On a Multiplicative Partition Function

Yifan Yang
Department of Mathematics
University of Illinois

Submitted: October 5, 2000; Accepted: April 12, 2000
MR Subject Classification: primary 11N60, secondary 05A18, 11P82

Abstract

Let $D(s)=\sum_{m=1}^{\infty} a_m m^{-s}$ be the Dirichlet series generated by the infinite product $\prod_{k=2}^{\infty}(1-k^{-s})$. The value of a_m for squarefree integers m with n prime factors depends only on the number n, and we let $f(n)$ denote this value. We prove an asymptotic estimate for $f(n)$ which allows us to solve several problems raised in a recent paper by M. V. Subbarao and A. Verma.

1 Introduction and Statements of Results

Let $D(s)=\sum_{m=1}^{\infty} a_m m^{-s}$ be the Dirichlet series generated by the infinite product $\prod_{k=2}^{\infty}(1-k^{-s})$. The coefficients a_m denote the excess of the number of (unordered) representations of m as a product of an even number of distinct integers > 1 over the number of representation of m as a product of an odd number of distinct integers > 1. The Dirichlet series $D(s)$ is closely related to the generating Dirichlet series in the "Factorisatio Numerorum" problem of Oppenheim (see [6]). Indeed, if we let b_m denote the number of (unordered) representations of m as a product of integers > 1, not necessarily distinct, then we have $\sum_{m=1}^{\infty} b_m m^{-s}=D(s)^{-1}$. Thus, by the Möbius inversion formula, the numbers a_m and b_m are related by the identity $a_m=\sum_{d|m}\mu(d)b_{m/d}$. Oppenheim [6] showed that

$$\frac{1}{x}\sum_{m\le x} b_m \sim \frac{e^{\sqrt{\log x}}}{2\sqrt{\pi}(\log x)^{3/4}}.$$

In [3], E. R. Canfield, P. Erdős and C. Pomerance proved that if m is an integer such that $b_n < b_m$ for all $n < m$, then

$$b_m = m\exp\left\{-(1+o(1))\log m\log_3 m/\log_2 m\right\},$$

where $\log_k$ denotes the k-times iterated logarithm.

In this paper, we consider the more difficult problem of investigating the asymptotic behavior of the numbers a_m. This problem was raised by M. V. Subbarao, who observed that $a_m = 0, \pm 1$ for all positive integers m with at most four prime factors and asked whether this is true for all m. It is easy to see that for a positive integer $m > 1$ the coefficient a_m depends only on the exponents $r_1, \ldots, r_n$ in the canonical prime factorization $m = p_1^{r_1} \ldots p_n^{r_n}$. In particular, for squarefree $m = p_1 \ldots p_n$, the value of a_m is a function of the number n of prime factors of m. We will denote this function by $f(n)$.

The function $f(n)$ can be interpreted as a set-partition function. Indeed, by identifying factors of $m = p_1 \ldots p_n$ with subsets of $\{1, 2, \ldots, n\}$, we see that $f(n)$ is equal to the excess of the number of ways to partition a set S of n elements into an even number of non-empty subsets over the number of ways to partition S into an odd number of non-empty subsets. Therefore, $f(n)$ can also be written as

$$f(n) = \sum_{k=1}^{n} (-1)^k S_2(n,k), \tag{1}$$

where the numbers $S_2(n,k)$ are the Stirling numbers of the second kind, which denote the number of partitions of an n-element set into k non-empty subsets (see, e.g., [8, Section 3.6]).

A further motivation for studying the function $f(n)$ is the following observation of D. Bowman [2]. For each integer $n > 0$ there exist exactly one integer b_n and a polynomial $P_n(x,y)$ such that

$$\sum_{k=0}^{m} (k^{n-1} + b_n)k! = P_n(m!, m)$$

holds for all integers m. It turns out that this integer b_n is equal to $f(n)$. By a simple proof by induction, we have $\sum_{k=0}^{m} k \cdot k! = (m+1)! - 1$, and hence $f(2) = b_2 = 0$. The case $n = 2$ is the only known case with $b_n = 0$. H. S. Wilf raised the question whether $b_n = 0$ (or equivalently $f(n) = 0$) infinitely often.

By (1) we have the trivial upper bound

$$|f(n)| \leq \sum_{k=1}^{n} S_2(n,k).$$

The numbers $B(n) = \sum_{k=1}^{n} S_2(n,k)$ are known as Bell numbers (see, e.g., [8, Section 1.6]). De Bruijn [4] gave a detailed asymptotic analysis of $B(n)$, using the saddle point method. In particular, de Bruijn [4, p. 108] showed that

$$\log B(n) = n\left(L - L_2 - 1 + \frac{L_2 + 1}{L} + \frac{L_2^2}{2L^2} + O\left(\frac{L_2^3}{L^3}\right)\right), \tag{2}$$

where $L = \log n$ and $L_2 = \log\log n$. Therefore we have the upper bound

$$\limsup_{n\to\infty} \frac{\log|f(n)|}{n \log n} \leq 1. \tag{3}$$

In a recent paper Subbarao and A. Verma [7] showed that in fact

$$\limsup_{n\to\infty} \frac{\log|f(n)|}{n \log n} = 1.$$

Thus the coefficients a_m in the Dirichlet series $\sum_{m=1}^{\infty} a_m m^{-s} = \prod_{k=2}^{\infty}(1-k^{-s})$ are not uniformly bounded. This answers the question of Subbarao mentioned earlier. (This result was also obtained by P. T. Bateman [1].)

