On the upper and lower chromatic numbers of BSQSs(16) *

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Abstract

A mixed hypergraph is characterized by the fact that it possesses $\mathcal{C}$-edges as well as $\mathcal{D}$-edges. In a colouring of a mixed hypergraph, every $\mathcal{C}$-edge has at least two vertices of the same colour and every $\mathcal{D}$-edge has at least two vertices coloured differently. The upper and lower chromatic numbers $\bar{\chi}$, χ are the maximum and minimum numbers of colours for which there exists a colouring using all the colours. The concepts of mixed hypergraph, upper and lower chromatic numbers are applied to $SQSs$. In fact a BSQS is an SQS where all the blocks are at the same time $\mathcal{C}$-edges and $\mathcal{D}$-edges. In this paper we prove that any $BSQS(16)$ is colourable with the upper chromatic number $\bar{\chi} = 3$ and we give new information about the chromatic spectrum of BSQSs(16).

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1 Introduction

A *mixed hypergraph* [11, 12] is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$, where X is a finite set of vertices, while $\mathcal{C}$ and $\mathcal{D}$ are two families of subsets of X. The elements of $\mathcal{C}$ and $\mathcal{D}$ are called *$\mathcal{C}$-edges* and *$\mathcal{D}$-edges* respectively. If $\mathcal{C} = \emptyset$ then $\mathcal{H}$ is called a *$\mathcal{D}$-hypergraph*, while if $\mathcal{D} = \emptyset$ then $\mathcal{H}$ is called a *$\mathcal{C}$-hypergraph*.

A *strict k-colouring* of $\mathcal{H}$ is a vertex colouring where any $\mathcal{C}$-edge has at least two vertices of the same colour and any $\mathcal{D}$-edge has at least two vertices coloured differently, and exactly k colours are used in it. If it is not necessary to know the number of used colours then a strict k-colouring will be called *strict colouring*.

The minimum (maximum) k for which there exists a strict k-colouring is called the *lower* (*upper*) *chromatic number* of $\mathcal{H}$ and is denoted by χ ($\bar{\chi}$). If there exists no strict colouring of $\mathcal{H}$, then $\mathcal{H}$ is said to be *uncolourable*.

Two strict colourings of $\mathcal{H}$ are *different* if there exist two vertices such that in one colouring they have different colours and with the same colour in the other one. Let us denote r_k as the number of different strict colourings of $\mathcal{H}$ using k colours; we will call the vector $R(\mathcal{H}) = (0, 0, \ldots, r_\chi, \ldots, r_{\bar{\chi}}, \ldots, 0, 0)$ the *chromatic spectrum* of $\mathcal{H}$ [12, 6].

The set $L \subseteq X$ is called a *$\mathcal{C}$-stable set* (*$\mathcal{D}$-stable set*) if it does not contain any $\mathcal{C}$-edges ($\mathcal{D}$-edges). If L does not contain $\mathcal{C}$-edges and $\mathcal{D}$-edges at the same time then it is called a *bi-stable set*.

A $\mathcal{D}$-hypergraph is a classical hypergraph [1] and its lower chromatic number is the chromatic number introduced by Erdős and Hajnal in 1966 [3].

In this paper the concepts of strict colouring and upper and lower chromatic numbers are applied to a particular *t-design* called a *Steiner quadruple system*.

Briefly, by a t-design $S_\lambda(t, k, v)$ we mean a pair (X,B) where X is a v-set of vertices and B is a collection of k-element subsets of X, usually called *blocks* such that every t-element subset of X occurs in exactly λ blocks of B. When $\lambda = 1$ the t-design is referred to as a Steiner system $S(t, k, v)$.

A Steiner Quadruple System (briefly SQS) is an $S(3, 4, v)$. In 1960 Hanani [5] proved that an $SQS(v)$ exists if and only if $v \equiv 2$ *or* $4 \pmod 6$.

A *BSQS* is a Steiner Quadruple System where all the blocks are at the same time $\mathcal{C}$-edges and $\mathcal{D}$-edges [8].

