Circular chromatic number of planar graphs of large odd girth

Xuding Zhu*
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan 80424
zhu@math.nsysu.edu.tw

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Abstract

It was conjectured by Jaeger that $4k$-edge connected graphs admit a $(2k+1,k)$-flow. The restriction of this conjecture to planar graphs is equivalent to the statement that planar graphs of girth at least $4k$ have circular chromatic number at most $2+\frac{1}{k}$. Even this restricted version of Jaeger's conjecture is largely open. The $k=1$ case is the well-known Grötzsch 3-colour theorem. This paper proves that for $k \geq 2$, planar graphs of odd girth at least $8k-3$ have circular chromatic number at most $2+\frac{1}{k}$.

1 Introduction

Let G be a graph and D an orientation of G. For positive integers $k \geq 2d$, a (k,d)-flow f of D is a mapping that assigns to each edge e of D an integer $f(e)$ such that (i): for every vertex x, $\Sigma_{e\in E^+(x)}f(e) - \Sigma_{e\in E^-(x)}f(e) = 0$, and (ii) for every edge e, $d \leq |f(e)| \leq k-d$. Here $E^+(x)$ is the set of edges incident to and oriented away from x, and $E^-(x)$ is the set of edges incident to and oriented towards x. A graph G is said to admit a (k,d)-flow if G has an orientation D that admits a (k,d)-flow. The following conjecture was proposed by Jaeger [7, 8]:

Conjecture 1.1 *For any integer $k \geq 1$, every $4k$-edge-connected graph admits an $(2k+1,k)$-flow.*

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Jaeger's conjecture is very strong. The $k = 1$ case is Tutte's 3-flow conjecture [16], and the $k = 2$ case implies Tutte's 5-flow conjecture [17]. Both the 3-flow conjecture and the 5-flow conjecture are long standing open problems [20]. Many difficult conjectures for flows have been proved for planar graphs. However, even restricted to planar graphs, Jaeger's conjecture remains largely open.

For planar graphs, the flow problem can be dualized to a colouring problem. For positive integers $k \geq 2d$, a (k, d)-colouring of a graph G is a mapping $f : V(G) \to \{0, 1, \cdots, k-1\}$ such that $d \leq |f(x) - f(y)| \leq k - d$ for every edge xy of G. The *circular chromatic number* $\chi_c(G)$ of G is defined as

$$\chi_c(G) = \min\{k/d : \text{ there exists a } (k, d)\text{-colouring of } G\}.$$

The circular chromatic number (also known as the "star chromatic number" [18]) has been studied extensively in the past decade. Readers are referred to [22] for a survey on this subject. A (k, d)-colouring of a planar graph G corresponds to a (k, d)-flow of the dual graph of G [4], [22]. Therefore, the restriction of Jaeger's conjecture to planar graphs is equivalent to the following:

Conjecture 1.2 [Jaeger's conjecture restricted to planar graphs] *Every planar graph G of girth at least $4k$ has circular chromatic number at most $2 + \frac{1}{k}$.*

It was proved by Galluccio, Goddyn and Hell [3] that for each $\epsilon > 0$, for every surface S, there exists an integer g such that every graph of girth at least g embedded in S has circular chromatic number at most $2 + \epsilon$. In particular, we seek the smallest integer $g(k)$ such that every planar graph G of girth at least $g(k)$ has $\chi_c(G) \leq 2 + \frac{1}{k}$. It follows from Grötzsch's Theorem (triangle free planar graphs are 3-colourable [5]) that $g(1) = 4$. For $k \geq 2$, the best known bounds at present are $4k \leq g(k) \leq 10k - 4$. The upper bound was proved by Galluccio, Goddyn and Hell [3], and the lower bound was proved by DeVos [2]. Conjecture 1.2 asserts that the lower bound is tight.

The *odd edge-connectivity* of a graph G is the size of a smallest odd edge cut of G. The *odd girth* of G is the length of a shortest odd cycle of G. Zhang [19] proposed a strengthing of Jaeger's conjecture, where the edge-connectivity condition is replaced by an odd edge-connectivity condition. Conjecture 1.3 below is the restriction of that conjecture to planar graphs (in the dual version).

