Values of Domination Numbers of the Queen's Graph

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Abstract

The queen's graph Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are adjacent if they are in the same row, column, or diagonal. Let $\gamma(Q_n)$ and $i(Q_n)$ be the minimum sizes of a dominating set and an independent dominating set of Q_n, respectively. Recent results, the Parallelogram Law, and a search algorithm adapted from Knuth are used to find dominating sets. New values and bounds:
(A) $\gamma(Q_n) = \lceil n/2 \rceil$ is shown for 17 values of n (in particular, the set of values for which the conjecture $\gamma(Q_{4k+1}) = 2k + 1$ is known to hold is extended to $k \leq 32$);
(B) $i(Q_n) = \lceil n/2 \rceil$ is shown for 11 values of n, including 5 of those from (A);
(C) One or both of $\gamma(Q_n)$ and $i(Q_n)$ is shown to lie in $\{\lceil n/2 \rceil, \lceil n/2 \rceil + 1\}$ for 85 values of n distinct from those in (A) and (B).

Combined with previously published work, these results imply that for $n \leq 120$, each of $\gamma(Q_n)$ and $i(Q_n)$ is either known, or known to have one of two values.

Also, the general bounds $\gamma(Q_n) \leq 69n/133 + O(1)$ and $i(Q_n) \leq 61n/111 + O(1)$ are established.

Keywords: dominating set, queen domination, queen's graph.

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1 Introduction

The queen's graph Q_n has the squares of the $n \times n$ chessboard as its vertices; two squares are adjacent if they are in the same row, column, or diagonal. A set D of squares of Q_n is a *dominating set* for Q_n if every square of Q_n is either in D or adjacent to a square in D. If no two squares of a set I are adjacent then I is an *independent set*. Let $\gamma(Q_n)$ denote the minimum size of a dominating set for Q_n; a dominating set of this size is a *minimum* dominating set. Let $i(Q_n)$ denote the minimum size of an independent dominating set for Q_n.

The problems of finding values of $\gamma(Q_n)$ and of $i(Q_n)$ are given as Problem C18 in [10], and have interested mathematicians for well over a century. De Jaenisch [13] considered these problems in 1862. In 1892, Rouse Ball [16] gave dominating sets and independent dominating sets of Q_n for $n \leq 8$. Ahrens [1] extended this in 1910 to $n \leq 13$ and $n = 17$ for $\gamma(Q_n)$ and to $n \leq 12$ for $i(Q_n)$. In most cases, proof that these sets were minimum dominating sets had to wait for recent work on lower bounds.

Spencer proved [7, 17] that

$$\gamma(Q_n) \geq (n-1)/2. \tag{1}$$

The only values for which equality is known to hold in (1) are $n = 3, 11$, so researchers have sought better bounds. Weakley showed [17] that

$$\gamma(Q_{4k+1}) \geq 2k + 1 \tag{2}$$

and that $i(Q_{4k+1}) = \gamma(Q_{4k+1}) = 2k + 1$ for $k \leq 6$ and $k = 8$. Other researchers [2, 5, 9] have gone further, showing $\gamma(Q_{4k+1}) = 2k+1$ for $k \leq 15$ and $k = 17, 19$. Later we extend this to $k \leq 32$.

Burger and Mynhardt showed [3] that $\gamma(Q_{4k+3}) \geq 2k + 2$ for $3 \leq k \leq 7$, with equality for $k = 4, 7$. Weakley proved [19] the following theorem.

Theorem 1 *Define a sequence of integers by $n_1 = 3$, $n_2 = 11$, and $n_i = 4n_{i-1} - n_{i-2} - 2$ for $i > 2$. If $\gamma(Q_n) = (n-1)/2$ then $n = n_i$ for some i.*

The first four values in the sequence of Theorem 1 are $3, 11, 39, 143$. As $\gamma(Q_{39}) = 20$ was shown in [19, Proposition 7], Theorem 1 and (1) imply the following bound.

Corollary 2 *If $n < 143$ and $n \neq 3, 11$, then $\gamma(Q_n) \geq n/2$.*

In this paper, we employ dominating sets called p-covers (defined below), with slight variations. A p-cover of Q_n is required to occupy certain lines (rows, columns, diagonals), and also occupies a few other lines if its size exceeds $(n-1)/2$. For these other lines, the Parallelogram Law (Theorem 4 below) allows us to greatly restrict the possibilities that need be considered. For $n \leq 35$, straightforward hand calculation suffices to either find corresponding dominating sets, or show that none exists. For larger n, we use computer search with the algorithm described in Section 4. The resulting dominating sets are given in Section 5, as are the values and bounds for $\gamma(Q_n)$ and $i(Q_n)$ implied by the sets and Corollary 2.

We devote considerable effort to proving for various n that one or both of $\gamma(Q_n)$, $i(Q_n)$ lies in $\{\lceil n/2\rceil, \lceil n/2\rceil+1\}$. Of course, exact values would be preferable. However, it is our belief that often $\lceil n/2\rceil+1$ is correct, but that it will be quite difficult to prove that $\lceil n/2\rceil$ cannot be achieved. For example, current knowledge that $\gamma(Q_n) > \lceil n/2\rceil$ for $n = 8, 14$–16 and that $i(Q_n) > \lceil n/2\rceil$ for $n = 4, 6, 8, 12, 14$–16 comes from exhaustive search, which is not feasible for large n.

Also, we use dominating sets found for Q_{131} and Q_{109} and work from [18] to give improved upper bounds for $\gamma(Q_n)$ and $i(Q_n)$.

2 Preliminaries

For odd positive integers n, we will identify the $n \times n$ chessboard with a square of side length n in the Cartesian plane, having sides parallel to the coordinate axes. We place the board with its center at the origin of the coordinate system, and refer to board squares by the coordinates of their centers. The square (x,y) is in *column* x and *row* y. Columns and rows will be referred to collectively as *orthogonals*. The *difference diagonal* (respectively *sum diagonal*) through square (x,y) is the set of all board squares with centers on the line of slope $+1$ (respectively -1) through the point (x,y). The value of $y-x$ is the same for each square (x,y) on a difference diagonal, and we will refer to the diagonal by this value. Similarly, the value of $x+y$ is the same for each square on a sum diagonal, and we associate this value to the diagonal. The *long* diagonals of Q_n are difference diagonal 0 and sum diagonal 0.

For even n, we obtain Q_n by adding a row and column to Q_{n-1}.

The square (x,y) is *even* if $x+y$ is even, *odd* if $x+y$ is odd. We divide the even squares of Q_n into two classes: (x,y) is *even-even* if both x and y are even, *odd-odd* if both are odd.

We now describe the dominating sets which have proven most useful in recent work [2, 3, 4, 5, 6, 8, 9, 14, 17, 18, 19] on queen domination.

Definitions. Let n be an odd positive integer, let D be a set of squares of Q_n, and let $p \in \{0,1\}$. Say that D is a *p-orthodox* set if every orthogonal of parity p contains a square of D.

If D is a 0-orthodox set and every odd-odd square of Q_n shares a diagonal with some square of D, we will say that D is a *0-cover*. If D is a 1-orthodox set and every even-even square shares a diagonal with some square of D, we say D is a *1-cover*.

It is clear from the definition that a p-cover dominates every even square, and every odd square is on one orthogonal of parity p, so is also covered: a p-cover is a dominating set.

Definitions. Let n be an odd positive integer and suppose that D is a set of squares of Q_n containing a square of each long diagonal.

Define $e = e(D)$ to be the largest integer such that for each i with $|i| \leq e$, D contains a square of difference diagonal $2i$.

Define $f = f(D)$ to be the largest integer such that for each i with $|i| \leq f$, D contains a square of sum diagonal $2i$.

Define $u = u(D)$ to be the largest integer such that for each i with $1 \leq i \leq u$, D contains squares of difference diagonals $\pm(2e+4i)$ and squares of sum diagonals $\pm(2f+4i)$.

The following characterization of p-covers was proved in [18].

Theorem 3 *Let n be an odd positive integer and let $p \in \{0,1\}$. Let D be a p-orthodox set for Q_n that contains at least one square from each of the long diagonals, and let $e = e(D)$, $f = f(D)$, and $u = u(D)$. The following are equivalent:*

(1) D is a p-cover of Q_n;

(2) Either (A) $e + f \equiv p \pmod 2$ and $e + f + 2u \geq (n-5)/2$,
or (B) $e + f \equiv 1 - p \pmod 2$ and $e + f \geq (n-3)/2$.

Definition. We say a p-cover is *type A* or *type B* depending on whether it satisfies condition (2A) or (2B) of Theorem 3.

Type A 0-covers have been used [6, 8, 17] to produce upper bounds for $\gamma(Q_n)$ and $i(Q_n)$. Also, most of the work [2, 5, 9, 14, 17] done to establish $\gamma(Q_{4k+1}) = 2k+1$ for $k \leq 15$ and $k = 17, 19$ used type A p-covers.