In [8, 9, 10] the necessary conditions for the existence of strict colourings for $BSQS$s were determined and the exact value of the upper chromatic number for $BSQS(8)$, $BSQS(10)$ was found in [9].

In [9] it was proved that for $BSQSs(16)$ obtained by doubling construction $\bar{\chi} = 3$ or they are uncolourable.

The aim of this paper is to prove that any $BSQS(16)$ is colourable with $\bar{\chi} = 3$ and we will show that only two kinds of chromatic spectrum can exist for a BSTS(16).

In the next section we will use the following theorem, proved in [4] for more general Steiner systems than SQSs.

Theorem 1 *In an SQS(v) (X,B), if $H \subseteq X$, $|H| = h$, T_H is the set of blocks included in H, and T_{X-H} is the set of blocks included in $X - H$, then*

1. $f(v,h) = |T_H| - |T_{X-H}|$, $f(v,h) = b_0 - \binom{v-h}{1} = b_1 + \binom{v-h}{2} b_2 - \binom{v-h}{3}$,
   where $b_i = \binom{v-i}{3-i} / \binom{4-i}{3-i}$ *for* $0 \leq i \leq 2$;

2. *If* $d_H(x) = |\{b \in B : \ x \in b \text{ and } b \subseteq H\}|$ *and*
   $\delta_H(x) = |\{b \in B : \ x \in b - \{x\} \text{ and } b \subseteq X - H\}|$, *then*

$$d_H(x) + \delta_H(x) = f(v,h) - f(v,h-1).$$

□

2 Upper chromatic number of BSQS(16)

Let us assume that a BSQS is colourable with the strict colouring $\mathcal{P}$ which uses h colours, the *colouring class* X_i is the vertex set coloured with the colour (i) and $|X_i| = n_i$. It is clear that $\mathcal{P}$ partitions the vertex set into a family of bi-stable sets. Let us consider the class of strict k-colourings which partition the vertex set into k colouring classes with cardinalities $n_1, \ n_2, \cdots, n_k$. We will denote this class with the h-tuple $(n_1, n_2, \cdots, n_h)$ where $n_i \leq n_{i+1}$ for $1 \leq i \leq h$. $\mathcal{P}$ colours the blocks of BSQS in three possible ways:

$i)$ Three vertices are coloured with one colour and the other vertex is coloured with another one;

$ii)$ Two vertices are coloured with one colour and the other two vertices are coloured with another one;

$iii)$ Two vertices are coloured with one colour and the other two vertices are coloured with two different colours different from the first.

If $I \subseteq \{1, 2, \cdots, h\}$, with $|I| \geq 2$, then let us define the set of vertices $\mathcal{S}_I \subseteq X$ as the union of $|I|$ colouring classes coloured with the colours *(i)* inside I. The number of triples of vertices coloured with different colours of $\mathcal{P}$ in $\mathcal{S}_I$, if $|\mathcal{S}_I| = s_I$, is:

$$c_I = \binom{s_I}{3} - \left[\sum_{j \in I} \binom{n_j}{3} + \sum_{j \in I} \binom{n_j}{2} (s_I - n_j) \right],$$

by the definitions of strict colouring and SQS we can give the following proposition.

Proposition 1 *If $\mathcal{P}$ is a strict colouring for a BSQS which uses h colours, then*
1) $\sum_{i=1}^{h} \binom{n_i}{3}$ *blocks are coloured as in i);*
2) ($c_{I'}$ / 2) *blocks are coloured as in iii), where* $I' = \{1, 2, \cdots, h\}$*;*
3) $|B| - \dfrac{c_{I'}}{2} - \sum_{i=1}^{h} \dbinom{n_i}{3}$ *blocks are coloured as in ii);*
4) all the c_I *have to be even.* □

The following theorem proves that any BSQS(16) is uncolourable with four colours.

Theorem 2 *If a* $BSQS(16)$ *is colourable then* $\bar{\chi} \leq 3$.