Conjecture 1.3 *[19] Every planar graph G of odd girth at least $4k + 1$ has circular chromatic number at most $2 + \frac{1}{k}$.*

It was proved by Klostermeyer and Zhang that for any $\epsilon > 0$, there is an integer f such that every planar graph of odd girth at least f has circular chromatic number at most $2 + \epsilon$. We seek the smallest odd integer $f(k)$ such that every planar graph of odd girth at least $f(k)$ has circular chromatic number at most $2 + \frac{1}{k}$. The best known bounds at present are $4k + 1 \leq f(k) \leq 10k - 3$. The upper bound was proved by Klostermeyer and Zhang, and lower bound follows from the lower bound for $g(k)$.

It follows from the definitions that $f(k) \geq g(k)+1$. So Conjecture 1.3 is stronger than Conjecture 1.2. In this paper, we improve the upper bound for $f(k)$, which also yields a better upper bound for $g(k)$.

Theorem 1.4 *For any integer $k \geq 2$, every planar graph G of odd girth at least $8k-3$ has circular chromatic number at most $2+\frac{1}{k}$.*

Note that Theorem 1.4 remains true for $k=1$. The $k=1$ case is exactly the Grötzsch 3-colour theorem [5]. However, the proof presented here does not work for this case.

The proof uses the discharging method. In Section 2, we shall give a family of unavoidable configurations in a counterexample of Theorem 1.4. In Section 3, we prove that any graph containing one of the unavoidable configurations cannot be a minimum counterexample.

2 Unavoidable configurations

Let G be a graph. A *thread* in G is a maximal subgraph of G which is a path whose internal vertices all have degree 2 in G. For $t \geq 1$, vertices x and y are *loosely t-adjacent* (or *loosely adjacent* if t is irrelevant) if G contains an x,y-path P of length t lies in a thread, i.e., P is a path whose internal vertices all have degree 2 in G. Let $d_l(x)$ be the number of vertices that are loosely adjacent to x, and let $d(x)$ the degree of x.

Lemma 2.1 *Let G be a planar graph of minimum degree at least 2. If each facial cycle of G has length at least $8k-3$, then one of the following holds:*

(a) *G has a vertex x such that $d_l(x) \geq 2k(d(x)-1)$.*

(b) *G has two loosely t-adjacent vertices x and y such that $d(x) = d(y) = 3$ and $d_l(x)+d_l(y) \geq 6k+t$.*

(c) *G has a vertex x of degree 3 which is loosely adjacent to a, b, c such that $d(a) = d(b) = d(c) = 3$ and $d_l(a) = d_l(b) = d_l(c) = 4k-1$.*

Proof. The proof uses the discharging method. Assume that G is a counterexample to Lemma 2.1. We first assign a charge $c(x)$ to each vertex x and prove that the total charge $\sum_{x \in V(G)} c(x)$ is negative. Then by two rounds of discharging, each vertex x gets a new charge $c^*(x)$. We prove that the total charge is unchanged and yet $c^*(x) \geq 0$ for each vertex x. This is an obvious contradiction, which shows that the counterexample G does not exist.

Since G is a counterexample, condition (a) does not hold. This implies that G has no thread of length at least $2k$. For otherwise, for any internal vertex x of a thread of length at least $2k$, we have $d_l(x) \geq 2k = 2k(d(x)-1)$. Thus we may assume that each cycle of G contains at least 3 vertices of degree at least 3 (otherwise G contains a cycle which contains at most two vertices of degree at least 3. By appropriately embed the graph, we may assume this cycle is a facial cycle, which then must have length at least $8k-3$. Hence G has a thread of length at least $2k$).