In this paper we provide a detailed asymptotic analysis of $f(n)$, which allows us to answer some open problems mentioned in [7]. Our main result is the following theorem, which gives an asymptotic estimate for $f(n)$.

Theorem 1 *Let z_n be the solution to the equation $ze^z = -n-1$ with the smallest positive imaginary part. Let $\phi_n(z) = -e^z - (n+1)\log z$, and let w_n be the solution of $w_n^2 = -2/\phi_n''(z_n)$ with $\pi/2 < \arg w_n < \pi$. Then we have*

$$f(n) = \operatorname{Im}\Phi(n) + O\left(\frac{\log n}{n}|\Phi(n)|\right),$$

where

$$\Phi(n) = \frac{n!e}{\sqrt{\pi}} w_n \exp\{\phi_n(z_n)\}.$$

Using estimates for z_n and w_n (see Lemma 1 below), we obtain the following asymptotic upper bound for $\log|f(n)|$, which sharpens (3). We recall here the notations

$$L = \log n, \qquad L_2 = \log\log n \tag{4}$$

introduced earlier.

Corollary 1 *We have, for $n \geq 3$,*

$$\log|f(n)| \leq n\left(L - L_2 - 1 + \frac{L_2+1}{L} + \frac{L_2^2-\pi^2}{2L^2} + O\left(\frac{L_2^3}{L^3}\right)\right).$$

Comparing this bound with the estimate (2) for the Bell numbers $B(n)$, we obtain the following corollary, which shows the cancellation effect occuring in the sum $f(n) = \sum_{k=1}^{n}(-1)^k S_2(n,k)$, when compared to $B(n) = \sum_{k=1}^{n} S_2(n,k)$.

Corollary 2 *We have, for $n \geq 3$,*

$$\log|f(n)| \leq \log B(n) - \frac{\pi^2 n}{2L^2} + O\left(\frac{nL_2^3}{L^3}\right).$$

By investigating the behavior of the argument of $\log\Phi(n)$, we can determine how often $f(n)$ changes signs. This is the content of the following two corollaries.

Corollary 3 *Let $\Phi(n)$ be defined as in Theorem 1. Then we have*

$$f(n) = |\Phi(n)|\left(\sin\theta(n) + O\left(\frac{\log n}{n}\right)\right),$$

where $\theta(t)$ is a differentiable function defined on $[3,\infty)$ satisfying

$$\theta(t) = -\frac{\pi t}{\log t} + O\left(\frac{t\log\log t}{(\log t)^2}\right), \tag{5}$$

$$\theta'(t) = -\frac{\pi}{\log t} + O\left(\frac{\log\log t}{(\log t)^2}\right), \tag{6}$$

and

$$\theta''(t) = \frac{\pi}{t(\log t)^2} + O\left(\frac{\log\log t}{t(\log t)^3}\right). \tag{7}$$

This result shows that $f(n)$ changes signs infinitely often and that $|f(n)|$ is not eventually monotone. This answers two questions raised by Subbarao and Verma [7].

The following result gives a precise estimate for the locations of the sign changes of $f(n)$.

Corollary 4 *Let $n_1 < n_2 < \ldots$ denote the sequence of integers at which $f(n)$ changes signs, i.e., at which $f(n_k) \le 0 < f(n_k+1)$ or $f(n_k) \ge 0 > f(n_k+1)$. Then*

$$n_k = k\log k + O(k\log\log k) \tag{8}$$

and

$$n_{k+1} - n_k = \log k + O(\log\log k). \tag{9}$$

Corollary 4 implies that the density of zeros of $f(n)$ is zero. In particular, we have

$$|\{n \le x : f(n) = 0\}| \ll \frac{x}{\log x}.$$

However, by a different approach, we can improve this bound.

Theorem 2 *We have*

$$|\{n \le x : f(n) = 0\}| \ll x^{2/3}.$$

This result provides a partial answer to the question mentioned above whether $f(n) = 0$ infinitely often.

To prove Theorem 1, we adapt the approach used by de Bruijn [4] to study the behavior of $B(n)$. We then use exponential sum estimates to prove Theorem 2.

2 Proof of Theorem 1

In this section we continue to use the notations L, L_2 given in (4). We first deduce some useful estimates for the quantities z_n, w_n and $\phi_n(z_n)$ defined in the statement of Theorem 1.