Type B p-covers are denser central packings than type A, and are less useful for producing small dominating sets, so we only use type A.

We require the following theorem from [19].

Theorem 4 (Parallelogram Law) *Let S be a set of k squares of Q_n occupying columns numbered $(x_i)_{i=1}^k$, rows $(y_i)_{i=1}^k$, difference diagonals $(d_i)_{i=1}^k$, and sum diagonals $(s_i)_{i=1}^k$. Then*

$$2\sum_{i=1}^{k} x_i^2 + 2\sum_{i=1}^{k} y_i^2 = \sum_{i=1}^{k} d_i^2 + \sum_{i=1}^{k} s_i^2.$$

Our general upper bounds derive from the next theorem, which was proved in [18].

Theorem 5 *Let n be an odd positive integer, let $p \in \{0,1\}$, and let D be a type A p-cover of Q_n that contains d squares, including a square of each long diagonal.*

If $p = 0$ and $n \equiv 1 \pmod 4$, or $p = 1$ and $n \equiv -1 \pmod 4$, then for all k, $\gamma(Q_k) \leq \frac{d+3}{n+2}k + O(1)$; if also D is independent, then for all k, $i(Q_k) \leq \frac{d+6}{n+2}k + O(1)$.

If $p = 1$ and $n \equiv 1 \pmod 4$, or $p = 0$ and $n \equiv -1 \pmod 4$, and D contains no edge squares, then for all k, $\gamma(Q_k) \leq \frac{d+1}{n}k + O(1)$; if also D is independent, then for all k, $i(Q_k) \leq \frac{d+2}{n}k + O(1)$.

Below we will give a type A 1-cover of size 66 for Q_{131} and an independent type A 0-cover of size 55 for Q_{109}. By Theorem 5, these imply the following bounds.

Corollary 6 *$\gamma(Q_n) \leq 69n/133 + O(1)$ and $i(Q_n) \leq 61n/111 + O(1)$.*

The best previously published upper bound for $\gamma(Q_n)$ is $8n/15 + O(1)$ from [4], and 69/133 is about 43% of the way from 8/15 to the 1/2 of the lower bound (1). The best previously published upper bound for $i(Q_n)$ is $19n/33 + O(1)$ from [18], and 61/111 is about 34% of the way from 19/33 to 1/2.

3 Constructions

Our constructions are all on Q_n with n odd; the long diagonals are occupied, so e and f are defined. By rotating the board if necessary, we will always take $e \geq f$. All of our dominating sets have size in $\{\lceil n/2 \rceil, \lceil n/2 \rceil + 1\}$, and all but one are type A p-covers. The following generalizes Theorem 3 of [5].

Proposition 7 *Let D be a type A p-cover of Q_n, with $(n+1)/2$ members. If $p = 0$, then either $e = f$ or $e = f + 2$. If $p = 1$, then $e = f + 1$.*

Proof. Counting occupied difference diagonals, we have $2e + 1 + 2u \leq (n+1)/2$. Subtracting this from inequality (2A) of Theorem 3 gives $e - f \leq 2$. Then $e \geq f$ and $e + f \equiv p \pmod 2$ imply the conclusion. □

For each construction we use, there are specific lines that must be occupied; we will refer to these as *required* lines, and to other lines occupied by the dominating set as *excess* lines. Complicating the picture slightly is the fact that a line may be occupied m times, with $m > 1$. If such a line is required, we also regard it as an excess line occurring $m - 1$ times.

Although the excess lines do not contribute to domination, their values are strongly restricted, both by the linear constraints

$$\left(\sum y_i\right) - \left(\sum x_i\right) = \sum(y_i - x_i) \text{ and } \left(\sum y_i\right) + \left(\sum x_i\right) = \sum(y_i + x_i), \tag{3}$$

and by the quadratic constraint due to the Parallelogram Law.

We now describe four constructions that gave us exact values of $\gamma(Q_n)$ or $i(Q_n)$; the first two constructions generalize those of [5]. As all the constructions are similar, we go into detail only for the first, which gave more exact values than the others. The constructions we used to establish bounds of the type $\gamma(Q_n)$ or $i(Q_n)$ in $\{\lceil n/2 \rceil, \lceil n/2 \rceil + 1\}$ are also like the ones described here.

Type A 0-cover of size $2k + 1$ for Q_{4k+1} with $e = f + 2$.
Here the required row numbers and column numbers are the members of $\{2j : -k \leq j \leq k\}$; as there are $2k + 1$ of these, we have no excess rows or columns. From $e = f + 2$, $n = 4k + 1$, and (2A) of Theorem 3, we see $u \geq k - e$. The number of required difference

diagonals is at most the size $2k+1$ of the dominating set, so $2e+1+2u \leq 2k+1$, which implies that $u = k-e$ and that there are no excess difference diagonals. Then the number of required sum diagonals is $2f+1+2u = 2k-3$, so there are four excess sum diagonals, which we will denote by $(s_i)_{i=1}^4$.

Let $D = \{(x_i, y_i)\}_{i=1}^{2k+1}$ be our 0-cover, with the squares numbered so that excess sum diagonal s_i is occupied by (x_i, y_i) for $1 \leq i \leq 4$. As the required sum diagonals other than 0 come in pairs with sum 0, we have $\sum_{i>4}(y_i+x_i) = 0$, so $\sum_{i\geq 1}(y_i+x_i) = \sum_{i=1}^4 s_i$. Looking at the required orthogonal numbers, we see $\sum_{i\geq 1} x_i = \sum_{i\geq 1} y_i = 0$, so $\sum_{i\geq 1}(y_i+x_i) = 0$ and thus

$$s_1 + s_2 + s_3 + s_4 = 0. \tag{4}$$

The Parallelogram Law implies

$$2\left(2\sum_{i=1}^{k}(2i)^2\right) + 2\left(2\sum_{i=1}^{k}(2i)^2\right) = \left(2\sum_{i=1}^{e}(2i)^2 + 2\sum_{i=1}^{u}(2e+4i)^2\right) + \left(s_1^2+s_2^2+s_3^2+s_4^2 + 2\sum_{i=1}^{f}(2i)^2 + 2\sum_{i=1}^{u}(2f+4i)^2\right),$$

and using $f = e-2$ and $u = k-e$, we can simplify this to

$$s_1^2+s_2^2+s_3^2+s_4^2 = 8 - 8(2k-1)\left[e^2 - (2k+1)e + 2k(k-1)/3\right]. \tag{5}$$

For each $k \leq 32$, we found all values of e and sequences $(s_i)_{i=1}^4$ satisfying (4) and (5). (This can easily be extended to larger k, but the remaining problem of finding D then has too large a search space for our approach.) It is interesting that for each k, e must be near $(1-\frac{1}{\sqrt{3}})k$; we sketch a proof.

Let $g(k,e)$ denote the right side of (5). From (5), we see that $g(k,e) \geq 0$, and then that the form of $g(k,e)$ implies $g(k,e) \geq 8$, so $e^2 - (2k+1)e + 2k(k-1)/3 \leq 0$. This implies

$$e \geq e_{\min} = k + 0.5 - \sqrt{[(k+2.5)^2 - 5.5]/3} > (1-\frac{1}{\sqrt{3}})k - \frac{5-\sqrt{3}}{2\sqrt{3}}.$$

Since no row, column, or difference diagonal can contain more than one square of D, (5) implies $(4k)^2 + 2(4k-2)^2 + (4k-6)^2 \geq g(k,e)$. For $k > 5$, this gives

$$e \leq e_{\max} = k + 0.5 - \sqrt{[(k-3.5)^2 - 4]/3},$$

and also $e_{\max} - e_{\min} < 2\sqrt{3} \approx 3.46$. Thus there are at most four values of e satisfying (4) and (5), all near $(1-\frac{1}{\sqrt{3}})k$. Quite similar bounds can be proved for the other constructions below.

The search method described in Section 4 allowed us to find the dominating sets for $k = 6, 11, 14, 17$–$20, 22$–$24, 27, 29$–32 given later. If the excess sum diagonals are all distinct and different from the required sum diagonals, as with those given for $k = 6, 11, 14, 17$–20, 22–24, 27, the resulting dominating set is independent.

As we show next, if the excess sum diagonals include certain values, we get information about Q_{4k+2} and possibly Q_{4k+3}.

Proposition 8 *Suppose there is a 0-cover D of size $2k+1$ for Q_{4k+1} with $e = f+2$. (Thus $\gamma(Q_{4k+1}) = 2k+1$.) Let S be the set of numbers of excess sum diagonals of D, and let $L = \{\pm(2f+2), \pm(4k-2f-4)\}$.*
(A) If $S \cap L$ has two members with the same sign, then
$\gamma(Q_{4k+2}) = 2k+1$. If also D is independent, then
$i(Q_{4k+2}) = 2k+1$.
(B) If $|S \cap L| \geq 1$ and D is independent, then
$i(Q_{4k+2}) \in \{2k+1, 2k+2\}$.
(C) If $|S \cap L| \geq 2$ and D is independent, then
$i(Q_{4k+3}) \in \{2k+2, 2k+3\}$.