Let v, e, f be the numbers of vertices, edges and faces of G respectively. Since each facial cycle of G has length at least $8k-3$, and the sum of the lengths of all the facial cycles is equal to $2e$, it follows that $(8k-3)f \leq 2e$. Plugging this into Euler's formula $v+f-e=2$, we have

$$(8k-5)e \leq (8k-3)(v-2) < (8k-3)v.$$

Assign to each vertex x of G a charge

$$c(x) = (8k-5)d(x) - (16k-6).$$

The total charge assigned to the vertices of G is

$$\begin{aligned} \Sigma_{x \in V(G)} c(x) &= \Sigma_{x \in V(G)}((8k-5)d(x) - (16k-6)) \\ &= 2e(8k-5) - (16k-6)v \\ &< 2(8k-3)v - (16k-6)v \\ &= 0. \end{aligned}$$

Discharging rule for the first round: *Transfer a charge of amount* 2 *from each vertex* x *of degree at least* 3 *to each vertex* y *of degree* 2 *that is loosely adjacent to* x.

If y has degree 2, then y is loosely adjacent to two vertices of degree at least 3. Thus the total amount of charge received by y is 4. So the new charge $c'(y)$ at y is

$$c'(y) = 2(8k-5) - (16k-6) + 4 = 0.$$

By definition, each vertex x is loosely adjacent to $d(x)$ vertices of degree at least 3. So x is loosely adjacent to $d_l(x) - d(x)$ vertices of degree 2. As condition (a) does not hold, $d_l(x) \leq 2k(d(x)-1) - 1$. If $d(x) \geq 3$, then the total amount of charge sent out from x is

$$2(d_l(x) - d(x)) \leq (4k-2)d(x) - (4k+2).$$

If $d(x) \geq 4$, then the new charge $c'(x)$ at x is

$$\begin{aligned} c'(x) &\geq d(x)(8k-5) - (16k-6) - ((4k-2)d(x) - (4k+2)) \\ &= d(x)(4k-3) - (12k-8) \\ &= d(x) + d(x)(4k-4) - (12k-8) \\ &\geq d(x). \end{aligned}$$

If x has degree 3, then as condition (a) does not hold, $d_l(x) \leq 4k-1$. The same calculation shows the following:

- If $d_l(x) = 4k-1$, then $c'(x) = -1$.
- If $d_l(x) = 4k-2$, then $c'(x) = 1$.
- If $d_l(x) \leq 4k-3$, then $c'(x) \geq 3$.

We call a vertex x *critical* if $d(x) = 3$ and $d_l(x) = 4k-1$. The critical vertices are the only vertices having negative charge at this moment. We apply a second round of discharging as follows:

Discharging rule for the second round: *Suppose x has degree at least 4 or x has degree 3 and $d_l(x) \leq 4k-3$. If x is loosely adjacent to a critical vertex y, then transfer a charge of amount 1 from x to y. Suppose x has degree 3 and $d_l(x) = 4k-2$. If x is loosely adjacent to a critical vertex y, then transfer a charge of amount $\frac{1}{2}$ from x to y.*

The new charge at vertex x after the second round of discharging is denoted by $c^*(x)$. We shall prove $c^*(x) \geq 0$ for all x.

If x has degree 2, then $c^*(x) = c'(x) = 0$.

If $d(x) \geq 4$, then $c'(x) \geq d(x)$. There are at most $d(x)$ critical vertices loosely adjacent to x, so the amount of charge sent out from x (at the second round) is at most $d(x)$. Therefore $c^*(x) \geq 0$.

If x has degree 3 and $d_l(x) \leq 4k-3$, then $c'(x) \geq 3$. The amount of charge sent out from x (at the second round) is at most 2. Therefore $c^*(x) \geq 0$.

Suppose x has degree 3 and $d_l(x) = 4k-2$. Then $c'(x) = 1$. As condition (c) does not hold, there are at most two critical vertices that are loosely adjacent to x. Hence the amount of charge sent out from x (at the second round) is at most 1. Therefore $c^*(x) \geq 0$.

It remains to consider critical vertices. If a critical vertex w is loosely adjacent to a vertex of degree at least 4 or loosely adjacent to a vertex a with $d(a) = 3$ and $d_l(a) \leq 4k-3$, then w receives at least an amount 1 of charge in the second round of discharging. Hence $c^*(w) \geq c'(w) + 1 = 0$.