Lemma 1 *Let z_n, w_n and $\phi_n(z)$ be defined as in the statement of Theorem 1. Then we have*

$$z_n = L - L_2 + \pi i + \frac{L_2}{L} - \frac{\pi i}{L} + O\left(\frac{L_2^2}{L^2}\right), \tag{10}$$

$$w_n = \sqrt{\frac{2L}{n}}\left(-\frac{\pi}{2L} + i - \frac{iL_2}{2L} - \frac{i}{2L} + O\left(\frac{L_2^2}{L^2}\right)\right), \tag{11}$$

$$\phi_n(z_n) = n\left(-L_2 + \frac{L_2+1}{L} - \frac{\pi i}{L} + \frac{L_2^2 - \pi^2}{2L^2} - \frac{\pi i L_2}{L^2} + O\left(\frac{L_2^3}{L^3}\right)\right). \tag{12}$$

Proof. By the definition of z_n, we have $e^{z_n} = -(n+1)/z_n$. This implies $|z_n| \ll L$, and by iteration we obtain

$$\begin{aligned}
z_n &= \log(n+1) - \log z_n + \pi i \\
&= L + \pi i - \log\left(L - \log z_n + \pi i\right) + O\left(\frac{1}{n}\right) \\
&= L - L_2 + \pi i + \frac{\log z_n}{L} - \frac{\pi i}{L} + O\left(\frac{L_2^2}{L^2}\right) \\
&= L - L_2 + \pi i + \frac{L_2}{L} - \frac{\pi i}{L} + O\left(\frac{L_2^2}{L^2}\right).
\end{aligned}$$

This proves estimate (10).

Similarly, since $\phi_n''(z) = -e^z + (n+1)/z^2$ and thus $\phi_n''(z_n) = (n+1)/z_n + (n+1)/z_n^2$, we have, by (10),

$$\begin{aligned}
w_n^2 &= -\frac{2}{\phi_n''(z_n)} = -\frac{2z_n}{n+1}\left(1 + \frac{1}{z_n}\right)^{-1} \\
&= -\frac{2L}{n}\left(1 - \frac{L_2}{L} + \frac{\pi i}{L} + O\left(\frac{L_2}{L^2}\right)\right)\left(1 - \frac{1}{L} + O\left(\frac{L_2}{L^2}\right)\right).
\end{aligned}$$

We then recall that, by the definition of w_n, $\pi/2 < \arg w_n < \pi$. Therefore

$$\begin{aligned}
w_n &= i\sqrt{\frac{2L}{n}}\left(1 - \frac{L_2}{2L} + \frac{\pi i}{2L} + O\left(\frac{L_2^2}{L^2}\right)\right)\left(1 - \frac{1}{2L} + O\left(\frac{L_2}{L^2}\right)\right) \\
&= \sqrt{\frac{2L}{n}}\left(-\frac{\pi}{2L} + i - \frac{iL_2}{2L} - \frac{i}{2L} + O\left(\frac{L_2^2}{L^2}\right)\right),
\end{aligned}$$

which is the claimed estimate (11). It remains to prove the estimate (12) for $\phi_n(z_n)$.

By (10) and the definitions of $\phi_n(z)$ and z_n, we have

$$\begin{aligned}
\phi_n(z_n) &= -e^{z_n} - (n+1)\log z_n = \frac{n+1}{z_n} - (n+1)\log z_n \\
&= \frac{n}{L - L_2 + \pi i + \frac{L_2}{L} - \frac{\pi i}{L} + O\left(\frac{L_2^2}{L^2}\right)} \\
&\qquad - n\log\left(L - L_2 + \pi i + \frac{L_2}{L} - \frac{\pi i}{L} + O\left(\frac{L_2^2}{L^2}\right)\right) + O(L_2) \\
&= n\left(\frac{1}{L} + \frac{L_2}{L^2} - \frac{\pi i}{L^2} + O\left(\frac{L_2^2}{L^3}\right)\right) \\
&\qquad - n\left(L_2 - \frac{L_2}{L} + \frac{\pi i}{L} + \frac{L_2}{L^2} - \frac{\pi i}{L^2} - \frac{1}{2L^2}(L_2 - \pi i)^2 + O\left(\frac{L_2^3}{L^3}\right)\right) \\
&= n\left(-L_2 + \frac{L_2+1}{L} - \frac{\pi i}{L} + \frac{L_2^2 - \pi^2}{2L^2} - \frac{\pi i L_2}{L^2} + O\left(\frac{L_2^3}{L^3}\right)\right).
\end{aligned}$$

This proves (12) and completes the proof of the lemma. □

Proof of Theorem 1. By the definition of $f(n)$, we have

$$f(n) = \sum_{\substack{0<n_1<\ldots<n_r \\ a_1,\ldots,a_r>0 \\ a_1n_1+\cdots+a_rn_r=n}} \frac{(-1)^{a_1+\cdots+a_r} n!}{a_1!\ldots a_r!(n_1!)^{a_1}\ldots(n_r!)^{a_r}}.$$