Proof. Two ways to obtain a copy of Q_{4k+2} from Q_{4k+1} are by adjoining either row and column $2k+1$ of Q_{4k+3} or row and column $-(2k+1)$ of Q_{4k+3}; adjoining all of these gives Q_{4k+3}. We ask which squares of these orthogonals are not covered by the required lines for a 0-cover D of size $2k+1$ for Q_{4k+1} with $e = f+2$; since the set of required lines of D is symmetric across each of the long diagonals, it suffices to examine row $2k+1$. Suppose square $s = (x, 2k+1)$ is a square of Q_{4k+3} not covered by D.

Since $|x| \leq 2k+1$, if x is even then column x is occupied by D, so we may conclude s is an odd-odd square, and thus lies on an empty difference diagonal with even number (say) m.

If $m \equiv 2e \pmod 4$, then by the definition of e and u, we have $m = 2e + 4u + 4 + 4i$ for some $i \geq 0$. Above it was shown that $u = k - e$, and we are assuming that $f = e - 2$, implying that s lies on sum diagonal $2f + 2 - 4i$, which is a required diagonal unless $i = 0$. Thus s is on sum diagonal $2f + 2$ in this case.

Otherwise $m \equiv 2e + 2 \pmod 4$, so $m = 2e + 2 + 4i$ for some $i \geq 0$, and a similar argument shows that s lies on sum diagonal $4k - 2f - 4$.

Thus if the excess sum diagonals of D include $2f+2$ and $4k-2f-4$, then D covers row and column $2k+1$ and therefore dominates a copy of Q_{4k+2}; likewise if $-(2f+2)$ and $-(4k-2f-4)$ are excess sum diagonals of D. This establishes (A).

Now suppose D is independent and S contains (say) one but not both of sum diagonals $2f+2$ and $4k-2f-4$. Then the other of these meets row $2k+1$ in a square D does not cover, and adding this square to D gives an independent dominating set of Q_{4k+2}. The rest is similar. □

Type A 0-cover of size $2k+1$ for Q_{4k+1} with $e = f$.
The required row and column numbers are as in the previous case, so again there are no excess rows or columns. By (2A) of Theorem 3, the minimum value of u for a 0-cover is $u = k - e - 1$; then for each kind of diagonal, there are $2e + 1 + 2u = 2k - 1$ required values, leaving excess difference diagonals d_1, d_2 and excess sum diagonals s_1, s_2. The linear constraints (3) give $d_1 + d_2 = 0$ and $s_1 + s_2 = 0$. The Parallelogram Law gives

$$d_1^2 + s_1^2 = -4[(2k-1)e^2 - (2k-1)^2 e + 2k(2k^2 - 9k + 1)/3].$$

This construction is much less versatile than the previous one. It led to the dominating sets for $k = 26, 28$ given later, with the one for $k = 26$ being independent.

Type A 1-cover of size $2k+1$ for Q_{4k+1} with $e = f + 1$.
Here the required row and column numbers are $\pm 1, \pm 3, \ldots, \pm(2k-1)$, so there is an excess column a and an excess column b. By (2A) of Theorem 3, the minimum possible value of u is $k - e$, and then there are no excess difference diagonals and two excess sum diagonals, say s_1 and s_2. From (3), we have $b - a = \sum y_i - \sum x_i = \sum(y_i - x_i) = 0$, so $a = b$, and also $s_1 + s_2 = 2a$. From the Parallelogram Law,

$$s_1^2 + s_2^2 - 4a^2 = -16\left[e^2 - (2k+1)e + (k+1)(2k+1)/3\right].$$

This method produced dominating sets for $k = 1$–$11, 14, 15, 17, 19$–$22, 25$–27. Of these, only the set for $k = 21$ gives a value not supplied by the previous constructions. However, if $|a| \neq 2k$, as with all of ours, these sets have no squares in edge rows or columns, thus giving information about Q_{4k} and Q_{4k-1}. For $k = 21, 25, 26$ we get new bounds.

Type A 1-cover of size $2k+2$ for Q_{4k+3} with $e = f + 1$.
The required row and column numbers are $\pm 1, \pm 3, \ldots, \pm(2k-1)$, so there are no excess rows or columns. By Theorem 3 (2A), the minimum possible value of u is $k - e$, which gives an excess difference diagonal d_1 and excess sum diagonals s_1, s_2, s_3. Then (3) implies $d_1 = 0$, so dominating sets of this kind are not independent, and $s_1 + s_2 + s_3 = 0$. The Parallelogram Law gives

$$s_1^2 + s_1 s_2 + s_2^2 = 4 - 8k\left[e^2 - (2k+1)e + (k+1)(2k-5)/3\right].$$

This yielded dominating sets for $k = 1, 2, 4, 5, 7$, confirming values already known, and also for new values $k = 17, 22, 28, 32$, as given later.

4 The Search Algorithm

We shall here describe a computer search for covers discussed in the previous section. Once we have chosen the size of the board and the type of cover, the theory developed is first used to find admissible values of the parameter e and positions of the excess lines. We shall first discuss the case of independent dominating sets, and then briefly look at the search for dependent dominating sets.

The precalculations give a set L of the lines to be occupied; queens can only be placed in squares that occupy four such lines. These squares are called *eligible.* We now list all eligible squares and to each square i associate the set S_i of the lines that it occupies. For independent dominating sets, the computational problem is now to find a set of eligible squares whose sets S_i partition L. This problem is known as the *exact cover problem.*

Knuth [15] has recently developed a very fast program for the exact cover problem. The program, which can be downloaded from Knuth's web page, in particular uses an idea of Hitotumatu and Noshita [12] for efficiently handling pointers in backtrack programs.

This approach is orders of magnitude faster than searching for dominating sets from scratch. Of course, the approach only works if there are dominating sets of the given types. The results of this work, however, are encouraging. We must remark, though, that for a given size of the board there are many parameter sets that can be tried, and we often had to try several of these before finding a solution (if any).

There is also a limit for this approach. For the smallest instances, a complete search only takes seconds. For somewhat larger instances, when a complete search is not feasible, one can hope that the first solution of the backtrack search is encountered within a reasonable time. We stopped the search at board sizes for which days of cpu time are needed to find a solution. (But note that although larger boards generally lead to longer computer runs, there is a big variation in the cpu time needed for different instances.)

If the parameter set admits, one can try to impose more structure on the dominating set, such as requiring a solution to have a 180° rotational symmetry. This approach, which reduces the search space considerably, was successful only in a few cases.

If we search for a dominating set that is not independent, we have a problem slightly different from the exact cover problem. Fortunately, Knuth's program can be slightly modified to handle such instances also. Without going into details, the main idea is to associate a positive integer to each element in L telling how many times the element must occur in the sets of a solution. These values are incremented and decremented during the search, and elements with value 1 are treated as in the original algorithm. Especially when all sets S_i have at least one element with value 1, which is the case in all our instances, this modification is straightforward.

5 The Dominating Sets

Below are given the dominating sets that, along with Corollary 2, establish new values or bounds for $\gamma(Q_n)$ and $i(Q_n)$. Each description begins with the new bounds or values implied (other than those due to the elementary fact $\gamma(Q_{n+1}) \leq \gamma(Q_n)+1$); any previously published value that is implied is given parenthetically. The kind of dominating set is given (all p-covers used are type A), then the values of e, f, u, the excess lines, and finally the squares of the set.

$i(Q_{26}) \in \{13, 14\}$ and $\gamma(Q_{27}) = 14$ (and $i(Q_{25}) = 13$): begin with a 0-cover of size 13 for Q_{25} with $e = 3$, $f = 1$, $u = 3$ and excess sum diagonals -22, -4, 8, 18. For even x from -12 to 12, y-values are 6, -12, -6, 8, 0, -8, 10, -2, 4, 12, -10, -4, 2. By Proposition 6(B), adding the square (-9,13) gives $i(Q_{26}) \in \{13, 14\}$. If instead we add the square (-9,-9), we obtain a 0-orthodox set that misses being a 0-cover of Q_{27} with $e = f = u = 3$ only by failing to occupy sum diagonal 4. However, every even-even square of sum diagonal 4 is orthogonally covered, and the only odd-odd squares of sum diagonal 4 not covered along their difference diagonals are (-9,13) and (13,-9), which are covered by (-9,-9). Thus the set dominates Q_{27}.

$i(Q_{27}), i(Q_{28}) \in \{14, 15\}$: 1-cover of size 15 for Q_{27} with $e = 6$, $f = 5$, $u = 0$, excess difference diagonals ± 24, excess sum diagonals -22, -16, 14, 20. For odd x from -13 to 13, y-values are -9, 13, -7, -1, 5, 9, -5, -11, 11, 3, -3, 1, -13, 7; additional square $(-2, -2)$. All squares of row and column 14 of Q_{29} are covered, so a copy of Q_{28} is dominated.