Assume that w is loosely adjacent to three vertices of degree 3, say a, b, c, such that $d_l(a), d_l(b), d_l(c) \geq 4k-2$. Suppose w is loosely t_1-adjacent to a, loosely t_2-adjacent to b and loosely t_3-adjacent to c. Then $d_l(w) = t_1 + t_2 + t_3 = 4k-1$. Since G is a counterexample, condition (b) does not hold, i.e., G does not have two loosely t-adjacent vertices x and y such that $d(x) = d(y) = 3$ and $d_l(x) + d_l(y) \geq 6k + t$. Therefore

$$\begin{aligned} d_l(a) + d_l(w) &\leq 6k + t_1 - 1, \\ d_l(b) + d_l(w) &\leq 6k + t_2 - 1, \\ d_l(c) + d_l(w) &\leq 6k + t_3 - 1. \end{aligned}$$

As $d_l(w) = 4k-1$ and $d_l(a), d_l(b), d_l(c) \geq 4k-2$, we conclude that $t_i \geq (4k-2) + (4k-1) - (6k-1) = 2k-2 \geq 2$ for $i = 1, 2, 3$. Hence $4k-1 = t_1 + t_2 + t_3 \geq 6k-6$, which implies that $k = 2$. As $t_i < 2k = 4$, we have $2 \leq t_i \leq 3$. Since $t_1 + t_2 + t_3 = 7$, without loss of generality, we may assume that $t_1 = 3$, $t_2 = t_3 = 2$. Now $d_l(b) + d_l(w) \leq 13$ implies that $d_l(b) \leq 6$, and hence $d_l(b) = 6$. Similarly, $d_l(c) = 6$. By our discharging rule, each of b and c transfer an amount of $\frac{1}{2}$ of charge to w. Hence $c^*(w) = c'(w) + 1 = 0$.

So each vertex x of G has a nonnegative new charge $c^*(x)$, contrary to the fact that the total charge is negative. ■

3 Reducibility of the unavoidable configurations

This section proves Theorem 1.4, by showing that any minimal counterexample does not contain any of those unavoidable configurations listed in Lemma 2.1. In the remainder of this section, $k \geq 2$ is a fixed integer, and we consider only $(2k+1, k)$-colouring of graphs. Thus the colour set is $C = \{0, 1, 2, \cdots, 2k\}$.

First we consider extending partial colourings of paths. Let P be an x, y-path of length n. Let S be a set of colours. Let $\phi(n, S) = \{j \in C :$ there exists a $(2k+1, k)$-colouring f of P such that $f(x) \in S$ and $f(y) = j\}$.

Lemma 3.1 *For any nonempty subset S of C, $|\phi(n, S)| \geq \min\{2k+1, |S| + n\}$.*

Proof. It suffices to prove the claim for $n = 1$, since applying that for each successive edge yields the full statement. Let $S' = \phi(1, S)$. We define an auxiliary bipartite graph H. Introduce a vertex a_i for each $i \in S$ and a vertex b_j for each $j \in S'$. Let $a_i b_j$ be an edge if and only if there is a $(2k+1, k)$-colouring f of the edge xy such that $f(x) = i$ and $f(y) = j$. In H, each a_i has degree 2 (if $i \in S$, then $i + k \bmod (2k+1)$, $i + k + 1 \bmod (2k+1) \in S'$), and each b_j has degree at most 2. Moreover, b_j has degree 2 for all $j \in S'$ only if $S' = C$. Therefore $|S'| \geq \min\{2k+1, |S|+1\}$. ■

Corollary 3.2 *Let $P_1, P_2, \cdots, P_n$ be paths that are pairwise disjoint except for one common endpoint y. Let x_i and t_i be the other endpoint and the length of P_i. If $2(j-1)k \leq \Sigma_{i=1}^{j} t_i \leq 2jk$ for $j = 2, 3, \cdots, n$, then any $(2k+1, k)$-colouring f of the vertices $x_1, x_2, \cdots, x_k$ can be extended to a $(2k+1, k)$-colouring of $\cup_{i=1}^{n} P_i$.*

