocks*. One partition $\pi_1$ is said to *refine* another $\pi_2$, denoted $\pi_1 \leq \pi_2$, provided every block of $\pi_1$ is contained in some block of $\pi_2$. The refinement relation is a partial ordering of the set $\Pi_n$ of all partitions of $[n]$. Given two partitions $\pi_1$ and $\pi_2$, their *meet*, $\pi_1 \wedge \pi_2$, (respectively *join*, $\pi_1 \vee \pi_2$) is the largest (respectively smallest) partition which refines (respectively is refined by) both $\pi_1$ and $\pi_2$. The meet has as blocks all nonempty intersections of a block from $\pi_1$ with a block from $\pi_2$. The blocks of the join are the smallest subsets which are exactly a union of blocks from both $\pi_1$ and $\pi_2$. Under these operations, the poset $\Pi_n$ is a lattice.

Recently Pittel has considered the number $M_n^{(t)}$ of $t$-tuples of partitions whose meet is the minimal partition $\{\{1\}, \{2\}, \ldots \{n\}\}$, and $J_n^{(t)}$ the number of $t$-tuples whose join is the maximal partition $\{\{1, 2, \ldots, n\}\}$. We shall prove

**Theorem 1** *Let $M_t(x)$ and $J_t(x)$ be the exponential generating functions for the sequences $M_n^{(t)}$ and $J_n^{(t)}$. Then*

$$M_t(e^x - 1) \quad = \quad \sum_{n=0}^{\infty} (B_n)^t \frac{x^n}{n!} \quad = \quad \exp\{J_t(x) - 1\}.$$

*where $B_n$ is the $n$-th Bell number, the total number of partitions of the set $[n]$.*

**Remark.** What about $n = 0$ and/or $t = 0$? The lattice $\Pi_0$ has exactly one element, and thus is isomorphic to $\Pi_1$. Generally, one takes the empty meet to be the maximal element and the empty join to be the minimal element. Thus, there is some logical justification to define

$$\begin{aligned} M_0^{(t)} &= J_0^{(t)} = 1 \text{ for all } t \\ M_n^{(0)} &= J_n^{(0)} = 1 \text{ for } n = 0, 1 \\ M_n^{(0)} &= J_n^{(0)} = 0 \text{ for all } n \geq 2 \end{aligned}$$

In particular, $M_0(x) = J_0(x) = 1 + x$, and $M_1(x) = J_1(x) = e^x$. These latter two when inserted in the theorem yield immediately recognized identities.

To prove the first equality of Theorem 1 we shall use the following known result:

**Theorem 2** *Let $E_n$ be the edge set of the complete graph $K_n$, $G_S$ the graph with vertex set $[n]$ and edge set $S \subseteq E_n$, and $\mathrm{c}(G)$ the number of connected components in the graph $G$. Then,*

$$\sum_{S \subseteq E_n} (-1)^{|S|} X^{\mathrm{c}(G_S)} = X(X-1)\cdots(X-n+1).$$

A consequence of Theorem 2 is our later formula (2) which gives $M_n^{(t)}$ as a sum of products of Bell number powers with Stirling numbers of the first kind. With the second equality of Theorem 1 we can prove

**Theorem 3**

$$J_n^{(2)} = (B_n)^2 \times (1 - \frac{r^2}{n} - \frac{2r^3 + 2r^4 + 2r^5 + r^6}{(r+1)^2 n^2} + \mathrm{O}(r^7/n^3))$$

*where $r$ is the positive real solution of the equation $re^r = n$.*

This improves on Pittel's estimate that $J_n^{(t)}$ is $(B_n)^t(1 + \mathrm{O}(r^{t+1}/n^{t-1}))$. The method by which we prove Theorem 3 yields in principle a complete asymptotic expansion of $J_n^{(t)}$ in descending powers of $n$, although the later terms are quite complicated. In the final section of our paper, we present a generalization of the first equality in Theorem 1.

# 2 Discussion of Theorem 2

We shall not give a proof of this theorem, since many are available. Indeed, using the Principle of Inclusion-Exclusion, the left side can be interpreted as the number of ways to color properly the complete graph $K_n$ with $X$ colors, which agrees with the right side. More generally, we may replace the graph $K_n$ on the left with an arbitrary graph $G$, and then on the right we replace the displayed polynomial with the chromatic polynomial of $G$. A good reference for this is [2].

Since the coefficients of $X(X-1)\cdots(X-n+1)$ are the (signed) Stirling numbers of the first kind, $s(n,k)$, Theorem 2 is equivalent to:

$$\sum_{\substack{S\subseteq E_n\\ \mathrm{c}(G_S)=k}} (-1)^{|S|} \quad = \quad s(n,k).$$

In this form the theorem states that among graphs of $n$ vertices and $k$ connected components, the excess of the number with an even number of edges over those with an odd number of edges is the signed Stirling number of the first kind $s(n,k)$. The case $k=1$ of this interesting interpretation appeared as a Monthly Problem a few years ago, and in the solution the generalization to larger $k$ was noted, [3].

