Spanning Trees of Bounded Degree

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Abstract

Dirac's classic theorem asserts that if $\mathbf{G}$ is a graph on n vertices, and $\delta(\mathbf{G}) \geq n/2$, then $\mathbf{G}$ has a hamilton cycle. As is well known, the proof also shows that if $\deg(x) + \deg(y) \geq (n-1)$, for every pair x, y of independent vertices in $\mathbf{G}$, then $\mathbf{G}$ has a hamilton path. More generally, S. Win has shown that if $k \geq 2$, $\mathbf{G}$ is connected and $\sum_{x \in I} \deg(x) \geq n-1$ whenever I is a k-element independent set, then $\mathbf{G}$ has a spanning tree $\mathbf{T}$ with $\Delta(\mathbf{T}) \leq k$. Here we are interested in the structure of spanning trees under the additional assumption that $\mathbf{G}$ does not have a spanning tree with maximum degree less than k. We show that apart from a single exceptional class of graphs, if $\sum_{x \in I} \deg(x) \geq n-1$ for every k-element independent set, then $\mathbf{G}$ has a spanning caterpillar $\mathbf{T}$ with maximum degree k. Furthermore, given a maximum path P in $\mathbf{G}$, we may require that P is the spine of $\mathbf{T}$ and that the set of all vertices whose degree in $\mathbf{T}$ is 3 or larger is independent in $\mathbf{T}$.

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1 Introduction

We consider only finite simple graphs and use the standard notation $\deg_{\mathbf{G}}(x)$ to denote the *degree* of a vertex in $\mathbf{G}$. We also use $\delta(\mathbf{G})$ and $\Delta(\mathbf{G})$ to denote respectively the *minimum* degree and *maximum* degree of a graph $\mathbf{G}$. The set of all vertices adjacent to a vertex u in $\mathbf{G}$ is denoted $N_{\mathbf{G}}(u)$.

Recall the now classic theorem of G. A. Dirac [3] which provides a sufficient condition for a graph to have a hamilton cycle.

Theorem 1.1 *Let* $\mathbf{G}$ *be a graph on* n *vertices. If* $\delta(G) \geq n/2$*, then* $\mathbf{G}$ *has a hamilton cycle.* $\square$

Dirac's theorem has lead to many new results and conjectures concerning paths and cycles in graphs. One theme to this research concentrates solely on hamilton cycles—investigating how the hypothesis of Theorem 1.1 can be weakened without allowing the graph to become non-hamiltonian. One well known example of this is the "closure" concept introduced by J. A. Bondy and V. Chvatàl [2].

A second direction is motivated by the fact that the proof of Dirac's theorem yields the following corollary [4].

Corollary 1.2 *Let* $\mathbf{G} = (V, E)$ *be a graph on* n *vertices. If* $\deg_{\mathbf{G}}(x) + \deg_{\mathbf{G}}(y) \geq n - 1$ *for every* $x, y \in V$ *with* $xy \notin E$*, then* $\mathbf{G}$ *has a hamilton path.* $\square$

Now a hamilton path is just a spanning tree with small maximum degree, so for integers n and k, it is natural to ask for the how the preceding theorem might be generalized to guarantee the existence of a spanning tree with maximum degree at most k. In 1975, S. Win [5] provided the following answer to this question.

Theorem 1.3 *Let* $k \geq 2$ *be an integer and let* $\mathbf{G}$ *be a connected graph so that*

$$\sum_{x \in I} \deg(x) \geq n - 1$$

for every k*-element independent set* $I \subset V$*. Then* $\mathbf{G}$ *has a spanning tree* $\mathbf{T}$ *with* $\Delta(\mathbf{T}) \leq k$*.* $\square$

Note that the technical condition on the degrees of vertices given in Theorem 1.3 is satisfied whenever $\delta(\mathbf{G}) \geq (n-1)/k$.

Along the lines of Theorem 1.3, there is a sequence of papers which study k-maximal trees. A k*-maximal* tree of a graph is a subtree that is maximal (by inclusion) among all subtrees having maximum degree at most k. The sequence culminates with the article of Aung and Kyaw [1], in which the authors obtain lower bounds for the size of a k-maximal tree and characterize graphs which meet those bounds.

The purpose of this paper is to investigate the *structure* of the spanning trees with small maximum degree. Recall that a tree $\mathbf{T}$ is called a *caterpillar* when there exists a path P in $\mathbf{T}$ so that every vertex of $\mathbf{T}$ which is not on the path P is adjacent to a point of P. The path P is called the *spine* of the caterpillar.

Our principal theorem will assert that graphs which satisfy the conclusion of Win's Theorem 1.3 with equality have spanning caterpillars, but there will be one exceptional class of graphs. Let n and k be positive integers and consider a sequence $\delta_1, \delta_2, \ldots, \delta_k$ of positive integers with $\sum_{i=1}^{k} \delta_i = n - 1$. Then form a graph $\mathbf{G}(\delta_1, \delta_2, \ldots, \delta_k)$ by taking k disjoint complete graphs, one of size δ_i for each $i = 1, 2, \ldots, k$ and then attaching a new vertex adjacent to all other vertices. Note that the only independent sets of size k consist of one point from each of the k cliques and that the sum of the degrees of the vertices in such a set is exactly $n - 1$. However, when three or more of the cliques have two or more points, the graph does not have a spanning caterpillar of maximum degree at most k.

Furthermore, note that if $\mathbf{G}$ has a spanning tree with maximum degree *less* than k, then in general it is difficult to say anything about the structure of a spanning tree $\mathbf{T}$ whose maximum degree is as small as possible, even when $\delta(\mathbf{G}) \geq (n-1)/k$. Here's why. Let $\mathbf{T}_0$ be any tree. Choose a positive integer δ and form a graph $\mathbf{G}$ as follows. For each edge $e = xy$ in $\mathbf{T}_0$, remove the edge e and add a complete subgraph $\mathbf{K}_e$ of δ new vertices with x and y both adjacent to all δ vertices in $\mathbf{K}_e$. It is easy to see that $\delta(\mathbf{G}) = \delta$, but that any spanning tree of $\mathbf{G}$ contains a homeomorph of $\mathbf{T}_0$.