Proof. For $i = 1, 2, \cdots, n$, let $S_i = \{j :$ there exists a $(2k+1, k)$-colouring g of P_i such that $g(x_i) = f(x_i)$ and $g(y) = j\}$. By Lemma 3.1, $|S_i| \geq 1 + t_i$. Obviously f can be extended to a $(2k+1, k)$-colouring of $\cup_{i=1}^{n} P_i$ if and only if $S_1 \cap S_2 \cap \cdots \cap S_n \neq \emptyset$. Since $|S_1 \cap S_2| = |S_1| + |S_2| - |S_1 \cup S_2|$ and $|S_1 \cup S_2| \leq 2k+1$, we know that $|S_1 \cap S_2| \geq t_1 + t_2 + 1 - 2k$, where by assumption $1 \leq t_1 + t_2 + 1 - 2k \leq 2k + 1$. By induction on i, it follows that

$$|S_1 \cap S_2 \cap \cdots \cap S_i| \geq t_1 + t_2 + \cdots + t_i + 1 - 2(i-1)k,$$

and

$$1 \leq t_1 + t_2 + \cdots + t_i + 1 - 2(i-1)k \leq 2k + 1.$$

■

Lemma 3.3 *If G has a vertex x such that $d_l(x) \geq 2k(d(x)-1)$, then G has proper subgraph G' such that any $(2k+1,k)$-colouring of G' can be extended to a $(2k+1,k)$-colouring of G.*

Proof. Assume that x is a vertex of G with $d_l(x) \geq 2k(d(x)-1)$. If $d(x)=2$, then x is the internal vertex of a thread P of length at least $2k$. Let G' be obtained from G by deleting the internal vertices of G. Corollary 3.2 implies that any $(2k+1,k)$-colouring of G' can be extended to a $(2k+1,k)$-colouring of G. Thus we may assume that $d(x)=n \geq 3$. Let $P_1, P_2, \cdots, P_n$ be the n threads of G ending at x. Let t_i be the length of P_i. Then $t_1+t_2+\cdots+t_n = d_l(x) \geq 2k(n-1)$. Since each thread of G has length at most $2k-1$, we have $2k(j-1) \leq t_1+t_2+\cdots+t_j \leq 2kj$ for $j=2,3,\cdots,n,$. Let G' be obtained from G by deleting x and the internal vertices of $P_1, P_2, \cdots, P_n$. The conclusion follows from Corollary 3.2. ∎

Lemma 3.4 *Assume x and y are two loosely t-adjacent vertices of G of degree 3. If $d_l(x)+d_l(y) \geq 6k+t$, then there is a proper subgraph G' of G such that any $(2k+1,k)$-colouring of G' can be extended to a $(2k+1,k)$-colouring of G.*

Proof. We may assume that G has no thread of length at least $2k$. Let P be the thread of length t joining x and y. Since $d(x)=3$, we may let x_1 and x_2 be vertices of degree at least 3 other than y that are loosely adjacent to x, with P_1 and P_2 being the threads joining them to x. Similarly, let y be loosely adjacent to vertices y_1 and y_2 of degree at least 3 via threads P_3 and P_4 other than P. Let G' be obtained from G by deleting x, y and the internal vertices of P_1, P_2, P, P_3, P_4. We shall prove that any $(2k+1,k)$-colouring f of G' can be extended to G.