$i(Q_{29})$, $i(Q_{30}) \in \{15, 16\}$: 0-cover of size 16 for Q_{29} with $e = 4$, $f = 2$, $u = 3$, excess difference diagonal 18, excess sum diagonals -24, -20, 10, 14, 20. For even x from -14 to 14, y-values are -10, -4, 6, 12, -14, -8, 4, -2, 14, -12, 8, 2, 10, 0, -6; additional square (-9,9). All squares of column -15 and row 15 of Q_{31} are covered, so a copy of Q_{30} is dominated.

$i(Q_{34}) \in \{17, 18\}$: 0-cover of size 18 for Q_{33} with $e = f = 4$, $u = 3$, excess difference diagonals -24, 24, 28, excess sum diagonals -32, 10, 24. For even x from -16 to 16, y-values are -16, -6, 4, 10, 16, -10, 2, -4, -12, 14, 6, 0, 12, -14, -8, -2, 8; additional square (-13,15). All squares of row and column 17 of Q_{35} are covered, so a copy of Q_{34} is dominated.

$i(Q_{35})$, $\gamma(Q_{35})$, $i(Q_{36})$, $\gamma(Q_{36}) \in \{18, 19\}$: 1-cover of size 19 for Q_{35} with $e = 6$, $f = 3$, $u = 3$, excess sum diagonals -32, -22, -8, 12, 22, 32. For odd x from -17 to 17, y-values are 3, -17, -9, 13, 7, -11, 1, 9, -7, 11, -13, -5, 15, -15, -1, 5, 17, -3; additional square (2,2). All squares of row and column -18 of Q_{37} are covered, so a copy of Q_{36} is dominated.

$i(Q_{37})$, $i(Q_{38}) \in \{19, 20\}$: 0-cover of size 20 for Q_{37} with $e = 5$, $f = 3$, $u = 4$, excess difference diagonal 12, excess sum diagonals -28, -26, 8, 26, 30. For even x from -18 to 18, y-values are -8, 6, -14, 14, 8, -10, 2, -18, -12, 4, -2, 18, 12, 0, -16, -6, 16, 10, -4; additional square (-1,11). All squares of Q_{39} except (11, -19) are covered, so a copy of Q_{38} is dominated.

$i(Q_{39})$, $i(Q_{40}) \in \{20, 21\}$: 1-cover of size 21 for Q_{39} with $e = 4$, $f = 3$, $u = 5$, excess difference diagonals ± 30, excess sum diagonals -20, 8, 12, 20. For odd x from -19 to 19, y-values are 9, -9, -3, 17, 5, 15, -13, 1, -19, 19, -1, -17, 7, -5, 13, 3, -15, 11, -7, -11; additional square (10,10). All squares of row and column 20 of Q_{41} are covered, so a copy of Q_{40} is dominated.

$i(Q_{41})$, $i(Q_{42}) \in \{21, 22\}$: 0-cover of size 22 for Q_{41} with $e = 5$, $f = 3$, $u = 5$, excess difference diagonal 34, excess sum diagonals -30, -28, 8, 20, 30. For even x from -20 to 20, y-values are 10, 4, -14, -4, -16, -12, 18, -20, 0, 16, 8, 2, -6, 20, 14, -8, -18, 6, -10, 12, -2; additional square (-17,17). All squares of Q_{43} except (13, -21) are covered, so a copy of Q_{42} is dominated.

$i(Q_{43})$, $\gamma(Q_{43})$, $i(Q_{44})$, $\gamma(Q_{44}) \in \{22, 23\}$: 1-cover of size 23 for Q_{43} with $e = 6$, $f = 3$, $u = 5$, excess sum diagonals -36, -30, 8, 12, 20, 30. For odd x from -21 to 21, y-values are -9, 13, 3, $-21 - 5$, -15, 15, 21, 5, -19, -3, 7, 19, -7, -17, 11, 1, 17, -13, 9, -1, -11; additional square (2,2). All squares of row and column 22 of Q_{45} are covered, so a copy of Q_{44} is dominated.

$i(Q_{46}) \in \{23, 24\}$ (and $i(Q_{45}) = \gamma(Q_{45}) = 23$): begin with a 0-cover of size 23 for Q_{45} from [2] with $e = 5$, $f = 3$, $u = 6$, excess sum diagonals -34, -12, 8, 38. For even x from -22 to 22, y-values are 12, -10, 0, 10, -12, -22, 20, 14, -6, 4, -20, -4, 6, -18, -2, 22, 16, -14, 8, 2, -16, 18, -8. By Proposition 8 (B), adding the square (15,-23) gives $i(Q_{46}) \in \{23, 24\}$.

$i(Q_{47})$, $\gamma(Q_{47})$, $i(Q_{48})$, $\gamma(Q_{48}) \in \{24, 25\}$: 1-cover of size 25 for Q_{47} with $e = 4$, $f = 3$, $u = 7$, excess difference diagonals 10, 22, excess sum diagonals ± 28, ± 12. For odd x from -23 to 23, y-values are -3, 15, -11, 7, -13, -21, -1, -9, 21, 17, -19, 5, -5, 19, 1, 11,

-23, 23, -15, 13, -7, -17, 9, 3; additional square (-16,16). All squares of column -24 and row 24 are covered, so a copy of Q_{48} is dominated.

$i(Q_{49})$, $i(Q_{50}) \in \{25, 26\}$: 0-cover of size 26 for Q_{49} with $e = 6$, $f = 4$, $u = 6$, excess difference diagonal 14, excess sum diagonals -36,-30, 10, 36, 40. For even x from -24 to 24, y-values are 12, 6, -12, -2, -20, -10, -18, 14, 24, 0, -24, 8, 2, 22, -6, 4, 20, -14, -4, -22, -16, 18, -8, 10, 16; additional square (3,17). All squares of row and column 25 of Q_{51} are covered, so a copy of Q_{50} is dominated.

$i(Q_{51})$, $\gamma(Q_{51})$, $i(Q_{52})$, $\gamma(Q_{52}) \in \{26, 27\}$: 1-cover of size 27 for Q_{51} with $e = 5$, $f = 4$, $u = 7$, excess difference diagonals -12, 20, excess sum diagonals -40, 10, 14, 40. For odd x from -25 to 25, y-values are -7, -13, 9, -5, 21, -25, 7, -9, 25, 15, -23, 23, 3, -3, -19, 5, -1, -17, 17, 1, -15, -21, 13, 19, -11, 11 additional square (8,16). All squares of Q_{53} except (16, -26) are covered, so a copy of Q_{52} is dominated.

$i(Q_{53})$, $i(Q_{54}) \in \{27, 28\}$: 0-cover of size 28 for Q_{53} with $e = 7$, $f = 5$, $u = 6$, excess difference diagonal 16, excess sum diagonals -46, -38, 12, 42, 44. For even x from -26 to 26, y-values are -12, -6, 16, -26, 4, 14, 20, -10, -24, -18, 6, 0, 24, -8, 10, -22, 2, -4, -20, 18, 12, 26, -16, -2, 22, -14, 8; additional square (-1,15). All squares of column -27 and row 27 of Q_{55} are covered, so a copy of Q_{54} is dominated.

$i(Q_{55})$, $\gamma(Q_{55})$, $i(Q_{56})$, $\gamma(Q_{56}) \in \{28, 29\}$: 1-cover of size 29 for Q_{55} with $e = 7$, $f = 6$, $u = 6$, excess difference diagonals -48, 54, excess sum diagonals -40, -14, 18, 38. For odd x from -27 to 27, y-values are 27, 9, 3, 17, 11, -19, -13, -27, 1, -15, -25, -9, -3, 21, 15, 7, -7, 25, 19, -23, 5, 23, -5, -11, -17, 13, -1, -21; additional square $(-2, 4)$. All squares of row and column -28 of Q_{57} are covered, so a copy of Q_{56} is dominated.

$i(Q_{57}) = 29$, $i(Q_{58}) \in \{29, 30\}$ (and $\gamma(Q_{57}) = 29$): 0-cover of size 29 for Q_{57} with $e = 7$, $f = 5$, $u = 7$, excess sum diagonals -42, -40, 28, 54. For even x from -28 to 28, y-coordinates are 14, 8, -10, -16, -20, -8, -26, 24, -18, 2, 10, 20, 4, 28, 22, -6, 0, -28, -14, -2, 16, -24, 18, -12, -22, -4, 6, 12, 26. By Proposition 8 (B), adding the square (17, -29) shows $i(Q_{58}) \in \{29, 30\}$.

$i(Q_{59})$, $\gamma(Q_{59})$, $i(Q_{60})$, $\gamma(Q_{60}) \in \{30, 31\}$: 1-cover of size 31 for Q_{59} with $e = 8$, $f = 5$, $u = 7$, excess sum diagonals -48, -42, -12, 20, 42, 44. For odd x from -29 to 29, y-values are -19, 1, -17, 17, -1, -15, 27, 21, 11, -27, 5, -5, -13, 29, -29, -9, 19, 3, -7, -23, 23, 7, -25, 13, 25, -3, -21, -11, 15, 9; additional square (2,2). All squares of row and column -30 of Q_{61} are covered, so a copy of Q_{60} is dominated.