With these remarks in mind, here is the statement of our principal result.

Theorem 1.4 *Let $k \geq 2$ be an integer and let $\mathbf{G} = (V, E)$ be a connected graph on n vertices satisfying:*

$$\sum_{x \in I} \deg(x) \geq n - 1$$

for every k-element independent set $I \subset V$. Then either:

1. *$\mathbf{G}$ has a spanning tree with maximum degree less than k;*

2. *$\mathbf{G} = \mathbf{G}(\delta_1, \delta_2, \ldots, \delta_k)$ for some sequence $\delta_1, \delta_2, \ldots, \delta_k$ of positive integers with at least three δ_is larger than 1; or*

3. *for every maximum length path P in $\mathbf{G}$, there is a spanning tree $\mathbf{T}$ of $\mathbf{G}$ such that:*

   a. *$\mathbf{T}$ is a caterpillar,*

   b. *$\Delta(\mathbf{T}) = k$,*

   c. *the spine of $\mathbf{T}$ is the path P, and*

   d. *the set $\{v \in V \mid \deg_{\mathbf{T}}(v) \geq 3\}$ is independent in $\mathbf{T}$.*

In addition, in Options 2 and 3, unless $\mathbf{G}$ is the star on $k+1$ vertices, $\mathbf{G}$ contains a dominating cycle.

Note that our theorem reduces to Corollary 1.2 when $k = 2$.

2 Proof of The Principal Result

We fix integers n and k with $k \geq 2$ and consider a connected graph $\mathbf{G} = (V, E)$ on n vertices satsfying:

$$\sum_{x \in I} \deg(x) \geq n - 1$$

for every k-element independent set $I \subset V$. Without loss of generality, we may assume that $k \geq 3$, for as noted previously, the case $k = 2$ is just Corollary 1.2. However, we will not assume Win's Theorem 1.3, so we do not assume that $\mathbf{G}$ has a spanning tree with maximum degree at most k. If $\mathbf{G}$ has a spanning tree $\mathbf{T}$ with maximum degree less than k, then Option 1 of our theorem holds. So we will assume that $\mathbf{G}$ does not have a spanning tree with maximum degree less than k.

Now let $P = (u_1, u_2, \ldots, u_t)$ be an arbitrary maximum path in $\mathbf{G}$ with $u_i u_{i+1} \in E$ for all $i = 1, 2, \ldots, t-1$. Since $k \geq 3$, we know that $\mathbf{G}$ does not have a hamilton path, so there is at least one vertex $v \notin P$. Since $\mathbf{G}$ is connected, we can choose v to be adjacent to a vertex of P. However, no vertex not on P can be adjacent to two consecutive vertices on P. Furthermore, $u_1 u_t \notin E$. Otherwise, if v is a vertex not on P and $vu_i \in E$, then $(u_{i+1}, u_{i+2}, \ldots, u_t, u_1, u_2, \ldots, u_i, v)$ is a longer path than P. More generally, $\mathbf{G}$ cannot contain *any* cycle of length t. The maximality of P also implies the following.

Fact 1. Let C be a cycle of length $t - 1$. Then

(a) C dominates $\mathbf{G}$,

(b) $V - C$ is independent, and

(c) no two consecutive vertices of C have a common neighbor in $V - C$.

It is natural to call u_1 and u_t the *left end point* and *right end point* of the path P, respectively. Moreover, if $1 \leq i < t$ and $u_1 u_{i+1} \in E$, then $(u_i, u_{i-1}, \ldots, u_1, u_{i+1}, u_{i+2}, \ldots, u_t)$ is also a maximum path in $\mathbf{G}$, and now u_i is the left end point. We define $X_L = \{u_i : i < t, u_1 u_{i+1} \in E\}$, and we call elements of X_L *potential left end points.* Dually, we call elements of $X_R = \{u_i : 1 < i, u_{i-1} u_t \in E\}$ *potential right end points.* Finally, we let $X = X_L \cup X_R$.

Fact 2. Suppose there is an independent $(k-2)$-set $I \subseteq V - P$ so that $\deg_{\mathbf{G}}(u_1) + \deg_{\mathbf{G}}(u_t) + \sum_{v \in I} \deg_{\mathbf{G}}(v) = n - 1$. Let $u_i \notin X$, with i minimum, and let $u_j \notin X$, with j maximum. Then

(a) $u_{i'} u_{j'} \notin E$ whenever $1 \leq i' < i \leq j < j' \leq t$,

(b) $\deg_{\mathbf{G}}(u_{i'}) \geq i - 1$ for all $1 \leq i' < i$, and

(c) $\deg_{\mathbf{G}}(u_{j'}) \geq t - j$ for all $j < j' \leq t$.