Suppose P_i has length t_i. Since $t_1+t_2+t+t_3+t_4 \geq 6k$ and G contains no thread of length at least $2k$, it follows that $t_1+t_2 \geq 2k+1$ and $t_3+t_4 \geq 2k+1$. Let $S=\{j :$ there is an extension g of f to $P_1 \cup P_2$ such that $g(x)=j\}$. By Corollary 3.2, $|S| \geq t_1+t_2+1-2k > 0$. Let $S'=\{j :$ there is an extension g of f to $P_1 \cup P_2 \cup P$ such that $g(y)=j\}$. By Lemma 3.1, $|S'| \geq t_1+t_2+t+1-2k$. Let $S''=\{j :$ there is an extension g of f to $P_3 \cup P_4$ such that $g(y)=j\}$. By Corollary 3.2, $|S''| \geq t_3+t_4+1-2k$. Obviously, f can be extended to G if and only if $S' \cap S'' \neq \emptyset$. This is so because $|S'|+|S''| \geq t_1+t_2+t+1-2k+t_3+t_4+1-2k \geq 2k+2$. ∎

Lemma 3.5 *Suppose G has a vertex x of degree 3 which is loosely adjacent to three critical vertices a, b, c. Then G has a proper subgraph G' such that any $(2k+1,k)$-colouring f of G' can be extended to a $(2k+1,k)$-colouring of G.*

Proof. Let $P_1, P_2, \cdots, P_9$ be the threads incident to a, b, c as shown in Figure 1, where $x_1, x_2, y_1, y_2, z_1, z_2$ are vertices of degree at least 3.

Let G' be obtained from G by deleting x, a, b, c and the internal vertices of $P_1, \cdots, P_9$. Let f be a $(2k+1,k)$-colouring of G'. Let $S_1=\{j :$ there is an extension g of f to $P_1 \cup P_2 \cup P_3$ such that $g(x)=j\}$, $S_2=\{j :$ there is an extension g of f to $P_4 \cup P_5 \cup P_6$

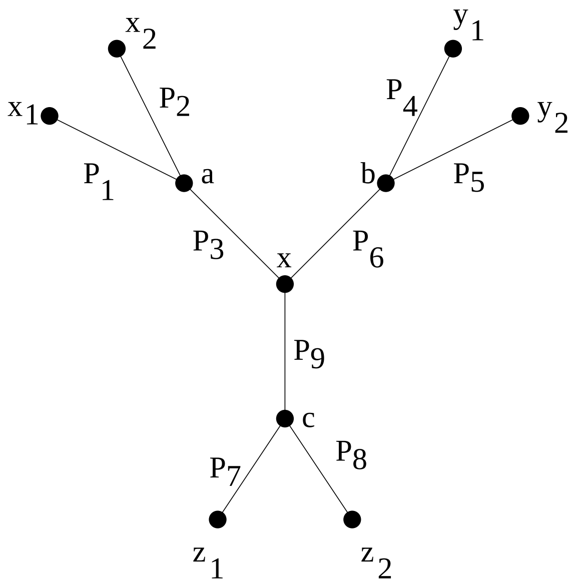

Figure 1: Threads incident to a, b, c

such that $g(x) = j\}$, $S_3 = \{j$: there is an extension g of f to $P_7 \cup P_8 \cup P_9$ such that $g(x) = j\}$. By the argument as in the proof of Lemma 3.4 (see the calculation of $|S'|$), $|S_i| \geq 2k$. It follows that $S_1 \cap S_2 \cap S_3 \neq \emptyset$. Therefore f can be extended to G. ■

To complete the proof of Theorem 1.4, we need one more lemma (the Folding Lemma) from [9].

Lemma 3.6 *[9] Suppose G is a planar graph of odd girth at least $8k - 3$. Then either every facial cycle of G has length at least $8k - 3$, or there exists a planar graph G' such that*

- $|V(G')| < |V(G)|$.
- *G admits a homomorphism to G'.*
- *G' has odd girth at least $8k - 3$ and, moreover, each facial cycle of G' has length at least $8k - 3$.*

Lemma 3.6 says that if G has a facial cycle of small length (which must be even), then that face can be "folded" without creating short odd cycles.

Proof of Theorem 1.4 If the claim does not hold, we may consider a smallest counterexample G. By Lemma 3.6, we may assume that each facial cycle of G has length at least $8k - 3$. By Lemma 2.1, one of the following holds:

- G has a vertex x such that $d_l(x) \geq 2k(d(x) - 1)$.