$i(Q_{62}) \in \{31, 32\}$: 0-cover of size 32 for Q_{61} with $e = f = u = 7$, excess difference diagonals -44, -16, 60, excess sum diagonals -44, 16, 58. For even x from -30 to 30, y-values are 30, -6, -16, 18, 12, -24, 20, -2, -8, -26, 16, 4, 24, -22, 6, -30, -10, -18, 10, 26, 2, -4, 8, -28, -20, 22, -12, 14, 0, -14, 28; additional square (15,15). All squares of Q_{63} except (-31, -15) are covered, so a copy of Q_{62} is dominated.

$i(Q_{63})$, $\gamma(Q_{63})$, $i(Q_{64})$, $\gamma(Q_{64}) \in \{32, 33\}$: 1-cover of size 33 for Q_{63} with $e = 8$, $f = 5$, $u = 8$, excess sum diagonals -46, -36, -16, 12, 44, 46. For odd x from -31 to 31, y-values are -7, 3, -19, 23, 13, -9, -17, 27, -27, -21, 29, 1, 7, 11, 25, -15, -5, 9, -23, -29, -1, 31, -11, 19, -31, -25, 17, 21, 5, -13, -3, 15; additional square (2,2). All squares of row and column 32 are covered, so a copy of Q_{64} is dominated.

$i(Q_{65})$, $i(Q_{66}) \in \{33, 34\}$: 0-cover of size 34 for Q_{65} with $e = 8$, $f = 6$, $u = 8$, excess difference diagonal 52, excess sum diagonals -48, -42, -14, 48, 58. For even x from -32 to 32, y-values are -4, 18, -16, -6, 0, -18, 20, 26, -32, -28, 2, 22, -20, 30, -10, 8, -2, -8, 6, 14, -24, -30, 28, -22, -12, 10, 16, -26, 24, 32, 4, -14, 12; additional square (-25,27). All squares of Q_{67} are covered except (33, -19), so a copy of Q_{66} is dominated.

$i(Q_{67})$, $\gamma(Q_{67})$, $i(Q_{68})$, $\gamma(Q_{68}) \in \{34, 35\}$: 1-cover of size 35 for Q_{67} with $e = 7$, $f = 6$, $u = 9$, excess difference diagonals -20, 24, excess sum diagonals -52, -14, 38, 52. For odd x from -33 to 33, y-values are 5, -1, -7, -21, -27, 11, 25, -25, 33, 27, 13, -9, -31, 17, 3, 9, -5, -11, -17, -3, -23, -33, 29, 23, -19, 31, 19, -29, -15, 7, 1, 15, 21, -13; additional square (10,14). All squares of row and column -34 of Q_{69} are covered, so a copy of Q_{68} is dominated.

$i(Q_{69}) = 35$ (and $\gamma(Q_{69}) = 35$): 0-cover with $e = 8$, $f = 6$, $u = 9$, excess sum diagonals ± 38, ± 56. For even x from -34 to 34, y-values are -22, -8, 18, 8, -10, -24, 30, -18, 22, 28, -30, -16, 10, 6, 26, -28, 4, -6, -12, 2, 34, -32, 0, -20, 24, 20, -34, 12, -26, 32, -2, 16, -14, -4, 14.

$i(Q_{70}) \in \{35, 36\}$: 0-cover of size 36 for Q_{69} with $e = f = 7$, $u = 9$, excess difference diagonals -52, 16, 40, excess sum diagonals -44, 16, 62. For even x from -34 to 34, y-values are -12, 6, -14, -2, 4, -26, 28, 22, 16, 30, 26, -22, -28, -34, 2, 14, 10, -8, -20, 0, -6, 8, 20, -18, -24, -30, 32, -32, 24, 18, 12, 34, -4, -10, -16; additional square (15,19). All squares of Q_{71} except $(\pm 19, 35)$ are covered, so a copy of Q_{70} is dominated.

$\gamma(Q_{71}) = 36$: 1-cover with $e = 6$, $f = 5$, $u = 11$, excess difference diagonal 0, excess sum diagonals 0, ± 2. For odd x from -35 to 35, y-values are -19, -13, -3, 15, 25, -25, 17, 35, -11, 19, 33, -29, 21, -17, -31, 5, -5, 1, 7, -7, -1, 31, -27, 11, -35, 27, -15, -21, 9, -33, 29, 23, -23, 3, 13, -9.

$i(Q_{71})$, $i(Q_{72}) \in \{36, 37\}$: 1-cover of size 37 for Q_{71} with $e = 7$, $f = 6$, $u = 10$, excess difference diagonals -16, 28, excess sum diagonals -56, 14, 42, 56. For odd x from -35 to 35, y-values are -9, 9, -21, -27, 11, 21, -5, 33, -29, 17, 35, -19, -25, 19, 15, -35, -3, 3, 31, -13, -7, -1, 5, -31, 27, -23, 25, -15, -33, 29, 23, 1, 7, 13, -17, -11; additional square (22,34). All squares of Q_{73} but (-36, 22) are covered, so a copy of Q_{72} is dominated.

$i(Q_{73}) = 37$ (and $\gamma(Q_{73}) = 37$): 0-cover with $e = 8$, $f = 6$, $u = 10$, excess sum diagonals ± 18, ± 60. For even x from -36 to 36, y-values are -24, 14, 24, -6, 0, 26, 12, -30, -20, 22, -28, -34, 32, -8, -2, 4, 10, 30, -32, -26, -10, -4, 2, 8, 28, 18, 36, -22, -36, -14, 20, 34, -16, 6, 16, -18, -12.

$i(Q_{74}) \in \{37, 38\}$: 0-cover of size 38 for Q_{73} with $e = f = 7$, $u = 10$, excess difference diagonals -36, 16, 24, excess sum diagonals -48, 16, 70. For even x from -36 to 36, y-values are -14, 8, 22, 0, -20, 20, -30, -12, -22, 32, 18, -32, 12, 6, 30, -16, -34, -6, 26, 14, 4, 24, -4, 2, -24, -28, -10, -36, 34, 28, -26, -8, -18, 16, 10, 36, -2; additional square (17,21). All squares of Q_{75} except $(\pm 21, -37)$ are covered, so a copy of Q_{74} is dominated.

$i(Q_{75})$, $\gamma(Q_{75})$, $i(Q_{76})$, $\gamma(Q_{76}) \in \{38, 39\}$: 1-cover of size 39 for Q_{75} with $e = 10$, $f = 7$, $u = 9$, excess sum diagonals -68, -54, -16, 36, 54, 60. For odd x from -37 to 37, y-values are -31, -15, -5, 13, 27, -19, -29, 25, 15, 33, 23, 3, -21, 21, 1, -23, -37, -13, -3, -27, 5, -1, 31, -9, -33, 17, -25, 29, 35, -35, 37, 11, 7, -7, 19, 9, -17, -11;

additional square (6,6). All squares of row and column -38 of Q_{77} are covered, so a copy of Q_{76} is dominated.

$i(Q_{77}) = 39$ (and $\gamma(Q_{77}) = 39$): 0-cover with $e = 9$, $f = 7$, $u = 10$, excess sum diagonals $-62, -48, 44, 66$. For even x from -38 to 38, y-values are -24, -18, -4, 2, 24, 14, -22, -26, 36, 30, -28, 22, 32, -30, -16, 4, -6, 12, 6, -34, -2, -12, 28, 34, 16, 0, -36, -14, -20, -38, -32, 26, 18, 38, 8, -10, 20, 10, -8.

$i(Q_{78}) \in \{39, 40\}$: 0-cover of size 40 for Q_{77} with $e = f = 8$, $u = 10$, excess difference diagonals -38, 18, 38, excess sum diagonals -68, 18, 74. For even x from -38 to 38, y-values are -30, -20, -10, -4, 22, -24, 30, 24, 10, 18, 26, -16, -34, 28, 4, -32, -14, 32, 8, -28, -12, -2, 12, 6, 0, -26, -38, 36, -22, -36, 34, 20, 14, -8, -18, 16, 2, 38, -6; additional square (3,21). All squares of Q_{79} except $(\pm 21, 39)$ are covered, so a copy of Q_{78} is dominated.

$i(Q_{79})$, $\gamma(Q_{79})$, $i(Q_{80})$, $\gamma(Q_{80}) \in \{40, 41\}$: 1-cover of size 41 for Q_{79} with $e = 10$, $f = 7$, $u = 10$, excess sum diagonals -64, -58, -16, 40, 44, 58. For odd x from -39 to 39, y-values are -25, -21, -15, -1, -23, 31, 25, -13, 33, 15, -27, 23, 29, -29, 7, 39, -19, -7, 3, 9, -17, -11, -35, 35, 13, 1, 37, -37, -31, 21, -39, -33, 19, -9, -3, 27, 5, 11, 17, -5; additional square (2,2). All squares of row and column -40 of Q_{81} are covered, so a copy of Q_{80} is dominated.