Thus the exponential generating function for $f(n)$ is given by

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{f(n)}{n!} z^n &= \sum_{\substack{n_1<\ldots<n_r \\ a_1,\ldots,a_r>0}} \frac{(-1)^{a_1+\cdots+a_r} z^{a_1n_1+\cdots+a_rn_r}}{a_1!\ldots a_r!(n_1!)^{a_1}\ldots(n_r!)^{a_r}} \\
&= \prod_{n=1}^{\infty} \left\{ \sum_{a=0}^{\infty} \frac{(-1)^a}{a!} \left(\frac{z^n}{n!}\right)^a \right\} \\
&= \exp(-z)\exp\left(-\frac{z^2}{2!}\right)\exp\left(-\frac{z^3}{3!}\right)\ldots \\
&= \exp\{-(e^z - 1)\}.
\end{aligned}$$

(For an alternative derivation of this identity see [7].) Using this generating function and Cauchy's formula, we obtain

$$\frac{f(n)}{n!e} = \frac{1}{2\pi i}\int_{\mathcal{C}} \exp(-e^z) z^{-n-1}\, dz,$$

where $\mathcal{C}$ is a simple closed curve encircling the origin. Since $\exp(-e^z)$ is uniformly bounded in any half-plane $\{z : \operatorname{Re} z \le \sigma\}$, the integration path $\mathcal{C}$ can be replaced by $\Gamma_1 \cup \Gamma_2$, where

$\Gamma_1 = \{z_n + w_n t : -\mathrm{Im}\, z_n/\mathrm{Im}\, w_n \le t < \infty\}$ and $\Gamma_2 = \{\bar{z}_n - \bar{w}_n t : -\infty < t < \mathrm{Im}\, z_n/\mathrm{Im}\, w_n\}$, i.e., Γ_1 is the straight line lying in the upper half-plane that passes through z_n in direction w_n, and the path Γ_2 is the reflection of Γ_1 with respect to the real axis, with direction $-\bar{w}_n$.

We now estimate the integral along Γ_1. Setting $z = z_n + w_n t$, we obtain

$$
\begin{aligned}
&\frac{1}{2\pi i}\int_{\Gamma_1} \exp(-e^z) z^{-n-1}\, dz \\
&= \frac{w_n}{2\pi i}\int_{-\mathrm{Im}\, z_n/\mathrm{Im}\, w_n}^{\infty} \exp\left\{\phi_n(z_n + w_n t)\right\}\, dt \\
&= \frac{w_n \exp\left\{\phi_n(z_n)\right\}}{2\pi i}\left\{\int_{-\mathrm{Im}\, z_n/\mathrm{Im}\, w_n}^{-1/|w_n|^{1/3}} + \int_{-1/|w_n|^{1/3}}^{1/|w_n|^{1/3}} + \int_{1/|w_n|^{1/3}}^{|z_n/w_n|} + \int_{|z_n/w_n|}^{\infty}\right\} \\
&\qquad \exp\left\{\phi_n(z_n + w_n t) - \phi_n(z_n)\right\}\, dt \\
&= \frac{w_n \exp\left\{\phi_n(z_n)\right\}}{2\pi i}\left\{I_1 + I_2 + I_3 + I_4\right\}.
\end{aligned}
$$

By estimates (10) and (11) of Lemma 1, we have, for $t \ge |z_n/w_n|$,

$$
\mathrm{Re}\, w_n t \le \left(-\frac{\pi}{2L} + O\left(\frac{L_2}{L^2}\right)\right)(L + O(L_2)) = -\frac{\pi}{2} + O\left(\frac{L_2}{L}\right),
$$

and thus

$$
\begin{aligned}
\mathrm{Re}\left(e^{z_n} - e^{z_n + w_n t}\right) &\le -\mathrm{Re}\left(\frac{n+1}{z_n}\right) + e^{\mathrm{Re}\,(z_n + w_n t)} \\
&\le -(1 - e^{-\pi/2})\frac{n}{L}\left(1 + O\left(\frac{L_2}{L}\right)\right).
\end{aligned} \tag{13}
$$

Furthermore, since, by the same lemma,

$$
\arg w_n - \arg z_n = \frac{\pi}{2} + O\left(\frac{L_2}{L}\right), \tag{14}
$$

we have $|z_n + w_n t| \ge |w_n t|$ for sufficiently large n and $t \ge |z_n/w_n|$. Using (13), it follows that

$$
\begin{aligned}
I_4 &\le \int_{|z_n/w_n|}^{\infty} \exp\left\{\mathrm{Re}\left(e^{z_n} - e^{z_n + w_n t} - (n+1)\log\left|\frac{z_n + w_n t}{z_n}\right|\right)\right\}\, dt \\
&\le \int_{|z_n/w_n|}^{\infty} \exp\left\{-\frac{c_1 n}{L} - n\log\left(\frac{|w_n|}{|z_n|}t\right)\right\}\, dt \\
&= \frac{|z_n|}{(n-1)|w_n|}\exp\left\{-\frac{c_1 n}{L}\right\} \ll \sqrt{\frac{n}{L^3}}\exp\left\{-\frac{c_1 n}{L}\right\}
\end{aligned} \tag{15}
$$

for sufficiently large n, where c_1 is a suitable positive constant.