Proof.

In [9] it was proved that $\bar{\chi} \leq 4$ and if a BSQS(16) is colourable with four colours then it is necessary to use one of the following strict colourings: $p = (2, 4, 5, 5)$ or $k = (2, 3, 5, 6)$. We will prove that the colourings p and k are not strict colourings and $\bar{\chi} = 3$. Let $X = \{1, 2, \ldots, 16\}$ be the set of vertices of $BSQS(16)$.

If $p = (2, 4, 5, 5)$ is a strict colouring, by theorem 1 we have $|T_{X_1 \cup X_2 \cup X_3}| = 25$. However, the number of 3-chromatic blocks coloured with the colours *(1)*, *(2)* and *(3)* is 20 and the number of bi-chromatic blocks coloured with the colours *(2)* and *(3)* is at least 7 and so $|T_{X_1 \cup X_2 \cup X_3}| \geq 27$ and this is absurd.

Let us consider a strict colouring $k = (2, 3, 5, 6)$.

We have $|T_{X_1 \cup X_2 \cup X_3}| = 15$, so all the blocks of $T_{X_1 \cup X_2 \cup X_3}$ are 3-chromatic. It is important to note that a monochromatic pair of vertices coloured with the colour *(1)*, *(2)* and *(3)* is present at least once in the blocks of $T_{X_1 \cup X_2 \cup X_3}$, and precisely 13 of the 14 above-mentioned monochromatic pairs are present exactly once while the remaining pair is present twice.

Let us define the colouring classes in this way: $X_1 = \{1, 2\}$, $X_2 = \{3, 4, 5\}$, $X_3 = \{6, 7, 8, 9, 10\}$ and $X_4 = \{11, 12, 13, 14, 15, 16\}$. All the blocks of $T_{X_1 \cup X_2 \cup X_3}$ contain the $\{1\}$ vertex or the $\{2\}$ vertex; more precisely: seven blocks contain $\{1\}$ and not $\{2\}$; seven blocks contain $\{2\}$ and not $\{1\}$ and exactly one block contains the pair $\{1, 2\}$. We may assume that $\{1, 2, 3, 6\} \in T_{X_1 \cup X_2 \cup X_3}$ and that the pair $\{3, 4\}$ appears in a block containing $\{1\}$, so without loss of generality we can share the blocks of $T_{X_1 \cup X_2 \cup X_3}$ in the following way:

$\{1,2,3,6\}$	$\{2,3,8,9\}$
$\{1,3,4,7\}$	$\{2,3,7,10\}$
$\{1,3,5,8\}$	$\{2,4,5,\cdot\}$
$\{1,3,9,10\}$	$\{2,4,\cdot,\cdot\}$
$\{1,4,6,9\}$	$\{2,4,\cdot,\cdot\}$
$\{1,4,8,10\}$	$\{2,5,\cdot,\cdot\}$
$\{1,5,7,9\}$	$\{2,5,\cdot,\cdot\}$
$\{1,5,6,10\}$	

Table 1

We cannot add the pairs $\{6,7\}$, $\{6,8\}$ and $\{7,8\}$ inside the incomplete blocks of *Table* 1 without violating the definition of SQS. □

The next theorem will prove that all BSQSs(16) are colourable with strict colourings which use three colours. This theorem is important because it shows that uncolourable BSQSs(16) do not exist.

Theorem 3 *For any $BSQS(16)$, $\bar{\chi} = 3$.*

Proof.

Let $(S,\mathcal{B})$ be a $BSQS(16)$. It is not difficult to see that we can find a bi-stable set N_1 with $|N_1| = 6$. Let us assume $N_1 = \{1,2,3,4,5,6\}$. If $H = \{7,8,9,\dots,16\}$ then $|T_H| - |T_{N_1}| = 15$ and so $|T_H| = 15$.