$i(Q_{81}) = 41$ and $i(Q_{82}) \in \{41, 42\}$ (and $\gamma(Q_{81}) = 41$): 0-cover with $e = 9$, $f = 7$, $u = 11$, excess sum diagonals $-62, -36, 28, 70$. For even x from -40 to 40, y-values are -22, 24, -10, 8, -26, 28, 6, 20, -12, 32, -30, -24, -38, 36, 26, -28, -4, 2, 12, -32, -8, -2, 34, -36, 22, -6, 0, 14, 38, 16, -18, -40, -34, -20, 30, 40, 18, -16, 10, 4, -14. By (B) of Proposition 8, adding the square (25, -41) shows $i(Q_{82}) \in \{41, 42\}$.

$i(Q_{83}), \gamma(Q_{83}), i(Q_{84}), \gamma(Q_{84}) \in \{42, 43\}$, $i(Q_{85}) = 43$ (and $\gamma(Q_{85})= 43$): 1-cover of size 43 for Q_{85} with $e = 10$, $f = 9$, $u = 11$, $a = 6$, excess sum diagonals -44, 56. For odd x from -41 to 41, y-values are -21, 25, 11, -3, -9, -15, 27, 33, -33, -27, 15, -25, -37, 29, 39, -5, 31, 3, 9, -31, -11, -1, 5, -13, -29, -39, 23, 41, 1, 21, 37, -19, -41, -35, 35, 13, 19, -23, -17, -7, 7, 17, and additional square (6,6). As the set has no squares on the edges of Q_{85}, it is also a dominating set of Q_{83} and Q_{84}.

$i(Q_{86}) \in \{43, 44\}$: 0-cover of size 44 for Q_{85} with $e = 10$, $f = 8$, $u = 11$, excess difference diagonal 26, excess sum diagonals -64, -62, 18, 64, 68. For even x from -42 to 42, y-values are 18, -8, -18, 8, -28, 16, -6, -16, -38, 40, 14, -12, -42, 36, 42, -40, 4, 10, -4, -36, 38, -14, 12, 2, 34, -10, 0, 6, -34, 32, -22, -32, 22, -20, 30, 24, -30, -24, -2, 28, -26, 20, 26; additional square (-1,25). All squares of Q_{87} except (25,-43) are covered, so a copy of Q_{86} is dominated.

$i(Q_{87})$, $\gamma(Q_{87})$, $i(Q_{88})$, $\gamma(Q_{88}) \in \{44, 45\}$: 1-cover of size 45 for Q_{87} with $e = 9$, $f = 8$, $u = 12$, excess difference diagonals ± 40, excess sum diagonals -68, 18, 22, 68. For odd x from -43 to 43, y-values are -9, -19, 15, 9, -21, 17, -1, 33, -37, -43, 43, 21, -25, 41, 23, -35, 5, 31, -33, 3, -7, -13, 13, -39, -3, 11, 19, -5, 39, -23, -29, 37, -41, 25, -15, -31, 35, 29, -17, 1, 7, -27, 27, -11; additional square (20,20). All squares of row and column 44 of Q_{89} are covered, so a copy of Q_{88} is dominated.

$i(Q_{89}) = \gamma(Q_{89}) = 45$ and $i(Q_{90}) \in \{45, 46\}$: 0-cover with $e = 11$, $f = 9$, $u = 11$, excess sum diagonals -80, -70, 66, 84. For even x from -44 to 44, y-values are -36,

-28, -22, -16, -2, -24, 22, 0, 38, 20, -18, 36, 42, 24, 34, -32, 4, -40, 12, 6, -8, -14, -34, 40, 30, -10, 8, 2, -30, -6, -42, -20, 10, -44, -38, 28, -26, 16, 26, 32, 14, -12, 44, -4, 18. By (B) of Proposition 8, adding the square $(-25, 45)$ shows $i(Q_{90}) \in \{45, 46\}$.

$\gamma(Q_{91}) = 46$: 1-cover with $e = 9$, $f = 8$, $u = 13$, excess difference diagonal 0, excess sum diagonals -70, 8, 62. For odd x from -45 to 45, y-values are 13, -27, -15, 27, 9, 15, -35, -9, 41, 35, 17, -21, -39, -45, -31, 19, -1, -41, 45, -3, 33, 5, -5, -37, 3, -7, 7, 1, 29, 43, -19, -33, -43, 31, -23, 39, 21, -13, 37, 11, 25, -29, 23, -17, -11, -25.

$i(Q_{91})$, $i(Q_{92}) \in \{46, 47\}$: 1-cover of size 47 for Q_{91} with $e = 11$, $f = 10$, $u = 11$, excess difference diagonals ± 88, excess sum diagonals -68, -22, 38, 68. For odd x from -45 to 45, y-values are 43, 15, -7, -17, 17, 27, -19, -1, 13, -33, -39, -45, 45, -21, 21, 1, -5, 39, -35, -15, 41, 9, 19, -11, 7, -41, -13, 5, 37, -37, -27, 35, 25, -9, 33, -29, 11, 31, -31, -25, -3, 3, 29, 23, -23, -43; additional square (8,8). All squares of row and column -46 of Q_{93} are covered, so a copy of Q_{92} is dominated.

$i(Q_{93}) = \gamma(Q_{93}) = 47$: 0-cover with $e = 11$, $f = 9$, $u = 12$, excess sum diagonals -82, -60, 52, 90. For even x from -46 to 46, y-values are -36, -22, -8, 30, -24, 6, -26, -14, 28, 38, 20, -30, 40, -38, 32, 22, -28, 42, 2, -4, 10, 16, -32, 0, -18, -42, -10, 34, -2, 4, 44, -34, -44, 18, -20, -46, -40, 24, 36, -6, 12, 26, -16, 14, 8, 46, -12.

$i(Q_{94}) \in \{47, 48\}$: 0-cover of size 48 for Q_{93} with $e = 11$, $f = 9$, $u = 12$, excess difference diagonal 74, excess sum diagonals -70, -48, -36, 70, 88. For even x from -46 to 46, y-values are -24, 6, -16, -6, 20, 10, 32, -30, 40, 34, 28, -26, -32, -46, -18, 4, -34, 30, 2, 22, -28, -12, 14, -4, -44, 12, 44, -8, -2, -42, -36, -22, 24, 38, 16, -38, 36, 42, -40, -10, 0, 18, 8, 26, 46, -14, -20; additional square $(-35,39)$. All squares of column -47 and row 47 of Q_{95} are covered, so a copy of Q_{94} is dominated.

$i(Q_{95})$, $\gamma(Q_{95})$, $i(Q_{96})$, $\gamma(Q_{96}) \in \{48, 49\}$: 1-cover of size 49 for Q_{95} with $e = 12$, $f = 9$, $u = 12$, excess sum diagonals -70, -64, -40, 20, 70, 92. For odd x from -47 to 47, y-values are 25, 19, 13, -5, 29, -33, -29, 27, -35, -21, -13, -37, -31, 31, -39, 23, 17, 35, -27, 9, 3, -11, 41, 21, -9, -45, -17, 5, 37, -7, -47, 1, 33, 43, -23, -41, 45, 39, -43, -1, -19, 7, -3, 15, -15, -25, 47, 11; additional square (4,4). All squares of row and column 48 of Q_{97} are covered, so a copy of Q_{96} is dominated.

$i(Q_{97}) = \gamma(Q_{97}) = 49$ and $i(Q_{98}) \in \{49, 50\}$: 0-cover with $e = 12$, $f = 10$, $u = 12$, excess sum diagonals -90, -78, 72, 96. For even x from -48 to 48, y-values are -42, -32, -24, -18, 0, -26, -20, -2, 36, -22, 24, 46, 32, 38, 44, 18, 28, -30, 6, 12, 40, -8, -44, -16, -10, -34, 14, 8, 2, -12, -6, 42, -36, -46, -28, -38, -48, 22, -40, 34, 20, 10, 4, 30, -4, -14, 16, 26, 48. By (B) of Proposition 8, adding the square $(-27, 49)$ shows $i(Q_{98}) \in \{49, 50\}$.

$\gamma(Q_{99}), \gamma(Q_{100}) \in \{50, 51\}$ and $\gamma(Q_{101}) = 51$: 1-cover of size 51 for Q_{101} with $e = 11$, $f = 10$, $u = 14$, $a = 29$, excess sum diagonals 8, 50. For odd x from -49 to 49, y-values are -27, 31, 17, -29, 5, -13, -31, 39, -23, 27, 37, -33, 45, -21, 21, -45, 33, 1, -35, 43, 11, -3, 3, -15, -39, 13, 41, -11, -1, -41, 7, 47, -5, 35, 49, 19, -47, 25, -19, -49, -43, 23, -7, -25, 9, -17, 29, -9, 29, 15; additional square $(29, -37)$. As the set has no squares on the edges of Q_{101}, it is also a dominating set for Q_{99} and Q_{100}.