Proof. Because of the choice of i we know that $u_{i'} \in X$ for all $1 \leq i' < i$. Suppose there is some $u_{i'} \in X_R$ with i' minimum. Then $u_{i'-1} \in X_L$ and $(u_1, \ldots, u_{i'-1}, u_t, \ldots, u_{i'})$

is a cycle of length t, a contradiction. Hence $u_{i'} \in X_L$ for all $1 \leq i' < i$. Likewise $u_{j'} \in X_R$ for all $j < j' \leq t$. Thus $\deg_{\mathbf{G}}(u_1) \geq |\{u_2, \ldots, u_i\}| = i - 1$, and $\deg_{\mathbf{G}}(u_t) \geq |\{u_j, \ldots, u_{t-1}\}| = t - j$. Now since $I \cup \{u_{i'}, u_t\}$ is independent for all $1 \leq i' < i$, we know that $\deg_{\mathbf{G}}(u_{i'}) + \deg_{\mathbf{G}}(u_t) + \sum_{v \in I} \deg_{\mathbf{G}}(v) \geq n - 1$, and so $\deg_{\mathbf{G}}(u_{i'}) \geq \deg_{\mathbf{G}}(u_1) \geq i - 1$ for each $1 \leq i' < i$. Likewise, $\deg_{\mathbf{G}}(u_{j'}) \geq \deg_{\mathbf{G}}(u_t) \geq t - j$ for each $j < j' \leq t$. Finally if $u_{i'}u_{j'} \in E$, with $1 \leq i' < i \leq j < j' \leq t$, then $(u_1, \ldots, u_{i'}, u_{j'}, \ldots, u_t, u_{j'-1}, \ldots, u_{i'+1})$ is a cycle of length t, a contradiction. $\diamond$

Case 1. $X_L \cap X_R \neq \emptyset$.

Let $u \in X_L \cap X_R$. Then $P - \{u\}$ contains a cycle C of length $t - 1$ that is formed using the edges of $P - \{u\}$ and those which witness $u \in X_L \cap X_R$. That is, $C = (u_1, u_2, \ldots, u_{j-1}, u_t, u_{t-1}, \ldots, u_{j+1})$, where $u = u_j$. By Fact 1(a), C is dominating. Label the vertices of $V - P$ as $v_1, v_2, \ldots, v_{n-t}$ so that $\deg_{\mathbf{G}}(v_i) \leq \deg_{\mathbf{G}}(v_j)$ when $1 \leq i < j \leq n - t$.

We now construct a spanning tree $\mathbf{T}$ using the following algorithm. Set $\mathbf{T}_0$ to be the tree consisting of P and its edges. Thereafter, for each $i = 1, 2, \ldots, n - t$, choose a vertex $w \in P$ with $wv_i \in E$ and $\deg_{\mathbf{T}_{i-1}}(w)$ minimum. Then add the vertex v_i and the edge wv_i to $\mathbf{T}_{i-1}$ to form $\mathbf{T}_i$.

Setting $\mathbf{T} = \mathbf{T}_{n-t}$, it is clear by Fact 1(b) that $\mathbf{T}$ is a caterpillar containing P as its spine. Moreover, the vertices of degree 3 or more in $\mathbf{T}$ are independent in $\mathbf{T}$. Indeed, u is not such a vertex, so if two such vertices are consecutive on P then they are consecutive on C, contradicting Fact 1(c) above. It remains only to show that $\Delta(\mathbf{T}) = k$.

To the contrary, suppose that $\Delta(\mathbf{T}) \neq k$. Then $\Delta(\mathbf{T}) > k$. Consider the first step at which a vertex of degree $k + 1$ is created. Suppose this occurs at step j when v_j is attached to a vertex w in P.

Suppose that $\deg_{\mathbf{G}}(v_j) = 1$ and note that $\deg_{\mathbf{G}}(v_{j'}) = 1$ for all $j' \leq j$. Let $I = \{u_1\} \cup \{v_{j'} \in V - P : j' \leq j,\, wv_{j'} \in E(\mathbf{T}_j)\}$. Then I is an independent set of size k in $\mathbf{G}$, and thus, by the original degree hypothesis,

$$\deg_{\mathbf{G}}(u_1) + \sum_{v \in I - \{u_1\}} \deg_{\mathbf{G}}(v) = \sum_{v \in I} \deg_{\mathbf{G}}(v) \geq n - 1 \ ,$$

from which we conclude

$$\deg_{\mathbf{G}}(u_1) \geq (n - 1) - (k - 1) = n - k \ .$$

However, this implies that $N_P(u_1) = P - \{u_1\}$. In particular, $u_1u_t \in E$, a contradiction.

On the other hand, suppose $\deg_{\mathbf{G}}(v_j) > 1$. The algorithm requires that for every $u_i \in N_{\mathbf{G}}(v_j)$ we have $\deg_{\mathbf{T}_{j-1}}(u_i) = k$. Now for each $u_i \in P$, let $W_i = \{u_{i-1}, u_i\} \cup \{v_{j'} \in V - P : 1 \leq j' < j, u_i v_{j'} \in E(\mathbf{T}_{j-1})\}$. Then $|W_i| = k$ for every $u_i \in N_{\mathbf{G}}(v_j)$. Furthermore, W_i and $W_{i'}$ are disjoint when u_i and $u_{i'}$ are distinct elements of $N_{\mathbf{G}}(v_j)$, and $(\cup_{u_i \in N_{\mathbf{G}}(v_j)} W_i) \cap \{v_j, u_t\} = \emptyset$. Fix $u_i, u_{i'} \in N_{\mathbf{G}}(v_j)$ and let $I = (W_i - \{u_i, u_{i-1}\}) \cup \{v_j, v_{j'}\}$ for some $v_{j'} \in N_{\mathbf{T}_{j-1}}(u_{i'})$. Note that $j^* < j$ for every $v_{j^*} \in I$, and so correspondingly $\deg_{\mathbf{G}}(v_{j^*}) \leq \deg_{\mathbf{G}}(v_j)$. Then I is an independent set of size k, and thus

$$k \deg_{\mathbf{G}}(v_j) \geq \sum_{x \in I} \deg_{\mathbf{G}}(x) \geq n - 1 > n - 2 \geq \sum_{u_{i'} \in N_{\mathbf{G}}(v_j)} |W_{i'}| = k \deg_{\mathbf{G}}(v_j) \ .$$

This contradiction completes the proof of Case 1.

Case 2. $X_L \cap X_R = \emptyset$.