We next estimate I_3. We first show that $\operatorname{Re}(e^{z_n} - e^{z_n+w_nt}) \ll t\sqrt{n/L^3}$ uniformly for all $t > 0$ and sufficiently large n. By the definition of z_n and (10), we have

$$\operatorname{Re} e^{z_n} = -\operatorname{Re}\frac{n+1}{z_n} = -\frac{n}{L}\left(1 + O\left(\frac{L_2}{L}\right)\right)$$

and

$$\operatorname{Im} e^{z_n} = -\operatorname{Im}\frac{n+1}{z_n} = -\frac{\pi n}{L^2}\left(1 + O\left(\frac{L^2}{L}\right)\right).$$

Using the inequality $0 < \sqrt{x^2+y^2} - x \leq y^2/(2x)$, which holds uniformly for all x and y with $0 < y \leq x$, we obtain

$$\left|\operatorname{Re} e^{z_n} + |e^{z_n}|\right| \leq \frac{1}{2}\left|\frac{(\operatorname{Im} e^{z_n})^2}{\operatorname{Re} e^{z_n}}\right| \leq \frac{c_2 n}{L^3},$$

where c_2 is a positive constant. Therefore if t is a real number satisfying $\operatorname{Re} w_n t < -2c_2/L^2$, i.e., $t > (4c_2/\pi + o(1))/(|w_n|L)$, then we have by (11)

$$\begin{aligned}\operatorname{Re}(e^{z_n} - e^{z_n+w_nt}) &\leq (|e^{z_n}| + \operatorname{Re} e^{z_n}) + \left(|e^{z_n}|e^{\operatorname{Re} w_n t} - |e^{z_n}|\right)\\ &\leq \frac{c_2 n}{L^3} - \frac{2c_2}{L^2}|e^{z_n}| \leq 0,\end{aligned}$$

On the other hand, if t is in the range $0 < t \leq (4c_2/\pi + o(1))/(|w_n|L)$, then, by (10) and (11),

$$\begin{aligned}\operatorname{Re}(e^{z_n} - e^{z_n+w_nt}) &= \operatorname{Re}(-e^{z_n}w_nt) + O(|e^{z_n}w_n^2|t^2)\\ &= \operatorname{Re}\left\{\frac{n}{L}\left(1 + \frac{L_2}{L} - \frac{\pi i}{L} + O\left(\frac{L_2}{L^2}\right)\right)\right.\\ &\qquad \left.\sqrt{\frac{2L}{n}}\left(-\frac{\pi}{2L} + i + \frac{iL^2 - i}{L} + O\left(\frac{L_2^2}{L^2}\right)\right)t\right\} + O(t^2)\\ &= \left(\frac{1}{\sqrt{2}} + o(1)\right)\sqrt{\frac{n}{L^3}}t + O(t^2) \leq c_3\sqrt{\frac{n}{L^3}}t\end{aligned}$$

for sufficiently large n, where c_3 is a positive constant. This proves the assertion that $\operatorname{Re} e^{z_n} - \operatorname{Re} e^{z_n+w_nt} \ll t\sqrt{n/L^3}$ uniformly for all $t > 0$ and sufficiently large n.

We now estimate I_3. For t in the interval $[1/|w_n|^{1/3}, |z_n|/|w_n|]$, the estimate (14) implies that

$$\log\left|1 + \frac{w_nt}{z_n}\right| \geq \frac{|w_n|}{4|z_n|}t$$

for sufficiently large n. It follows that, by Lemma 1,

$$\begin{aligned}I_3 &\leq \int_{1/|w_n|^{1/3}}^{|z_n/w_n|} \exp\left\{c_3\sqrt{\frac{n}{L^3}}t - \frac{n}{4}\frac{|w_n|}{|z_n|}t\right\}\,dt\\ &\leq \frac{|z_n|}{|w_n|}\exp\left\{-\left(\frac{1}{4} + o(1)\right)\sqrt{\frac{n}{L}}|w_n|^{-1/3}\right\} \ll \exp\left\{-\frac{c_4 n^{2/3}}{L^{2/3}}\right\}\end{aligned} \tag{16}$$

for some suitable positive constant c_4. The same bound holds for I_1. It remains to estimate I_2.