$i(Q_{99})$, $i(Q_{100}) \in \{50, 51\}$: 1-cover of size 51 for Q_{99} with $e = 12$, $f = 9$, $u = 13$, excess sum diagonals -74, -52, -36, 20, 74, 84. For odd x from -49 to 49, y-values are 27, 17, -29, 29, -1, -27, 31, 1, -37, -21, 19, 33, -33, 21, 35, -35, -19, 37, -49, 11, -41, -5, 13, -13, -45, -39, 9, -9, -15, 41, -7, 7, 43, -43, 23, 45, 3, -23, 47, -47, 39, 5, 49, 25, 15, -31, -25, -11, -17, -3; additional square (8,8). All squares of row and column 50 of Q_{101} are covered, so a copy of Q_{100} is dominated.

$i(Q_{101}), i(Q_{102}) \in \{51, 52\}$: 0-cover of size 52 for Q_{101} with $e = f = 10$, $u = 14$, excess difference diagonals -66, 22, 46, excess sum diagonals -62, 22, 80. For even x from -50 to 50, y-values are -26, -12, -18, 16, -20, 0, -6, 36, -34, 44, 14, 20, 30, -8, -50, 48, 28, -36, 50, -44, 42, -2, 4, 10, -46, 18, -16, -10, -4, 6, -30, -48, 8, -40, 26, -32, -14, -42, 46, 40, 34, 24, -38, 12, 22, -24, 38, 32, -22, -28, 2; additional square (19,21). All squares of row and column 51 of Q_{103} are covered, so a copy of Q_{102} is dominated.

$i(Q_{103}), \gamma(Q_{103}), i(Q_{104}), \gamma(Q_{104}) \in \{52, 53\}$, $i(Q_{105}) = \gamma(Q_{105}) = 53$: 1-cover of size 53 for Q_{105} with $e = 12$, $f = 11$, $u = 14$, $a = 4$, excess sum diagonals -52, 60. For odd x from -51 to 51, y-values are -27, 31, -7, -29, 13, 7, -31, 35, 29, -13, -21, 39, -39, 51, 37, -37, 25, -25, -47, 9, 3, 43, -43, -11, 15, -19, -1, -41, -17, -3, 11, 47, 41, 1, 49, 23, 17, -49, -35, 33, -51, -45, 45, -33, -15, -9, -23, 19, 5, 27, 21, -5; additional square (4,4). As the set has no squares on the edges of Q_{105}, it is also a dominating set for Q_{103} and Q_{104}.

$i(Q_{106}) \in \{53, 54\}$: 0-cover of size 54 for Q_{105} with $e = 12$, $f = 10$, $u = 14$, excess difference diagonal 26, excess sum diagonals -84, -62, 22, 80, 92. For even x from -52 to 52, y-values are -32, 30, 16, -14, 32, -34, 0, 34, -36, -28, 36, -38, -24, 26, -40, -26, 40, 4, 28, -42, 44, 8, -10, 42, 6, -8, -44, 38, 18, -12, 10, -4, 2, -50, -6, 46, -48, -30, -16, 50, -52, -46, 48, 22, -20, 14, 52, -18, 24, -2, 12, -22, 20; additional square (11,37). All squares of row and column 53 of Q_{107} are covered, so a copy of Q_{106} is dominated.

$i(Q_{107}), \gamma(Q_{107}), i(Q_{108}), \gamma(Q_{108}) \in \{54, 55\}$: 1-cover of size 55 for Q_{107} with $e = 10$, $f = 9$, $u = 16$, excess difference diagonals 22, 42, excess sum diagonals -36, -28, 20, 48. For odd x from -53 to 53, y-values are 23, 33, -33, 21, 35, 5, -5, -35, -25, 37, -45, -39, 27, -31, -41, 29, -15, -9, -37, -1, 47, 11, 31, -43, 13, 41, -7, -13, 45, 7, -11, -51, 9, 3, -49, 49, 39, -27, 51, 25, -17, 53, -53, -47, 43, -19, 15, 1, -29, 17, -21, -3, 19, -23; additional square (-30,34). All squares of column -54 and row 54 of Q_{109} are covered, so a copy of Q_{108} is dominated.

$i(Q_{109}) = \gamma(Q_{109}) = 55$: 0-cover with $e = 13$, $f = 11$, $u = 14$, excess sum diagonals -94, -86, 72, 108. For even x from -54 to 54, y-values are -40, -34, -28, 34, -4, 2, 36, -30, -36, 26, 40, -26, -32, -38, 24, 46, 32, 38, 48, -22, -6, 0, -44, 8, 14, -12, 22, -46, -10, 42, 10, 4, -14, -8, 44, -50, 52, -42, -52, 50, -24, -54, -48, 18, -20, 6, 16, -2, -16, 28, 20, 30, 12, -18, 54.

$i(Q_{110}) \in \{55, 56\}$: 0-cover of size 56 for Q_{109} with $e = f = 11$, $u = 15$, excess difference diagonals -60, 36, 44, excess sum diagonals -84, 24, 86. For even x from -54 to 54, y-values are -16, -10, -34, 18, 32, 6, 16, -14, -4, -30, 48, 22, -48, -54, 4, 46, -52, 54, -28, 20, 30, 0, 52, -50, -44, 42, -2, 8, -18, -12, -6, 12, 2, -46, 10, 34, 40, -40, -24, -42, 28, -26, -20, -38, 44, 50, 36, 26, -36, 38, 24, 14, -32, -22, -8; additional square (3,23). All squares of row and column 55 of Q_{111} are covered, so a copy of Q_{110} is dominated.

$i(Q_{111})$, $\gamma(Q_{111})$, $i(Q_{112})$, $\gamma(Q_{112}) \in \{56, 57\}$: 1-cover of size 57 for Q_{111} with $e = 13$, $f = 12$, $u = 14$, excess difference diagonals -100, 108, excess sum diagonals -84, -26, 50, 84. For odd x from -55 to 55, y-values are 27, 55, -5, -19, 11, 33, 19, -23, 31, 37, 15, -27, -49, -55, 39, -47, -53, 3, -21, 25, 1, -13, 9, -43, -15, -1, -41, 53, 13, -51, -37, -17, 47, 7, -7, 49, 5, 41, 23, -39, 51, 21, 43, -35, 17, 45, -33, 29, -9, -3, -29, -11, 35, -31, -25, -45; additional square (8,16). All squares of row and column -56 of Q_{113} are covered, so a copy of Q_{112} is dominated.

$\gamma(Q_{113}) = 57$: 0-cover with $e = f = 11$, $u = 16$, excess difference diagonals ± 88, excess sum diagonals ± 20. This set is symmetric by a half-turn about board center, so we only give the y-values for even x from 0 to 56: 0, -48, -12, 44, -4, 52, 8, 6, -42, -2, 22, 56, 50, -20, -46, 36, -54, 24, 18, -40, -30, 28, -10, -16, -34, -38, -14, 32, 26.

$i(Q_{113})$, $i(Q_{114}) \in \{57, 58\}$: 0-cover of size 58 for Q_{113} with $e = 13$, $f = 11$, $u = 15$, excess difference diagonal 90, excess sum diagonals -92, -86, 36, 76, 86. For even x from -56 to 56, y-values are -36, -32, 34, -24, 14, -12, 6, 16, 30, 36, -26, -20, 50, -40, -54, 20, 54, -56, 22, -48, -34, 52, -4, 44, 8, 18, -42, -6, 4, -22, -16, -10, 0, -52, 24, 2, 46, 56, -50, -44, -18, 32, 26, 48, -46, 12, -14, 38, 42, 28, -30, 40, -38, -8, -2, -28, 10; additional square (-35,55). All squares of column -57 and row 57 of Q_{115} are covered, so a copy of Q_{114} is dominated.

$\gamma(Q_{115}) = 58$: 1-cover with $e = 11$, $f = 10$, $u = 17$, excess difference diagonal 0, excess sum diagonals 0, ± 54. For odd x from -57 to 57, y-values are -31, -25, -19, 15, 29, -9, 13, 43, -35, 11, -17, 55, -15, 31, 53, -57, -27, 51, -43, 27, -1, -53, 57, -3, -51, 47, -5, 9, 3, -7, -47, 5, 49, -11, 7, 1, -55, -41, 33, -33, 45, -21, 21, -37, 41, -49, 19, 39, 25, -45, 17, -29, 37, -13, -39, 35, -23, 23.