When $\mathbf{T}$ is a spanning tree of $\mathbf{G}$ which contains P, we let $\mathrm{dist}_{\mathbf{T}}(x, y)$ denote the *distance* from x to y in $\mathbf{T}$, i.e., the number of edges in the (unique) path from x to y in $\mathbf{T}$. Also, we let $\mathrm{dist}_{\mathbf{T}}(x, P) = \min\{\mathrm{dist}_{\mathbf{T}}(x, u) : u \in P\}$, so that $\mathrm{dist}_{\mathbf{T}}(x, P) = 0$ if and only if $x \in P$. We let $Q_{\mathbf{T}}(x)$ denote the unique shortest path in $\mathbf{T}$ from x to a vertex in P. Of course $Q_{\mathbf{T}}(x)$ is trivial when $x \in P$. When $\mathrm{dist}_{\mathbf{T}}(x, P) > 0$, we let $S_{\mathbf{T}}(x)$ denote the unique vertex y which is adjacent to x in $\mathbf{T}$ with $\mathrm{dist}_{\mathbf{T}}(y, P) = \mathrm{dist}_{\mathbf{T}}(x, P) - 1$. When $a \in V$ is not a leaf of $\mathbf{T}$, the set of vertices belonging to components of $\mathbf{T} - \{a\}$ which do not intersect P is denoted $F(a)$.

In this case, we select a spanning tree $\mathbf{T}$ by applying the following five "tie-breaking" rules. These rules are applied sequentially in the order listed to narrow the set of trees from which $\mathbf{T}$ must be drawn.

Rule 1. $\mathbf{T}$ must contain P and its edges.

Rule 2. Minimize $\Delta = \max\{\deg_{\mathbf{T}}(x) : x \in V\}$.

Rule 3. Minimize $m = |\{x \in V : \deg_{\mathbf{T}}(x) = \Delta\}|$.

Rule 4. Maximize $q = \max\{\mathrm{dist}_{\mathbf{T}}(a, P) : \deg_{\mathbf{T}}(a) = \Delta\}$.

Rule 5. Maximize $s = \max\{\sum_{x \in F(a)} \mathrm{dist}_{\mathbf{T}}(x, a) : \deg_{\mathbf{T}}(a) = \Delta,\ \mathrm{dist}_{\mathbf{T}}(a, P) = q\}$.

Now let $\mathbf{T}$ be any spanning tree selected according to these five rules. Choose a vertex a_0 with $\deg_{\mathbf{T}}(a_0) = \Delta$ (recall $\Delta \geq k$), $\mathrm{dist}_{\mathbf{T}}(a_0, P) = q$, $|F(a_0)| = f$, and $\sum_{x \in F(a_0)} \mathrm{dist}_{\mathbf{T}}(x, a_0) = s$. Label the vertices of $Q(a_0) = (a_0, a_1, \ldots, a_q)$ so that $a_{i-1}a_i$ is an edge of $\mathbf{T}$ for $1 \leq i \leq q$ and so that $a_q \in P$. We denote the number of components of $F(a_0)$ by r and we label these components by $F_1, F_2, \ldots, F_r$, noting that r is either $\Delta - 2$ or $\Delta - 1$ depending on whether a_0 belongs to P or not, respectively. (This subtle note will be used in Conclusion 1 of Subcase B below, where we deduce that $a_0 \in P$ after learning that $r = \Delta - 2$.)

For each $i = 1, 2, \ldots, r$, let x_i be a vertex in F_i for which $\mathrm{dist}_{\mathbf{T}}(x, a_0)$ is maximum. Then x_i is a leaf in the tree $\mathbf{T}$. Also, for each $i = 1, 2, \ldots, r$, let y_i be the unique vertex of F_i which is adjacent to a_0 in $\mathbf{T}$. Note that $x_i = y_i$ if and only if the component F_i is trivial.

Claim 1. If $x \in F(a_0)$, then $\deg_{\mathbf{T}}(x) < \Delta$.

Proof. This follows immediately from the definition of a_0. $\diamond$

Claim 2. Let $i \in \{1, 2, \ldots, r\}$. Then all neighbors of x_i in $\mathbf{G}$ belong to $Q(x_i) \cup P$.

Proof. Suppose to the contrary that $x_i y \in E$ and $y \notin Q(x_i) \cup P$. Then either $y \in F_j$ for some $j \neq i$, $y \in F_i - Q(x_i)$ or $y \in V - (F(a_0) \cup Q(a_0) \cup P)$. Suppose first that $y \in F_j$ with $i \neq j$. Then form a new tree $\mathbf{S}$ by removing the edge $a_0 y_j$ and adding the edge $x_i y$.

Then $\mathbf{S}$ wins by Rule 2, 3 or 4. The contradiction shows that no leaf x_i has a neighbor in $F(a_0) - F_i$.

Next, suppose that $y \in F_i - Q(x_i)$. Then $y \neq y_i$. Form $\mathbf{S}$ by removing the edge $yS(y)$ and adding the edge $x_i y$. Now, because of the choice of x_i, $\mathbf{S}$ wins by Rule 5. The contradiction shows that no leaf x_i has a neighbor in $F_i - Q(x_i)$.

Finally, suppose that $y \in V - \big(F(a_0) \cup Q(a_0) \cup P\big)$. Now form the tree $\mathbf{S}$ by removing the edge $yS(y)$ and adding the edge $x_i y$.

Now $\mathbf{S}$ wins either by Rule 3 or by Rule 5. The contradiction completes the proof of the claim. $\diamond$

Claim 3. Let $i \in \{1, 2, \ldots, r\}$. If $q > 0$ then $x_i a_1 \notin E$.