- G has two loosely t-adjacent vertices x and y such that $d(x) = d(y) = 3$ and $d_l(x) + d_l(y) \geq 6k + t$.

- G has a vertex x of degree 3 which is loosely adjacent to three critical vertices.

Since G is a minimal counterexample, any proper subgraph of G is $(2k+1,k)$-colourable. However, by Lemmas 3.3, 3.4, and 3.5, in each of the above cases G has a proper subgraph G' such that any $(2k+1,k)$-colouring of G' can be extended to a $(2k+1,k)$-colouring of G. This is a contradiction.

Corollary 3.7 *If G is a planar graph of girth at least $8k-4$, then $\chi_c(G) \leq 2 + \frac{1}{k}$.*

4 A remark on the relation between $g(k)$ and $f(k)$

We observed earlier that it follows from the definition that $f(k) \geq g(k)+1$. It is unknown if equality holds for all k.

Question 4.1 *For the functions $g(k), f(k)$ defined above, is it true that $f(k) = g(k) + 1$?*

There is some evidence supporting a positive answer. (Also, Conjecture 1.3 implies a positive answer). Firstly, all the presently known methods for proving upper bounds for $g(k)$ are based on Euler's formula. The girth requirement is only used in the sense that every facial cycle has length at least $g(k)$. By the "folding lemma" of Klostermeyer and Zhang [9], if a planar graph G has odd girth at least $f(k)$, then (for the purpose of investigating circular chromatic number) we may assume that it has large facial cycles as well (cf. the proof of Theorem 1.4).

Secondly, based on a good understanding of the relationship between the circular chromatic number and the girth (as well as the odd girth) of series-parallel graphs [1, 6, 11, 12], we can show that the analogue of Conjecture 4.1 for series-parallel graphs is true. (Series-parallel graphs are graphs obtained from K_2 by repeatedly applying two operations: subdividing an edge and duplicating an edge.) The following results were proved by Pan and Zhu [11, 12].

Theorem 4.2 *[11] Suppose G is a series-parallel graph. If G has odd girth at least $6k-1$, then $\chi_c(G) \leq 8k/(4k-1)$. If G has odd girth at least $6k+1$, then $\chi_c(G) \leq (4k+1)/2k$. If G is has odd girth at least $6k+3$, then $\chi_c(G) \leq (4k+3)/(2k+1)$.*

Theorem 4.3 *[12] Let $k \geq 1$ be an integer, and let $\epsilon > 0$. There exists a series-parallel graph G of girth $6k-1$ with $\chi_c(G) > 8k/(4k-1) - \epsilon$. There exists a series-parallel graph G of girth $6k+1$ with $\chi_c(G) > (4k+1)/2k - \epsilon$. There exists a series-parallel graph G of girth $6k+3$ with $\chi_c(G) > (4k+3)/(2k+1) - \epsilon$.*

For any $0 < \epsilon < 1$, let $g^*(\epsilon)$ be the smallest integer such that every series-parallel graph G of girth at least $g^*(\epsilon)$ has $\chi_c(G) \leq 2 + \epsilon$; let $f^*(\epsilon)$ be the smallest odd integer such that every series-parallel graph G of odd girth at least $f^*(\epsilon)$ has $\chi_c(G) \leq 2 + \epsilon$.

Theorem 4.4 *For any* $0 < \epsilon < 1$, $f^*(\epsilon) = g^*(\epsilon) + 1$.