We close this section with a useful inclusion/exclusion enumeration formula based on Theorem 2.

**Corollary.** Let $X$ be a set of combinatorial objects which may have properties corresponding to the pairs $E_n$, $n \geq 1$. Suppose that for $S \subseteq E_n$, the number of objects which have at least all the properties of $S$ depends only on $\mathrm{c}(G_S)$, the number of connected components of the graph $G_S$ determined by the pairs $S$. If this number is $f(\mathrm{c}(G_S))$, then,

$$\#\{x \in X : x \text{ has no property}\} \quad = \quad \sum_{k=1}^{n} s(n,k)\, f(k).$$

# 3 An Application

We shall now use the above inclusion/exclusion formula to give another proof of the beautiful formula found by Boris Pittel [5]. The formula is

$$M_n^{(t)} \quad = \quad e^{-t} \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \frac{(i_1 \cdots i_t)_n}{i_1! \cdots i_t!}, \tag{1}$$

where, again, $M_n^{(t)}$ is the number of $t$-tuples of partitions satisfying

$$\pi_1 \wedge \pi_2 \wedge \cdots \wedge \pi_t \quad = \quad \{\{1\},\{2\},\ldots,\{n\}\}.$$

A striking feature of Pittel's formula is its resemblance to Dobinski's formula (see [6])

$$B_n \quad = \quad e^{-1} \sum_{i=1}^{\infty} \frac{i^n}{i!},$$

or its $t$-th power:

$$(B_n)^t \quad = \quad e^{-t} \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \frac{(i_1 \cdots i_t)^n}{i_1! \cdots i_t!}.$$

A collection of $t$ partitions will have nontrivial meet precisely when there is at least one pair of integers $i$ and $j$ which belong to the same block in all $t$ of the partitions. Let $X$ be the set of all $t$-tuples of partitions, and let $(i,j)$ be the property that when the meet of a $t$-tuple is formed, elements $i$ and $j$ are still in the same block. Then, by the Corollary of the previous section,

$$M_n^{(t)} \quad = \quad \sum_{k=1}^{n} s(n,k)(B_k)^t. \tag{2}$$

Herb Wilf pointed out that the previous identity is equivalent to, (1). Indeed,

$$\begin{aligned} e^{-t} \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \frac{(i_1 \cdots i_t)_n}{i_1! \cdots i_t!} \quad &= \quad e^{-t} \sum_{i_1=1}^{\infty} \cdots \sum_{i_t=1}^{\infty} \frac{\sum_{k=1}^{n} s(n,k)(i_1 \cdots i_t)^k}{i_1! \cdots i_t!} \\ &= \quad \sum_{k=1}^{n} s(n,k) \left( e^{-1} \sum_{i=1}^{\infty} \frac{i^k}{i!} \right)^t \\ &= \quad \sum_{k=1}^{n} s(n,k)(B_k)^t, \end{aligned}$$

proving the theorem.

# 4 Proof of the First Equality in Theorem 1

Since (see for example [1])

$$\sum_{n \geq 0} s(n,k) \frac{x^n}{n!} \quad = \quad \frac{(\log(1+x))^k}{k!},$$

equation (2) is equivalent to

$$[\frac{x^n}{n!}]\, M_t(x) \quad = \quad \sum_{k \geq 0} (B_k)^t \, [\frac{x^n}{n!}] \frac{(\log(1+x))^k}{k!}.$$

The linear operator $[\frac{x^n}{n!}]$, "take the coefficient of $\frac{x^n}{n!}$," can be moved outside the summation on the right. Then, we may drop the $[\frac{x^n}{n!}]$ from both sides, leaving an identity. The identity is exactly the first equality in Theorem 1, after substituting $e^x - 1$ for $x$.

# 5 Proof of the Second Equality in Theorem 1

There is a Basic Principle of Exponential Generating Functions which says that if $J(x)$ is the egf of certain labeled combinatorial objects, then $\exp\{J(x)-1\}$ is the egf for partitions of $n$ with a $J$-object built on each block. A very good account of this exponential formula is given in [7], Chapter 3. It suffices, therefore, to establish a bijection

$$\underbrace{\Pi_n \times \Pi_n \times \cdots \times \Pi_n}_{t \text{ factors}} \quad \longleftrightarrow \quad Q_{n\,t} \tag{3}$$