Proof. Suppose that $q > 0$ and $x_i a_1 \in E$. Form $\mathbf{S}$ by removing the edge $a_0 a_1$ and adding the edge $x_i a_1$. In $\mathbf{S}$, the degree of x_i is 2, and the degree of a_0 is $\Delta - 1$. However, the degree of a_1 is the same in both trees, so $\mathbf{S}$ wins either by Rule 2 or by Rule 3. $\diamond$

Since P is a maximum path in $\mathbf{G}$, no point of $V - P$ can be adjacent to two consecutive points of P. Here is a somewhat analogous claim for the path $Q(a_0)$.

Claim 4. Let $i \in \{1, 2, \ldots, r\}$. If $q > 0$ then x_i is not adjacent in $\mathbf{G}$ to consecutive vertices of the path $Q(a_0)$.

Proof. Suppose to the contrary that a leaf x_i is adjacent to both a_j and a_{j+1}. Form $\mathbf{S}$ from $\mathbf{T}$ by inserting x_i between a_j and a_{j+1}, i.e., remove the edges $x_i S(x_i)$ and $a_j a_{j+1}$, and add the edges $x_i a_j$ and $x_i a_{j+1}$. Then $\mathbf{S}$ wins by Rule 2, 3 or 4. $\diamond$

Claim 5. If $i \in \{1, 2, \ldots, r\}$ and the leaf $x_i \in F_i$ is adjacent in $\mathbf{G}$ to a vertex $v \in Q(a_0) \cup P$, then $\deg_{\mathbf{T}}(v) \geq \Delta - 1$.

Proof. Suppose to the contrary that $x_i v \in E$, $v \in Q(a_0) \cup P$ but $\deg_{\mathbf{T}}(v) < \Delta - 1$. Then $v \neq a_0$. Form $\mathbf{S}$ by removing the edge $y_i a_0$ and adding the edge $v x_i$. Then $\mathbf{S}$ wins either by Rule 2 or by Rule 3. $\diamond$

Without loss of generality, we may assume that the components of $F(a_0)$ have been labelled so that

$$\deg_{\mathbf{G}}(x_1) - \mathrm{dist}_{\mathbf{T}}(x_1, a_0) \geq \deg_{\mathbf{G}}(x_i) - \mathrm{dist}_{\mathbf{T}}(x_i, a_0)$$

for all $i = 1, 2, \ldots, r$.

At this point, our argument for Case 2 splits into two subcases.

Subcase A. $\deg_{\mathbf{G}}(x_1) - \mathrm{dist}_{\mathbf{T}}(x_1, a_0) \leq 0$.

In this case, we know that $\deg_{\mathbf{G}}(x_i) - \mathrm{dist}_{\mathbf{T}}(x_i, a_0) \leq 0$ for all $i = 1, 2, \ldots, r$. Now consider the k-element independent set $I = \{u_1, u_t\} \cup \{x_1, x_2, \ldots, x_{k-2}\}$. Then

$$\deg_{\mathbf{G}}(u_1) + \deg_{\mathbf{G}}(u_t) + \sum_{i=1}^{k-2} \deg_{\mathbf{G}}(x_i) \geq n - 1. \tag{1}$$

However, $\deg_{\mathbf{G}}(u_1) + \deg_{\mathbf{G}}(u_t) \leq t-1$, since $\deg_{\mathbf{G}}(u_1) = |X_L|$, $\deg_{\mathbf{G}}(u_t) = |X_R|$, $X_L \cap X_R = \emptyset$ and $a_q \notin X_L \cup X_R$. Also, $\deg_{\mathbf{G}}(x_i) \leq \mathrm{dist}_{\mathbf{T}}(x_i, a_0) \leq |F_i|$ for each $i = 1, 2, \ldots, k-2$. Thus

$$\deg_{\mathbf{G}}(u_1) + \deg_{\mathbf{G}}(u_t) + \sum_{i=1}^{k-2} |F_i| \leq |P| - 1 + |F(a_0)| \leq n - 1. \tag{2}$$

Inequalities (1) and (2) force equalities (1) and (2). Thus $r = k-2$, $\deg_{\mathbf{G}}(u_1)+\deg_{\mathbf{G}}(u_t) = t-1$, and $\deg_{\mathbf{G}}(x_i) = \mathrm{dist}_{\mathbf{T}}(x_i, a_0) = |F_i|$ for all $i = 1, 2, \ldots, k-2$. In particular, $a_0 \in P$ and each F_i is a path. Furthermore, a_0 is the only point on P which is not a potential end point, so that no point of $F(a_0)$ can be adjacent in $\mathbf{G}$ to any point of $P - \{a_0\}$.

Now let $i \in \{1, 2, \ldots, k-2\}$, let $f_i = |F_i|$, and let $x_i = z_1, z_2, \ldots, z_{f_i} = y_i, z_{f_i+1} = a_0$ be a listing of the points of the path $F_i \cup \{a_0\}$. Then we know that $z_1 z_j \in E$ for all $j = 2, 3, \ldots, f_i + 1$. Now let j be any integer with $2 \leq j \leq f_i$. Form a new tree $\mathbf{S}$ by removing the edge $z_j z_{j+1}$ and adding $z_1 z_{j+1}$. Now $\mathbf{S}$ ties $\mathbf{T}$ on each of the tiebreaking rules. Since z_j is a leaf, we know as above that $z_j z_{j'} \in E$ for all $j' = 1, \ldots, j-1, j+1, \ldots, f_i+1$. Thus each $F_i \cup \{a_0\}$ is a clique.

Choose $u_i \notin X$ with i minimum, and $u_j \notin X$ with j maximum. Here, $u_i = a_0 = u_j$. Because of equalities (1) and (2), we may apply Fact 2. Parts (a) and (b) imply that $\{u_1, \ldots, u_i\}$ is a clique, and parts (a) and (c) imply that $\{u_j, \ldots, u_t\}$ is a clique. But these remarks then imply that $\mathbf{G}$ is the exceptional graph $\mathbf{G}(\deg_{\mathbf{G}}(u_1), \deg_{\mathbf{G}}(u_t), f_1, f_2, \ldots, f_{k-2})$. If $\mathbf{G}$ is not a star, that is, if not all of its parameters are 1, then $\mathbf{G}$ has a dominating cycle. If at most two of its parameters are 1, then $\mathbf{G}$ satisfies Option 3 of the theorem; otherwise it satisfies Option 2. This completes the argument in this subcase.