$i(Q_{115})$, $i(Q_{116}) \in \{58, 59\}$: 1-cover of size 59 for Q_{115} with $e = 14$, $f = 11$, $u = 15$, excess sum diagonals -104, -86, -24, 56, 72, 86. For odd x from -57 to 57, y-values are -47, -31, -25, -19, -33, -27, 43, 1, -5, 25, 31, -23, 47, 53, 19, 33, -37, -43, 51, 57, 35, 41, -11, -39, -3, 15, 9, -17, -53, -9, -41, -21, 5, -13, -45, 7, 55, -1, 37, 29, 49, -55, -49, 13, -57, -51, 27, -35, -29, 45, 39, 21, 11, -15, 3, -7, 23, 17; additional square (0,0). All squares of row and column -58 of Q_{117} are covered, so a copy of Q_{116} is dominated.

$\gamma(Q_{117}) = 59$: 0-cover with $e = 13$, $f = 11$, $u = 16$, excess sum diagonals -80, -78, 58, 100. For even x from -58 to 58, y-values are -16, -22, -24, -6, -12, 18, 4, 26, 44, 50, -32, 42, 28, -18, -56, -52, 56, -58, -44, 54, 36, -30, -2, 12, 48, -46, 14, -4, 6, -20, -14, -8, 10, -42, 2, 8, -40, -54, 40, 46, -36, 34, 24, -50, 32, 22, 52, -38, -48, -26, 58, 38, 0, 30, 20, -10, -28, -34, 16.

$i(Q_{117})$, $i(Q_{118}) \in \{59, 60\}$: 0-cover of size 60 for Q_{117} with $e = f = 12$, $u = 16$, excess difference diagonals -102, 26, 92, excess sum diagonals -62, 34, 50. For even x from -58 to 58, y-values are -30, 36, -22, -32, 38, 4, -10, -36, -20, 0, -34, 48, 34, 40, -38, 20, 54, -28, 42, 56, -46, -6, 46, 14, 8, -52, -42, 52, 12, -18, -8, 10, -16, -44, -4, 6, 58, -48, 16, 44, 22, 26, -54, -12, -26, -56, -50, -40, 50, 28, 18, -58, 30, -24, 2, 32, -14, 24, -2; additional square (3,19). All squares of column -59 and row 59 of Q_{119} are covered, so a copy of Q_{118} is dominated.

$i(Q_{119})$, $\gamma(Q_{119})$, $i(Q_{120})$, $\gamma(Q_{120}) \in \{60, 61\}$: 1-cover of size 61 for Q_{119} with $e = 14$, $f = 11$, $u = 16$, excess sum diagonals -90, -84, -24, 32, 76, 90. For odd x from -59 to 59, y-values are -31, 35, 21, -33, -15, -35, 37, -37, 45, 3, 25, 43, -39, 27, -27, -49, -43, 47, 33, -29, 49, 31, -9, 9, -51, -11, -5, 13, 7, -45, -55, -19, -47, 59, -17, 5, -1, 39, 57, 1, 11, 55, 51, -57, 41, 19, -59, -53, 53, -41, 17, -25, -3, 15, -23, -13, -7, -21, 29, 23; additional square (0,0). All squares of row and column -60 of Q_{121} are covered, so a copy of Q_{120} is dominated.

$\gamma(Q_{121}) = 61$: 0-cover with $e = 13$, $f = 11$, $u = 17$, excess sum diagonals -74, -64, 40, 98. For even x from -60 to 60, y-values are 26, 36, 18, 24, 38, -40, -6, -32, 2, 40, 30, -36, -28, 28, -42, -56, -30, -44, -58, 44, 34, -48, 0, 6, -50, 14, 50, -14, -46, 10, -4, 52, -16, -52, 8, 48, -12, -2, 4, 22, 54, 16, 42, 56, -26, -20, 58, -60, -54, 60, 46, 20, -38, 32, -18, -24, -34, -8, -22, 12, -10.

$\gamma(Q_{125}) = 63$: 0-cover with $e = 13$, $f = 11$, $u = 18$, excess sum diagonals -68, -40, 18, 90. For even x from -62 to 62, y-values are -32, -26, -20, 22, -36, 30, 20, -6, 0, 42, 24, -18, 56, 62, 28, -30, 60, -12, -44, -50, -46, -62, 6, 58, -52, -8, 48, 4, 14, -4, 52, -16, 10, 54, -2, -58, -10, 8, 2, -54, -60, -34, 32, 50, 40, 18, -56, -42, -24, -14, 44, -22, 36, -38, 16, 46, -48, 38, -40, 34, 12, 26, -28.

$\gamma(Q_{129}) = \gamma(Q_{130}) = 65$: 0-cover of Q_{129} with $e = 14$, $f = 12$, $u = 18$, excess sum diagonals -100, -26, 0, 126. For even x from -64 to 64, y-values are 28, 22, -16, 14, -24, -2, -12, -22, 0, 54, 24, 46, -56, 58, -64, -58, 32, 26, -60, -34, 36, -62, 56, 62, -52, -8, 10, 4, 18, 12, -10, -50, -26, -20, -14, 16, -6, 8, 2, 38, -48, -42, -32, 42, 60, -54, -44, -38, 48, 34, 40, 50, 20, 30, 52, -46, 44, 6, -4, -30, -40, -18, -28, 64, -36. We may apply Proposition 6(A) with $k = 32$; since there are excess sum diagonals numbered -100 and -26, this set dominates Q_{130}.

$\gamma(Q_{131}) = 66$: 1-cover with $e = 14$, $f = 13$, $u = 18$, excess difference diagonal 0, excess sum diagonals -130, 2, 128. Take the set on Q_{129} just given, add 1 to each row and column number, and include the square $(-65, -65)$.

Below we summarize established values and bounds for $\gamma(Q_n)$ and $i(Q_n)$; values from [19] are in italic, those from [14] are underlined, those from this work are in bold, and older ones are in roman type.

$i(Q_n) = \gamma(Q_n) = (n-1)/2$ for $n = 3, 11$.

$\gamma(Q_n) = \lceil n/2 \rceil$ for n = 1, 2, 4–7, 9, 10, 12, 13, 17–19, 21, *23*, 25, **27**, 29–31, 33, 37, *39*, 41, 45, 49, 53, 57, 61, <u>65</u>, 69, **71**, <u>73</u>, 77, <u>81</u>, <u>85</u>, **89**, **91**, **93**, **97**, **101**, **105**, **109**, **113**, **115**, **117**, **121**, **125**, **129–131**.

$\gamma(Q_n) = \lceil n/2 \rceil + 1$ for n = 8, <u>14</u>, <u>15</u>, <u>16</u>.

$\gamma(Q_n) \in \{\lceil n/2 \rceil, \lceil n/2 \rceil + 1\}$ for n = 20, 22, 24-26, **28**, 32, 34, **35**, **36**, 38, *40*, 42, **43**, **44**, 46, **47**, **48**, 50, **51**, **52**, 54, **55**, **56**, 58, **59**, **60**, 62, **63**, **64**, <u>66</u>, **67**, **68**, 70, **72**, <u>*74*</u>, **75**, **76**, 78, **79**, **80**, <u>82</u>, **83**, **84**, <u>86</u>, **87**, **88**, **90**, **92**, **94-96**, **98-100**, **102-104**, **106-108**, **110-112**, **114**, **116**, **118-120**, **122**, **126**, **132**.

$i(Q_n) = \lceil n/2 \rceil$ for $n =$ 1, 2, 5, 7, 9, 10, 13, 17, 21, 25, 33, 45, **57**, 61, **69**, **73**, **77**, **81**, **85**, **89**, **93**, **97**, **105**, **109**.

$i(Q_n) = \lceil n/2 \rceil + 1$ for $n =$ 4, 6, 8, 12, 14–16, <u>18</u>.

$i(Q_n) \in \{\lceil n/2 \rceil, \lceil n/2 \rceil + 1\}$ for $n =$ 19, 20, <u>22</u>, 23, 24, **26-30**, 31, 32, **34-44**, **46-56**, **58-60**, **62-68**, **70-72**, **74-76**, **78-80**, **82-84**, **86-88**, **90-92**, **94-96**, **98-104**, **106-108**, **110-120**.

6 Concluding Remarks

Much previous work has depended on finding dominating sets symmetric by a half-turn about board center. This is not always possible, even when the line sets have the desired symmetry. Specifically, the dominating sets given above to establish the values of $i(Q_{69})$ and $i(Q_{73})$ each have the property that their sets of occupied lines have 180° symmetry, but an exhaustive search showed there are no minimum dominating sets with 180° symmetry that occupy these line sets.

From (1) and the definitions of γ and i, we have $(n-1)/2 \le \gamma(Q_n) \le i(Q_n)$ for all n; our results lead us to add the following conjecture.

Conjecture For all n, $i(Q_n) \le \lceil n/2 \rceil + 1$.

It would be very interesting to know if equality occurs in (1) for any n other than 3 and 11. It is only necessary to examine the members of the sequence (n_i) defined in Theorem 1. The smallest open case is $n_4 = 143$; it is shown in [19] that if $\gamma(Q_{143}) = 71$, then any minimum dominating set of Q_{143} is an independent 0-cover with $e = f = 15$, $u = 20$, and no excess diagonals.

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