where $Q_{n\,t}$ consists of all sets of $t$-tuples of partitions

$$\{\ (x_{1\,1}, x_{1\,2}, \ldots x_{1\,t}), \quad \ldots \quad , \ (x_{\ell\,1}, x_{\ell\,2}, \ldots x_{\ell\,t})\ \} \tag{4}$$

with the property that each join

$$x_{i\,1} \;\vee\; x_{i\,2} \;\vee\; \cdots \;\vee\; x_{i\,t}$$

is a one-block partition $\{S_i\}$, where $S_i \subseteq [n]$ and $\{S_i : 1 \le i \le \ell\}$ is a partition of $[n]$. To repeat for clarity, each member of $Q_{n\,t}$ is a nonempty set (whose size is denoted here $\ell \ge 1$), each element of which is a $t$-tuple $(x_{i\,1}, \ldots, x_{i\,t})$. The various $x_{i\,j}$ are themselves partitions of a set $S_i \subseteq [n]$; the join (over $j$) of the $x_{i\,j}$ equals $\{S_i\}$; and $\pi = \{S_i : 1 \le i \le \ell\}$ is a partition of $[n]$.

Once the definition of the set $Q_{n\,t}$ has been comprehended, the bijection (3) with the Cartesian product $(\Pi_n)^t$ is fairly natural. In the direction $\longrightarrow$, let a $t$-tuple of partitions $(\pi_1, \ldots, \pi_t)$, be given. Let $\pi = \{S_i : 1 \le i \le \ell\}$ be their join. The partitions $x_{i\,j}$, $(1 \le i \le \ell, 1 \le j \le t)$, are the nonempty intersections of the blocks of $\pi_j$ with the set $S_i$.

In the other direction $\longleftarrow$, let $T$ be a set of the form (4), consisting of $t$-tuples of partitions $x_{i\,j}$. We know that each join $\vee_{j=1}^{t} x_{i\,j}$ is a one-block partition $\{S_i\}$. Since $x_{i\,j}$ is a partition of $S_i$, and $\{S_i : 1 \le i \le \ell\}$ is itself a partition of $[n]$, it follows that

$$\pi_j \;=\; x_{1\,j} \;\cup\; x_{2\,j} \;\cup\; \cdots \;\cup\; x_{\ell\,j}$$

is a partition of $[n]$. The $t$-tuple $(\pi_1, \pi_2, \ldots \pi_t)$ so formed is the one to be associated by the bijection with the initially given set $T$.

## 6 Calculations

The equation (2) yields efficient calculation of $M_n^{(2)}$. By differentiating the second equality of Theorem 1, we obtain, by a familiar technique, the recursion

$$J_{n+1}^{(t)} \;=\; (B_{n+1})^t \;-\; \sum_{j=1}^{n} \binom{n}{j} (B_j)^t J_{n-j+1}^{(t)}, \quad n \ge 0, \tag{5}$$

and this permits efficient calculation of $J_n^{(t)}$. By these means we determine the following table for $t = 2$.

| $n$ | $M_n^{(2)}$ | $J_n^{(2)}$ |
|---|---:|---:|
| 0 | 1 | 1 |
| 1 | 1 | 1 |
| 2 | 3 | 3 |
| 3 | 15 | 15 |
| 4 | 113 | 119 |
| 5 | 1153 | 1343 |
| 6 | 15125 | 19905 |
| 7 | 245829 | 369113 |
| 8 | 4815403 | 8285261 |
| 9 | 111308699 | 219627683 |
| 10 | 2985997351 | 6746244739 |

# 7 Proof of Theorem 3

To simplify and avoid proliferation of cases, we take $t = 2$ and accuracy $n^{-2}$; the method can be adapted for any fixed $t \geq 2$, and any desired accuracy. It is an iterative method, and we need an initial estimate. From [5] we know

$$J_n^{(2)} \quad = \quad (B_n)^2\,(1 + \mathrm{O}(r^3/n)), \tag{6}$$

where $r$ is the positive real solution of $re^r = n$. By the Moser-Wyman method [4] we have

$$\frac{B_{n+1}}{B_n} \quad = \quad \frac{n+1}{r}\,(1 + \mathrm{O}(n^{-1})), \tag{7}$$

and from the recursion (5),

$$\frac{J_{n+1}^{(2)}}{(B_{n+1})^2} \quad = \quad 1 \; - \; n\frac{J_n^{(2)}}{(B_{n+1})^2} \; - \; \frac{1}{(B_{n+1})^2}\sum_{j=2}^{n}\binom{n}{j}(B_j)^2 J_{n+1-j}^{(2)}.$$

We bound the summation above by replacing $J_{n+1-j}^{(2)}$ with $(B_{n+1-j})^2$. The resulting convolution can be further bounded as in the proof of Theorem 5 in [5]; namely, it is the terms at the extreme ends of the sum which dominate:

$$\frac{1}{(B_{n+1})^2}\sum_{j=2}^{n}\binom{n}{j}(B_j B_{n+1-j})^2 \quad = \quad \mathrm{O}(r^4/n^2).$$