Subcase B. $\deg_{\mathbf{G}}(x_1) - \mathrm{dist}_{\mathbf{T}}(x_1, a_0) > 0$.

In this subcase, vertex x_1 has at least one neighbor in $\mathbf{G}$ which does not belong to $F_1 \cup \{a_0\}$.

Let $N_1 = (N_{\mathbf{G}}(x_1) \cap Q(a_0)) - \{a_0, a_q\}$ and $N_2 = (N_{\mathbf{G}}(x_1) \cap P) - \{a_o\}$. By Claim 2 $N_{\mathbf{G}}(x_1)$ is contained in the disjoint union $F_1 \uplus \{a_0\} \uplus N_1 \uplus N_2$. In this subcase, we are assuming that $|N_1| + |N_2| > 0$.

For each $a_j \in N_1$, let $W_j = (N_{\mathbf{T}}(a_j) - \{a_{j+1}\}) \cup \{a_j\}$. By Claim 5, $|W_j| = \deg_{\mathbf{T}}(a_j) \geq \Delta - 1$ for every $a_j \in N_1$. Furthermore, by Claim 4, $W_{j_1} \cap W_{j_2} = \emptyset$ for all $a_{j_1}, a_{j_2} \in N_1$ with $j_1 \neq j_2$. Also, note that $a_1 \notin N_1$ by Claim 3, and so

$$\big(X \cup \{a_0\} \cup F(a_0)\big) \cap W_j = \emptyset$$

for all $a_j \in N_1$.

For each $u_j \in N_2$, let $Z_j = \{u_j\} \cup (N_{\mathbf{T}}(u_j) - P)$. As above, by Claim 5, $|Z_j| \geq \deg_{\mathbf{T}}(u_j) - 1 \geq \Delta - 2$ for each $j \in N_2$, and since P is maximum $Z_{j_1} \cap Z_{j_2} = \emptyset$ for all $u_{j_1}, u_{j_2} \in N_2$ with $j_1 \neq j_2$. Likewise, note that

$$\big(X \cup \{a_0\} \cup F(a_0)\big) \cap Z_j = \emptyset$$

for all $u_j \in N_2$.

It follows that $V \supseteq X \uplus \{a_0\} \uplus F(a_0) \uplus (\uplus_{a_j \in N_1} W_j) \uplus (\uplus_{u_j \in N_2} Z_j)$, and so

$$n-1 \geq |X| + |F(a_0)| + |N_1|(\Delta-1) + |N_2|(\Delta-2)\ . \tag{3}$$

Now $|X_L| = \deg_{\mathbf{G}}(u_1)$, $|X_R| = \deg_{\mathbf{G}}(u_t)$, $X_L \cup X_R = X$ and $X_L \cap X_R = \emptyset$. Thus

$$|X| = \deg_{\mathbf{G}}(u_1) + \deg_{\mathbf{G}}(u_t). \tag{4}$$

Noting that $|F_i| \geq \mathrm{dist}_{\mathbf{T}}(x_i, a_0)$ for each $i = 1, 2, \ldots, r$, we have

$$|F(a_0)| \geq \sum_{i=1}^{r} \mathrm{dist}_{\mathbf{T}}(x_i, a_0). \tag{5}$$

Furthermore, because of Claim 2 we have

$$N_{\mathbf{G}}(x_1) \subseteq (F_1 \cap Q(x_1)) \uplus \{a_0\} \uplus N_1 \uplus N_2\ ,$$

and because $|F_1 \cap Q(x_1)| = \mathrm{dist}_{\mathbf{T}}(x_1, a_0)$, we obtain

$$|N_1| + |N_2| \geq \deg_{\mathbf{G}}(x_1) - \mathrm{dist}_{\mathbf{T}}(x_1, a_0). \tag{6}$$

It follows that inequality 3 can be rewritten and relaxed to

$$\begin{aligned} n-1 \geq{}& \deg_{\mathbf{G}}(u_1) + \deg_{\mathbf{G}}(u_t) + \sum_{i=1}^{r} \mathrm{dist}_{\mathbf{T}}(x_i, a_0) \\ &+ |N_1| + (\Delta-2)\big(\deg_{\mathbf{G}}(x_1) - \mathrm{dist}_{\mathbf{T}}(x_1, a_0)\big). \end{aligned} \tag{7}$$

On the other hand, consider the k-element independent set $I = \{u_1, u_t\} \cup \{x_1, x_2, \ldots, x_{k-2}\}$. Then

$$n-1 \leq \deg_{\mathbf{G}}(u_1) + \deg_{\mathbf{G}}(u_t) + \sum_{i=1}^{k-2} \deg_{\mathbf{G}}(x_i). \tag{8}$$

Recall that the components of $F(a_0)$ were labelled so that

$$\deg_{\mathbf{G}}(x_i) \leq \deg_{\mathbf{G}}(x_1) - \mathrm{dist}_{\mathbf{T}}(x_1, a_0) + \mathrm{dist}_{\mathbf{T}}(x_i, a_0) \tag{9}$$

for each $i = 1, 2, \ldots, r$. It follows that

$$\begin{aligned} n-1 \leq{}& \deg_{\mathbf{G}}(u_1) + \deg_{\mathbf{G}}(u_t) \\ &+ \sum_{i=1}^{k-2} \big(\deg_{\mathbf{G}}(x_1) - \mathrm{dist}_{\mathbf{T}}(x_1, a_0) + \mathrm{dist}_{\mathbf{T}}(x_i, a_0)\big). \end{aligned} \tag{10}$$