Let $h_1 \in H$ be such that $d_H(h_1) = max\{d_H(h) : h \in H\}$ and let us assume $h_1 = 7$. By *2.* of theorem 1 we have $d_H(7) = 6 + x$, with $0 \leq x \leq 2$. If $H' = H - \{7\}$ then $|T_{H'}| = 9 - x$. Let $h_2 \in H'$ such that $d_{H'}(h_2) = max\{d_{H'}(h) : \ h \in H'\}$ and suppose $h_2 = 8$. We have $d_{H'}(8) \geq \frac{4(9-x)}{9} = 4 - \frac{4}{9}x$. Let $d_{H'}(8) = 4 + y$. The pair $\{7,8\}$ is obviously contained in at least one block of T_H and so $0 \leq y \leq 3$. Let us denote with $\mathcal{B}^\star$ the set of blocks b of T_H such that $b \cap \{7,8\} = \emptyset$, obviously $|\mathcal{B}^\star| = 5 - (x + y)$. Let us consider the following cases.

Case 1: $x + y = 5$. The sets of vertices N_1, $\{7,8\}$, $\{9,10,\dots,16\}$ give a strict colouring for $BSQS(16)$.

Case 2: $x + y = 4$. If $b = \{9,10,11,12\}$ is the only block of $\mathcal{B}^\star$, then N_1, $\{7,8,9\}$, $\{10,11,\dots,16\}$ give a strict colouring for $BSQS(16)$.

Case 3: $x + y = 3$. Let b_1, b_2 be the blocks of $\mathcal{B}^\star$. If $b_1 \cap b_2 = \emptyset$, then we can assume that $b_1 = \{9,10,11,12\}$ and $b_2 = \{13,14,15,16\}$. Let $\{7,8,9,k\} \in \mathcal{B}$, then it is possible to find $t \in b_2$, $t \neq k$, such that N_1, $\{7,8,9,t\}$, $\{10,11,\ldots,16\} - \{t\}$ give a strict colouring for $BSQS(16)$.

Case 4: $x+y=2$. Let $\mathcal{B}^\star = \{b_1, b_2, b_3\}$. We can assume that there exists an element, let us say 9, such that $9 \in b_1 \cap b_2$. If $9 \in b_3$ then N_1, $\{7,8,9\}$ and $\{10,11,\ldots,16\}$ give a strict colouring for $BSQS(16)$; otherwise, let $b_3 = \{10,11,12,13\}$ and $\{7,8,9,k\} \in \mathcal{B}$, then it is possible to find $t \in b_3$, $t \neq k$, such that N_1, $\{7,8,9,t\}$, $\{10,11,\ldots,16\} - \{t\}$ give a strict colouring for $BSQS(16)$.

Case 5: $x+y=1$. Let $\mathcal{B}^\star = \{b_1, b_2, b_3, b_4\}$. If $b_1 \cap b_2 \cap b_3 \cap b_4 \neq \emptyset$ (let us assume $9 \in b_1 \cap b_2 \cap b_3 \cap b_4$) then N_1, $\{7,8,9\}$, $\{10,11,\ldots,16\}$ give a strict colouring for $BSQS(16)$. If $b_1 \cap b_2 \cap b_3 \cap b_4 = \emptyset$, then we have two possibilities.

i) There are three blocks of $\mathcal{B}^\star$, let us say b_1, b_2, b_3, such that $b_1 \cap b_2 \cap b_3 \neq \emptyset$. We can assume that $9 \in b_1 \cap b_2 \cap b_3$ $b_4 = \{10,11,12,13\}$. If $\{7,8,9,k\} \in \mathcal{B}$, then it is possible to find $t \in b_4$, $t \neq k$, such that N_1, $\{7,8,9,t\}$, $\{10,11,\ldots,16\} - \{t\}$ give a strict colouring for $BSQS(16)$.