With

$$\frac{J_n^{(2)}}{(B_{n+1})^2} \quad = \quad \frac{J_n^{(2)}}{(B_n)^2}\,(\frac{B_n}{B_{n+1}})^2,$$

the bound for the summation, (6), and (7) we have

$$\frac{J_n}{(B_n)^2} \quad = \quad 1 \; - \; \frac{r^2}{n} \; + \; \mathrm{O}(r^5/n^2).$$

(When we replace $n$ by $n-1$, we must replace $r$ by $r + \mathrm{O}(n^{-1})$.) We now repeat the process. This time we substitute into

$$\begin{aligned}\frac{J_{n+1}}{(B_{n+1})^2} \quad &= \quad 1 \; - \; n\frac{J_n^{(2)}}{(B_{n+1})^2} \; - \; 4\binom{n}{2}\frac{J_{n-1}^{(2)}}{(B_{n+1})^2} \; - \; J_1^{(2)}\,(\frac{B_n}{B_{n+1}})^2 \\ &\quad - \; \frac{1}{(B_{n+1})^2}\sum_{j=3}^{n-1}\binom{n}{j}(B_j)^2 J_{n+1-j}^{(2)},\end{aligned}$$

using in place of (7) the more accurate

$$\frac{B_{n+1}}{B_n} \quad = \quad \frac{n+1}{r}\,(1 - \frac{2+4r+r^2}{2(r+1)^2 n} + \mathrm{O}(r^2/n^2)),$$

and

$$\frac{1}{(B_{n+1})^2}\sum_{j=3}^{n-1}\binom{n}{j}(B_j B_{n+1-j})^2 \quad = \quad \mathrm{O}(r^6/n^3).$$

The result, after some algebra, including this time a replacement of $n$ by $n-1$ and of $r$ by $r - r/(1+r)n + \mathrm{O}(n^{-2})$, is the formula stated as Theorem 3.

## 8 A Generalization in Terms of Whitney Numbers

In the lattice $\Pi_n$, 0 is the finest partition $\{\{1\},\ldots,\{n\}\}$, and 1 is the coarsest $\{\{1,\ldots,n\}\}$. The intervals of $\Pi_n$ have an interesting recursive structure. Consider first an interval of the form $[\pi,1]$ Observe that the latter interval is isomorphic to $\Pi_k$, where $\pi$ has $k$ blocks. Now, if we take any $t$-tuple of partitions, and form their meet, we obtain some partition $\pi$. Thus, we can count all $t$-tuples according to their meet, as follows:

$$(B_n)^t = \sum_k S(n,k)M_k^{(t)}.$$

This provides, by inversion, another proof and further understanding of equation 2. We can formalize this as follows.

**Theorem 4** *Let $L_n$ be a sequence of lattices with rank$(1)=n$. Assume that each interval $[x,1]\subseteq L_n$ is isomorphic to $L_k$ if $x\in L_n$ and rank$(x)=n-k$. If $M_{L\ n}^{(t)}$ equals the number of $t$-tuples of points in $L_n$ whose meet is $0$, then*

$$|L_n|^t = \sum_k W_{n-k}M_{L\ k}^{(t)},$$

*where $W_k$ are the Whitney numbers of the second kind, the number of elements of rank $k$.*

As an example, consider the lattice $B_n$ of subsets of $[n]$. Theorem 4 tells us

$$2^{nt} = \sum_k \binom{n}{k} M_{B\ k}^{(t)}.$$

By inversion, we conclude there are

$$\sum_k (-1)^k \binom{n}{k} 2^{tk} = (2^t-1)^n$$

$t$-tuples of subsets of $[n]$ whose intersection is empty.

A similar remark can be made for the join operation in $\Pi_n$. Namely, the interval $[0,\pi]$ is isomorphic to a Cartesian product of $\lambda_1$ copies $\Pi_1$ with $\lambda_2$ copies $\Pi_2$, etc., where the shape of partition $\pi$ is $1^{\lambda_1},\ldots n^{\lambda_n}$. Hence,

$$(B_n)^t = \sum_{\lambda\vdash n} \frac{n!}{\prod_i (i!)^{\lambda_i}\lambda_i!}\prod_i (J_i^{(t)})^{\lambda_i}.$$

In this equation, the fraction on the right is the well known [1] formula for the number of partitions of shape $\lambda$. This identity is equivalent to the second equality of Theorem 1. We will not formulate a generalization, since no examples other than $\Pi_n$ come to mind!

**Acknowledgment.** I would like to thank Konrad Engel, Ira Gessel, and David Jackson for helpful comments on earlier versions of the paper.

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