In the range $-1/|w_n|^{1/3} \le t \le 1/|w_n|^{1/3}$, we have, by Lemma 1,

$$\phi_n^{(3)}(z_n) = -e^{z_n} - \frac{2(n+1)}{z_n^3} = \frac{n+1}{z_n} + O\left(\frac{n}{L^3}\right) \ll \frac{n}{L},$$

$$\phi_n^{(4)}(z_n + w_n t) = -e^{z_n + w_n t} + \frac{6(n+1)}{(z_n + w_n)^3} \ll \frac{n}{L} e^{|w_n|^{2/3}} + \frac{n}{L^3} \ll \frac{n}{L},$$

and thus

$$\phi_n^{(3)}(z_n)(w_n t)^3 \ll \frac{n}{L}|w_n|^2 \ll 1,$$

$$\phi_n(z_n + w_n t) - \phi_n(z_n) - \phi_n'(z_n) w_n t - \frac{\phi_n''(z_n)}{2}(w_n t)^2 - \frac{\phi_n^{(3)}(z_n)}{6}(w_n t)^3 \ll \frac{n}{L}|w_n t|^4 \ll 1.$$

Since, by the definition of z_n and w_n, $\phi_n'(z_n) = 0$ and $\phi_n''(z_n) w_n^2/2 = -1$, it follows that

$$\begin{aligned}
I_2 &= \int_{-1/|w_n|^{1/3}}^{1/|w_n|^{1/3}} \exp\left\{ -t^2 + \frac{\phi_n^{(3)}(z_n)}{6}(w_n t)^3 + O\left(\frac{n}{L}|w_n t|^4\right) \right\} dt \\
&= \int_{-1/|w_n|^{1/3}}^{1/|w_n|^{1/3}} e^{-t^2} \left(1 + \frac{\phi_n^{(3)}(z_n) w_n^3}{6} t^3 + O\left(|\phi_n^{(3)}(z_n)^2 w_n^6 | t^6\right) + O\left(\frac{n|w_n|^4}{L} t^4\right) \right) dt \\
&= \sqrt{\pi} + O\left(\exp\left\{-|w_n|^{-2/3}\right\}\right) + O\left(|\phi_n^{(3)}(z_n)^2 w_n^6|\right) + O\left(\frac{n|w_n|^4}{L}\right) \\
&= \sqrt{\pi} + O\left(\frac{L}{n}\right).
\end{aligned}$$

Combining this estimate, (15) and (16), we obtain

$$\int_{\Gamma_1} \exp\{\phi_n(z)\}\, dz = w_n \exp\{\phi_n(z_n)\} \left(\sqrt{\pi} + O\left(\frac{L}{n}\right)\right).$$

Since $\int_{\Gamma_2} = -\overline{\int_{\Gamma_1}}$, it follows that

$$\begin{aligned}
f(n) = \frac{n!e}{2\pi i}\left(\int_{\Gamma_1} + \int_{\Gamma_2}\right) &= \operatorname{Im} \frac{n!e}{\sqrt{\pi}} w_n \exp\{\phi_n(z_n)\} + O\left(\frac{L}{n} n! |w_n \exp\{\phi_n(z_n)\}|\right) \\
&= \operatorname{Im} \Phi(n) + O\left(\frac{L}{n}|\Phi(n)|\right).
\end{aligned}$$

This completes the proof of Theorem 1. $\square$

3 Proofs of Corollaries

Throughout this section, L will denote $\log n$ or $\log t$, and L_2 will denote $\log\log n$ or $\log\log t$, depending on the context.

Proof of Corollary 1. By Theorem 1, we have

$$|f(n)| \leq \frac{n!e}{\sqrt{\pi}} |w_n \exp\{\phi_n(z_n)\}| \left(1 + O\left(\frac{L}{n}\right)\right).$$

By Lemma 1 and the Stirling formula for $n!$, it follows that

$$\begin{aligned} \log|f(n)| &\leq (n+1/2)\log n - n + \operatorname{Re}\phi_n(z_n) + O(1) \\ &= n\left(L - L_2 - 1 + \frac{L_2+1}{L} + \frac{L^2-\pi^2}{2L^2} + O\left(\frac{L_2^3}{L^3}\right)\right). \end{aligned}$$

This proves Corollary 1. Corollary 2 is an immediate consequence of Corollary 1. □

Proof of Corollary 3. We first note that the domains of the functions z_n, w_n, $\phi_n(z)$ and $\Phi(n)$ can be extended from the set of positive integers to the set of positive real numbers, and the asymptotic formulas in Lemma 1 remain valid with n replaced by a positive real number t. From Theorem 1 we deduce that

$$f(n) = |\Phi(n)| \left(\sin\theta(n) + O\left(\frac{L}{n}\right)\right),$$

where

$$\theta(t) = \operatorname{Im}\left(\phi_t(z_t) + \log w_t\right). \tag{17}$$

By Lemma 1, we have

$$\operatorname{Im}\log w_t = \frac{\pi}{2} + O\left(\frac{1}{L}\right)$$

and

$$\operatorname{Im}\phi_t(z_t) = -\frac{\pi t}{L} + O\left(\frac{tL_2}{L^2}\right).$$

The claimed estimate (5) for $\theta(t)$ follows by inserting these estimates into (17).