Thus

$$n-1 \leq \deg_{\mathbf{G}}(u_1) + \deg_{\mathbf{G}}(u_t) + \sum_{i=1}^{k-2} \mathrm{dist}_{\mathbf{T}}(x_i, a_0) \tag{11}$$

$$+(k-2)\big(\deg_{\mathbf{G}}(x_1)-\mathrm{dist}_{\mathbf{T}}(x_1,a_0)\big).$$

Comparing inequalities 7 and 11, we obtain

$$\begin{aligned} \deg_{\mathbf{G}}(u_1)+\deg_{\mathbf{G}}(u_t)+\sum_{i=1}^{r}\mathrm{dist}_{\mathbf{T}}(x_i,a_0) \\ +|N_1|+(\Delta-2)\big(\deg_{\mathbf{G}}(x_1)-\mathrm{dist}_{\mathbf{T}}(x_1,a_0)\big) \\ \leq\ \deg_{\mathbf{G}}(u_1)+\deg_{\mathbf{G}}(u_t)+\sum_{i=1}^{k-2}\mathrm{dist}_{\mathbf{T}}(x_i,a_0) \\ +(k-2)\big(\deg_{\mathbf{G}}(x_1)-\mathrm{dist}_{\mathbf{T}}(x_1,a_0)\big)\ , \end{aligned} \tag{12}$$

which reduces to

$$\sum_{i=k-1}^{r}\mathrm{dist}_{\mathbf{T}}(x_i,a_0)+|N_1|+(\Delta-k)\big(\deg_{\mathbf{G}}(x_1)-\mathrm{dist}_{\mathbf{T}}(x_1,a_0)\big)\leq 0\ . \tag{13}$$

Recalling that in this subcase we have $\deg_{\mathbf{G}}(x_1)-\mathrm{dist}_{\mathbf{T}}(x_1,a_0)>0$, we conclude that equality must hold in (3)-(13), from which we draw the following string of conclusions.

Conclusion 1. From equality in (13) it is clear that the summation must be empty; that is, $r=k-2$. Recall that this implies that $a_0\in P$; i.e., $q=0$. Moreover, we also learn from (13) that $\Delta=k$ and, of course, N_1 is empty, which implies that N_2 is nonempty in this subcase.

For each $u_j\in N_2$, let $Z'_j=Z_j-\{u_j\}$, and set $Z=\cup_{u_j\in N_2}Z'_j$. Also, define the set $M=N_2\cup\{a_0\}$. By the maximality of P, the vertex x_1 cannot have internally disjoint paths to consecutive vertices of P. Hence M is independent.

Conclusion 2. The path P is partitioned into $X_L\uplus X_R\uplus M$. Moreover, the set of vertices V is partitioned into $P\uplus Z\uplus F(a_0)$. Indeed, both assertions follow from equality in (3).

Conclusion 3. $F_i\cup\{a_0\}$ is a path of length $\mathrm{dist}_{\mathbf{T}}(x_i,a_0)$ for each $i=1,2,\ldots,k-2$. This is because we obtain $|F_i|=\mathrm{dist}_{\mathbf{T}}(x_i,a_0)$ for all i from equality in (5).

Equalities in (6) and (9), along with Conclusion 1, imply that $\deg_{\mathbf{G}}(x_i)-\mathrm{dist}_{\mathbf{T}}(x_i,a_0)=|N_2|$ for all $i=1,2,\ldots,k-2$. The next conclusion follows easily from this fact.

Conclusion 4. For each $i=1,2,\ldots,k-2$, we have $N_{\mathbf{G}}(x_i)=(F_i-\{x_i\})\uplus(N_2\cup\{a_0\})=(F_i-\{x_i\})\uplus M$.

Another simple consequence of equality in (3) is that, for every $u_j\in N_2$, we have $|Z'_j|=k-3$. In other words, $\deg_{\mathbf{T}}(u_j)=k-1$ for each $u_j\in N_2$. This observation implies the following.

Conclusion 5. The vertex a_0 is the unique vertex whose degree in $\mathbf{T}$ is k.

Our final conclusion is merely the statement of equality in (8).

Conclusion 6. $n-1=\deg_{\mathbf{G}}(u_1)+\deg_{\mathbf{G}}(u_t)+\sum_{i=1}^{k-2}\deg_{\mathbf{G}}(x_i)$.

We first show that if $k\geq 4$ then $\mathbf{T}$ satisfies Option 3. We already know that $\Delta(\mathbf{T})=k$ from Conclusion 1, and that M is independent. From Conclusion 2 and the maximality of P, we know that the only vertices $u\in P$ with $\deg_{\mathbf{T}}(u)\geq 3$ are in M. Thus it suffices to show that $\mathbf{T}$ is a caterpillar with spine P. First we prove two claims.

Claim 6. Suppose $k\geq 4$. Then for every $v\in Z$ we have $N_{\mathbf{G}}(v)\subseteq M$.

Proof. To the contrary, suppose that $v'v\in E$ with $v'\notin M$. Of course, $v'\notin X$, so it must be that $v'\in Z\cup F(a_0)$, by Conclusion 2. In particular, $v'\notin P$. We will modify the tree $\mathbf{T}$ to create a tree $\mathbf{T}'$ as follows. Let $u\in M$ be the vertex adjacent to v in $\mathbf{T}$, and let $i\in\{1,\ldots,k-2\}$ be chosen so that $v'\notin F_i$. Now define $\mathbf{T}'=\mathbf{T}-\{vu,y_ia_0\}+\{vv',x_iu\}$. Note that $\deg_{\mathbf{T}'}(a_0)<\deg_{\mathbf{T}}(a_0)$, and $\deg_{\mathbf{T}'}(w)\leq\deg_{\mathbf{T}}(w)$ for all $w\in V-\{x_i,v'\}$. Also, $\deg_{\mathbf{T}'}(x_i)\leq\deg_{\mathbf{T}}(x_i)+1\leq 2$, and $\deg_{\mathbf{T}'}(v')=\deg_{\mathbf{T}}(v')+1\leq k$, by the definition of a_0 and Conclusion 1. Therefore, since $\mathbf{G}$ has no spanning tree of maximum degree less than k, it must be that $\deg_{\mathbf{T}'}(v')=k$. However, this contradicts Rule 4 because of Conclusion 1. $\diamond$

Claim 7. Suppose $k\geq 4$. Then for every $i=1,2,\ldots,k-2$ we have $|F_i|=1$.