Proof. For $k = 0, 1, \cdots$, let $s_{3k} = \frac{4k+3}{2k+1}$, $s_{3k+1} = \frac{8(k+1)}{4(k+1)-1}$, $s_{3k+2} = \frac{4(k+1)+1}{2(k+1)}$. Then $s_0, s_1, s_2, \cdots$ is a strictly decreasing sequence. For any $0 < \epsilon < 1$, let i be the integer such that $s_i \leq 2+\epsilon < s_{i-1}$. Assume $i = 3k$ for some $k \geq 0$. By Theorem 4.3 4.2, $f^*(\epsilon) \leq 6k+3$. By Theorem, $g^*(\epsilon) > 6k + 1$. Hence $g^*(\epsilon) \geq 6k + 2$ and $f^*(\epsilon) \leq g^*(\epsilon) + 1$. Therefore $f^*(\epsilon) = g^*(\epsilon) + 1$. The case $i = 3k + 1$ or $i + 3k + 2$ can be discusses similarly. ■

References

[1] J. Chien and X. Zhu, *The circular chromatic number of series-parallel graphs of large girth*, J. Graph Theory, 33(2000), 185-198.

[2] M. DeVos, *Communication at workshop on flows and cycles*, Simon Fraser University, 2000.

[3] A. Galluccio, L. Goddyn and P. Hell, *High girth graphs avoiding a minor are nearly bipartite*, J. Combinatorial Theory (B), to appear.

[4] L. A. Goddyn, M. Tarsi, and C. Q. Zhang, *On (k, d)-colorings and fractional nowhere-zero flows*, J. Graph Theory, 28(1998), 155-161.

[5] H. Grötzsch, *Ein Dreifarbensatz für dreikreisfreie Netze auf der Kugel*, Wissenschaftliche Zeitschrift der Martin-Luther-Universität Halle-Wittenberg, *Mathematisch-Naturwissenschaftliche Reihe*, 8 (1958/1959), 109-120

[6] P. Hell and X. Zhu, *The circular chromatic number of series-parallel graphs*, J. Graph Theory, 33(2000), 14-24.

[7] F. Jaeger, *On circular flows in graphs*, Finite and Infinite Sets (Eger, 1981), Colloquia Mathematica Societatis János Bolyai 37, North Holland, (1984) 391-402.

[8] F. Jaeger, *Nowhere-zero flow problems*, Selected Topics in Graph Theory 3, (L. W. Beineke and R. J. Wilson eds.), Academic Press, London, (1988) 71-95.

[9] W. Klostermeyer and C.Q.Zhang, *$(2+\epsilon)$-coloring of planar graphs with large odd girth*, J. Graph Theory, 33(2000), 109-119.

[10] D. Moser, *The star chromatic number of planar graphs*, J. Graph Theory, 24 (1997), 33-43

[11] Z. Pan and X. Zhu, *Circular chromatic number of series-parallel graphs of large odd girth*, Discrete Mathematics, to appear.

[12] Z. Pan and X. Zhu, *Tight relation between the circular chromatic and girth of series-parallel graphs*, manuscript, 2000.

[13] J. Nešetřil and X. Zhu, *On Bounded Tree-Width Duality of Graphs*, J. Graph Theory, 23 (1996), 151-162

[14] E. Steffen and X. Zhu, *On the star chromatic numbers of graphs*, Combinatorica, 16(1996), 439-448.

[15] C. Thomassen, *Grötzsch's 3-colour theorem and its counterparts for torus and the projective plane*, J. Combinatorial Theory (B) 62 (1994), 268-279

[16] W. T. Tutte, *A geometrical version of the four color problem*, Combinatorial Mathematics and its Applications, (eds. R.C.Bose and T.A.Dowling), University of North Carolina Press, Chapel Hill, (1967).

[17] W. T. Tutte, *A contribution on the theory of chromatic polynomial*, Canad. J. Math., 6(1954), 80-91.

[18] A. Vince, *Star chromatic number*, J. Graph Theory 12 (1988), 551-559.

[19] C.Q.Zhang, *Circular flows of nearly eulerian graphs and vertex splitting*, manuscript, 2000.

[20] C.Q.Zhang, **Integer flows and cycle covers of graphs**, Marcel Dekker Inc. New York, (1997).

[21] X. Zhu, *Planar graphs with circular chromatic numbers between* 3 *and* 4, J. Combinatorial Theory (B), 76(1999), 170-200.

[22] X. Zhu, *Circular chromatic number: a survey*, Discrete Mathematics, 229 (1-3) (2001), 371-410.