We now prove estimate (6). By the definition of z_t, we have $z_t e^{z_t} + (t+1) = 0$. Thus, the chain rule yields

$$\frac{dz_t}{dt} = -\frac{1}{e^{z_t}(z_t+1)} = \frac{z_t}{(t+1)(z_t+1)}. \tag{18}$$

Since

$$w_t^2 = -\frac{2}{\phi_t''(z_t)} = -\frac{2}{-e^{z_t} + (t+1)/z_t^2} = -\frac{2}{(t+1)/z_t + (t+1)/z_t^2},$$

by estimate (10) of Lemma 1 and (18), we have

$$\frac{1}{w_t}\frac{dw_t}{dt} = -\frac{1}{2}\frac{\frac{d}{dt}\left(\frac{t+1}{z_t}+\frac{t+1}{z_t^2}\right)}{\left(\frac{t+1}{z_t}+\frac{t+1}{z_t^2}\right)} = -\frac{1}{2(t+1)} + \frac{z_t^{-2}+2z_t^{-3}}{2(z_t^{-1}+z_t^{-2})}\frac{dz_t}{dt} \ll \frac{1}{t}. \tag{19}$$

Similarly, we have

$$\begin{aligned}\frac{d\phi_t(z_t)}{dt} &= \frac{d}{dt}\left(-e^{z_t}-(t+1)\log z_t\right)\\ &= -\log z_t - \left(e^{z_t}+\frac{t+1}{z_t}\right)\frac{dz_t}{dt} = -\log z_t\end{aligned} \tag{20}$$

and thus, by (10),

$$\begin{aligned}\operatorname{Im}\frac{d\phi_t(z_t)}{dt} &= -\operatorname{Im}\left\{\log\left(L-L_2+\pi i+\frac{L_2}{L}-\frac{\pi i}{L}+O\left(\frac{L_2^2}{L^2}\right)\right)\right\}\\ &= -\frac{\pi}{L}+O\left(\frac{L_2}{L^2}\right).\end{aligned} \tag{21}$$

Combining this estimate and (19), we obtain

$$\theta'(t) = \operatorname{Im}\frac{1}{w_t}\frac{dw_t}{dt} + \operatorname{Im}\frac{d}{dt}\phi_t(z_t) = -\frac{\pi}{L}+O\left(\frac{L_2}{L^2}\right).$$

This proves the estimate (6).

The proof of (7) is essentially the same as that of (6). By (18) and (20), we have

$$\frac{d^2}{dt^2}\phi_t(z_t) = -\frac{d}{dt}\log z_t = -\frac{1}{z_t}\frac{z_t}{(t+1)(z_t+1)} = -\frac{1}{t\left(L-L_2+\pi i+1+O(L_2/L)\right)}$$

and hence

$$\operatorname{Im}\frac{d^2}{dt^2}\phi_t(z_t) = \frac{\pi}{tL^2}+O\left(\frac{L_2}{tL^3}\right).$$

Similarly, we have, by (18) and (19),

$$\frac{d^2}{dt^2}\log w_t = \frac{1}{(t+1)^2} + \frac{1}{2}\frac{d}{dt}\left\{-\frac{z_t^{-2}+2z_t^{-3}}{z_t^{-1}+z_t^{-2}}\frac{z_t}{(t+1)(z_t+1)}\right\} \ll \frac{1}{t^2}.$$

Thus we conclude that

$$\theta''(t) = \operatorname{Im}\frac{d^2}{dt^2}\phi_t(z_t) + \operatorname{Im}\frac{d^2}{dt^2}\log w_t = \frac{\pi}{tL^2}+O\left(\frac{L_2}{tL^3}\right).$$

This completes the proof of Corollary 3. $\square$

Proof of Corollary 4. Let $\theta(t)$ be defined as in Corollary 3. Let k be a positive integer, and let t_k be the solution of $\theta(t) = -k\pi$. By Corollary 3, t_k satisfies

$$k = \frac{t_k}{\log t_k} + O\left(\frac{t_k \log\log t_k}{(\log t_k)^2}\right).$$

Hence, we obtain

$$t_k = k\log t_k + O\left(\frac{t_k \log\log t_k}{\log t_k}\right) = k\log k + O(k\log\log k). \tag{22}$$

From this estimate we deduce that $n_k = t_k + O(1)$, and therefore estimate (8) holds.

To prove the second part of the corollary, we note that, by the mean value theorem, $\pi = \theta(t_k) - \theta(t_{k+1}) = (t_k - t_{k+1})\theta'(\xi_k)$, where ξ_k is a real number between t_k and t_{k+1}. The estimate (22) implies that

$$\xi_k = t_k + O(t_{k+1} - t_k) = t_k + O(k\log\log k).$$

Hence, by (6) in Corollary 3, we have

$$\begin{aligned} n_{k+1} - n_k = t_{k+1} - t_k + O(1) &= -\frac{\pi}{\theta'(\xi_k)} \\ &= \frac{\pi}{\dfrac{\pi}{\log t_k} + O\left(\dfrac{\log\log t_k}{(\log t_k)^2}\right)} = \log k + O(\log\log k), \end{aligned}$$

which is the claimed result. □

4 Proof of Theorem 2

We will use the following well-known exponential sum estimate (see, e.g., [5, p. 17]).