ii) For every $z \in \{9,10,\ldots,16\}$ there are exactly two blocks of $\mathcal{B}^\star$ containing z. Then we can assume that

$$b_1 = \{9,\cdot,\cdot,\cdot\}, \quad b_2 = \{9,\cdot,\cdot,\cdot\}$$

$$b_3 = \{10,\cdot,\cdot,\cdot\}, \quad b_4 = \{10,\cdot,\cdot,\cdot\}$$

If $\{7,8,9,10\} \notin \mathcal{B}$, then N_1, $\{7,8,9,10\}$, $\{11,12,\ldots,16\}$ give a strict colouring for $BSQS(16)$. Let $\{7,8,9,10\} \in \mathcal{B}$, we can assume, without loss of generality, that

$$b_3 = \{10,11,12,13\}$$

$$b_4 = \{10,14,15,16\}$$

and so,

$$b_1 = \{9,11,12,14\}$$

$$b_2 = \{9,13,15,16\}.$$

If $\{7,8,11,15\} \notin \mathcal{B}$, then N_1, $\{7,8,11,15\}$, $\{9,10,12,13,14,16\}$ give a strict colouring for $BSQS(16)$; otherwise N_1, $\{7,8,11,16\}$, $\{9,10,12,13,14,15\}$ give a strict colouring for $BSQS(16)$.

Case 6: $x+y=0$. From the hypothesis on d_7 and d_8 we have:

1. $d_z = 6$, for $\forall z \in H$;

2. any pair of distinct elements of H is contained exactly in two blocks of T_H.

So T_H is one of the three nonisomorphic $S_2(2,4,10)$ designs [7].

I	II	III
$\{7,8,9,10\}$	$\{7,8,9,10\}$	$\{7,8,9,10\}$
$\{7,8,11,12\}$	$\{7,8,11,12\}$	$\{7,8,11,12\}$
$\{7,9,11,13\}$	$\{7,9,11,13\}$	$\{7,9,13,14\}$
$\{7,10,14,15\}$	$\{7,10,14,15\}$	$\{7,10,15,16\}$
$\{7,12,14,16\}$	$\{7,12,14,15\}$	$\{7,11,13,15\}$
$\{7,13,15,16\}$	$\{7,13,15,16\}$	$\{7,12,14,16\}$
$\{8,9,14,15\}$	$\{8,9,14,16\}$	$\{8,9,15,16\}$
$\{8,10,13,16\}$	$\{8,10,11,16\}$	$\{8,10,13,14\}$
$\{8,11,14,16\}$	$\{8,11,14,15\}$	$\{8,11,14,16\}$
$\{8,12,13,15\}$	$\{8,12,13,15\}$	$\{8,12,13,15\}$
$\{9,10,12,16\}$	$\{9,10,12,15\}$	$\{9,10,11,12\}$
$\{9,11,15,16\}$	$\{9,11,15,16\}$	$\{9,11,14,15\}$
$\{9,12,13,14\}$	$\{9,12,13,14\}$	$\{9,12,13,16\}$
$\{10,11,12,15\}$	$\{10,11,12,16\}$	$\{10,11,13,16\}$
$\{10,11,13,14\}$	$\{10,11,13,14\}$	$\{10,12,14,15\}$

In any case N_1, $\{7,8,9,11,14\}$, $\{10,12,13,15,16\}$ give a strict colouring for these $BSQSs(16)$. □

In [2] it is shown that it is possible to construct BSQSs(16) which contain blocking sets and BSQSs(16) which do not contain any blocking set. So BSQSs(16) colourable with strict colourings which use two and three colours exist and only "3-strict" colourable BSQSs(16) also exist.

There are two classes of BSQS(16): the first class contains BSQSs(16) with $\chi = \bar{\chi} = 3$; the second one contains BSQSs(16) with $\chi = 2$ or $\bar{\chi} = 3$.

The next theorem completes theorem 3 and gives important information about the chromatic spectrum of BSQSs(16).

Theorem 4 *The upper chromatic number for all BSQSs(16) is $\bar{\chi} = 3$; the lower chromatic number for a BSQS(16) can be $\chi = 3$ or $\chi = 2$.* □

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