Proof. As above, define the set $I=\{u_1,u_t,x_1,\ldots,x_{k-2}\}$. Choose a vertex $u\in N_2$ and fix a value for i. Since $k\geq 4$, by Claim 5 there is a vertex $v\in Z$ which is adjacent to u in $\mathbf{T}$. Now consider the k-element independent set $I'=I\cup\{v\}-\{x_i\}$. Because of Conclusion 6 and our original hypothesis, we must have

$$\sum_{x\in I}\deg_{\mathbf{G}}(x)=n-1\leq\sum_{x\in I'}\deg_{\mathbf{G}}(x)\ .$$

By cancelling common terms and using Conclusion 4 and Claim 6, we have

$$|M|+|F_i|-1=\deg_{\mathbf{G}}(x_i)\leq\deg_{\mathbf{G}}(v)\leq|M|\ .$$

Hence $|F_i|\leq 1$. $\diamond$

By Conclusion 2, $V-P=Z\cup F(a_0)$. By definition, every element of Z is adjacent in $\mathbf{T}$ to a vertex in P, and by Claim 8, every vertex in $F(a_0)$ is adjacent to $a_0\in P$. Thus $\mathbf{T}$ is a catepillar with spine P. By our earlier remarks, $\mathbf{T}$ satisfies Option 3.

Next we consider the particular case $k=3$. For this value of k, we cannot obtain as strong a result as in Claim 7, but instead, we settle for Claim 8 below. In this special case we will be able to modify $\mathbf{T}$ if necessary to obtain a spanning tree that satisfies Option 3. As in the previous case, we know that $\Delta(\mathbf{T})=k$ and that the only vertices $u\in P$ with $\deg_{\mathbf{T}}(u)\geq 3$ are in the independent set M. In fact, by Conclusion 5, a_0 is the unique such vertex.

Claim 8. Suppose $k=3$. Then $|F_1|\leq 2$.

Proof. Suppose to the contrary that $|F_1|\geq 3$, and let $a_0=u_j$. By Conclusion 3, the vertices of $F_1\cup\{a_0\}$ form the path $(x_1,y_1',\ldots,y_1,a_0)$. Conclusion 1 states that $N_2\neq 0$. Choose $u_{j'}\in N_2$ so that $|j'-j|$ is minimal. Without loss of generality we shall assume that $j'<j$.

We know that both $u_{j'+1}, u_{j-1} \notin M$ because of the maximality of P. Also, $u_{j'+1} \notin X_L$ since otherwise the path $(y_1, \ldots, y_1', x_1, u_{j'}, u_{j'-1}, \ldots, u_1, u_{j'+2}, \ldots, u_t)$ is longer than P. Likewise, $u_{j-1} \notin X_R$. By Conclusion 2, $u_{j'+1} \in X_R$ and $u_{j-1} \in X_L$.

Now there must be some h, $j'+1 \leq h < j-1$, so that $u_h \in X_R$ and $u_{h+1} \in X_L$. However, now we see that the path $(x_1, y_1', \ldots, y_1, u_j, u_{j+1}, \ldots, u_t, u_{h-1}, u_{h-2}, \ldots, u_1, u_{h+2}, \ldots, u_{j-1})$ is longer than P. (Of course, if $h = j-2$ then the path actually is $(x_1, y_1', \ldots, y_1, u_j, u_{j+1}, \ldots, u_t, u_{h-1}, u_{h-2}, \ldots, u_1)$.) This contradiction completes the proof. $\diamond$

If $|F_1| = 1$, then $\mathbf{T}$ is a caterpillar and all requirements of Option 3 are satisfied as in the case $k \geq 4$. In the case that $|F_1| = 2$, we define $\mathbf{T}' = \mathbf{T} - x_1y_1 + x_1u$ for some $u \in N_2$. Note that $\deg_{\mathbf{T}'}(w) \leq \deg_{\mathbf{T}}(w)$ for all $w \in V - \{u\}$. Since $\deg_{\mathbf{T}}(w) \leq 2$ for all $w \in V - \{a_0\}$, we have $\deg_{\mathbf{T}'}(u) \leq 3$ and $\deg_{\mathbf{T}'}(w) < 3$ unless $w \in \{u, a_o\} \subset M$, an independent set. Thus $\mathbf{T}'$ satisfies Option 3.

Finally, we show that $\mathbf{G}$ contains a dominating cycle. Choose $u_i \notin X$ with i minimum, and $u_j \notin X$ with j maximum. Then Conclusion 6 allows us to apply Fact 2. Parts (a) and (b) imply that every $u_{i'}$, $1 \leq i' < i$, has a neighbor in $\{u_i, u_{i+1}, \ldots, u_j\}$, while parts (a) and (c) imply that every $u_{j'}$, $j < j' \leq t$, has a neighbor in $\{u_i, u_{i+1}, \ldots, u_j\}$. Notice that Conclusions 1 and 2 imply that $i < j$ and that a_0 is on the subpath of P from u_i to u_j. Hence $C = (x_1, u_i, u_{i+1}, \ldots, u_j)$ is a cycle and is dominating. This completes the proof of Theorem 1.4. $\square$

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