Lemma 2 *Let a and b be integers with $a < b$, and let g be twice differentiable on $[a, b]$ with $g''(x) \geq \rho > 0$ or $g''(x) \leq -\rho < 0$ for some positive real number ρ and all $x \in [a, b]$. Then*

$$\left|\sum_{n=a}^{b} e^{ig(n)}\right| \ll (|g'(b) - g'(a)| + 1)(\rho^{-1/2} + 1).$$

Proof of Theorem 2. It suffices to show that

$$\left|\left\{x^{1/2} < n \leq x : f(n) = 0\right\}\right| \ll x^{2/3}.$$

In light of Corollary 3, we have for sufficiently large x

$$\left|\left\{x^{1/2} < n \leq x : f(n) = 0\right\}\right| \leq \left|\left\{x^{1/2} < n \leq x : \|\theta(n)\| < \frac{c_1 \log n}{n}\right\}\right|$$

for some positive constant c_1, where $\theta(t)$ is the function occuring in the statement of Corollary 3 and $\|\theta(n)\|$ denotes the distance from $\theta(n)$ to the closest integer multiple of π. On the other hand, if $H = H(x)$ is an integer-valued function satisfying $H(x) \leq (\pi/(2c_1))x^{1/2}/\log x$, then, for $x^{1/2} < n \leq x$, the condition $\|\theta(n)\| < c_1 \log n/n$ implies that

$$\left(\frac{\sin((H+1)\theta(n))}{\sin\theta(n)}\right)^2 \geq \frac{(H+1)^2}{\pi^2}.$$

We therefore have

$$\left|\left\{x^{1/2} \leq n \leq x : f(n) = 0\right\}\right| \ll \frac{1}{H^2} \sum_{x^{1/2}<n\leq x} \left(\frac{\sin((H+1)\theta(n))}{\sin\theta(n)}\right)^2. \tag{23}$$

Using the identity

$$\left(\frac{\sin((H+1)t)}{\sin t}\right)^2 = \sum_{h=-H}^{H} (H+1-|h|)e^{2iht},$$

the right-hand side of (23) becomes

$$\frac{1}{H^2}\sum_{h=-H}^{H}(H+1-|h|)S_h \leq \frac{x(H+1)}{H^2} + \frac{2}{H^2}\sum_{h=1}^{H}(H+1-h)|S_h|, \tag{24}$$

where $S_h = \sum_{x^{1/2}<n\leq x} e^{2ih\theta(n)}$.

We now estimate the exponential sum S_h. By Corollary 3, we have

$$|h\theta'(\lfloor x\rfloor) - h\theta'(\lceil x^{1/2}\rceil)| \ll \frac{h}{\log x}$$

and

$$h\theta''(t) \gg \frac{h}{x(\log x)^2}$$

for $x^{1/2} < t \leq x$. Thus, applying Lemma 2 with $g(t) = h\theta(t)$, we obtain

$$|S_h| \ll \left(\frac{h}{\log x} + 1\right)\left((\log x)\sqrt{\frac{x}{h}} + 1\right).$$

Inserting this estimate in (24), we arrive at

$$\begin{aligned}\left|\left\{x^{1/2} \leq n \leq x : f(n) = 0\right\}\right| &\ll \frac{x}{H} + \frac{1}{H}\sum_{h=1}^{H}\left(\sqrt{hx} + (\log x)\sqrt{\frac{x}{h}} + \frac{h}{\log x} + 1\right) \\ &\ll \frac{x}{H} + \sqrt{Hx} + (\log x)\sqrt{\frac{x}{H}} + \frac{H}{\log x} + 1.\end{aligned}$$

Taking $H = \lfloor x^{1/3}\rfloor$, we finally obtain

$$\left|\left\{x^{1/2} \leq n \leq x : f(n) = 0\right\}\right| \ll x^{2/3}.$$

This completes the proof. $\square$

5 Acknowledgment

The author would like to thank Professor Hildebrand of University of Illinois for helpful advice on the exposition of the paper. The author would also like to thank the referee for thorough reading of the manuscript and for correcting a crucial mistake in the proof of Theorem 2.

References

[1] P. T. Bateman, Private communication.

[2] D. Bowman, Private communication.

[3] E. R. Canfield, P. Erdős and C. Pomerance, *On a problem of Oppenheim concerning "Factorisatio Numerorum"*, J. Number Theory **17** (1983), 1–28.

[4] N. G. de Bruijn, *Asymptotic methods in analysis*, Corrected reprint of the third edition, Dover Publications, Inc., New York, 1981.

[5] L. Kuipers and H. Niederreiter, *Uniform distribution of sequences*, Pure and Applied Mathematics, Wiley-Interscience [John Wiley & Sons], New York-London-Sydney, 1974.

[6] A. Oppenheim, *On an arithmetic function*, J. London Math. Soc. **1** (1926), 205-211.

[7] M. V. Subbarao and A. Verma, *Some remarks on a product expansion: an unexplored partition function*, preprint.

[8] H. S. Wilf, *Generatingfunctionology*, Second Edition, Academic Press, Inc., Boston, 1994.