## The hexad game.

Joseph Kahane,
Dept of Mathematics,
Queens College, CUNY
Flushing, NY 11367,
e-mail: jokqc@qcunix1.qc.edu

and

Alexander J. Ryba,
Dept of Mathematics,
Queens College, CUNY
Flushing, NY 11367,
e-mail: ryba@qcunix1.qc.edu

Submitted: February 25, 2000; Accepted: May 18, 2000.

**Abstract.**

We give a non-enumerative proof that the $P$-positions of the hexad game give the Steiner system $S(5,6,12)$. We show that the distributions of nim sums and Welter values of these $P$-positions have simple and surprising regularities.

### 1. Introduction.

The hexad game, also known as mathematical blackjack, is played on a linear board ruled into twelve squares, labelled $0, 1, 2, \ldots, 10, 11$. Six coins are initially placed on distinct squares whose labels have a sum of at least 21. Each player at his turn selects a coin and moves it to a smaller unoccupied square, possibly skipping over several coins, with the restriction that the sum of the 6 occupied squares must remain greater than or equal to 21. The first player unable to move — when the sum of the occupied squares has just reached 21 — is the loser.

The hexad game was first investigated by Conway and Ryba; their result that the $P$-positions form the hexads of the shuffle numbered Steiner system $S(5,6,12)$ is reported in [CS]. We shall give a new proof of this result, our proof is the only known argument that does not simply calculate the Grundy values of all 905 legal positions in the hexad game. Our proof makes use of certain properties of the Steiner system, we will review these in Section 2.

There is a close similarity between the hexad game and Welter's game [We1], [We2]. Indeed, if we strike the clauses that mention the number 21 from our earlier description of the hexad game, we obtain a description of a case of Welter's game. This similarity leads us to investigate the Welter values and nim sums of the Steiner hexads. We discover two surprising regularities in the distributions of these values. These regularities seem as striking as the known regularity in the distribution of sums of Steiner hexads.

## 2. The Steiner system.

A detailed description of the Steiner system $S(5,6,12)$ is available in Sections 11.16 to 11.18 of [CS2]. We give a brief summary in order to establish notation and terminology for our paper. Proofs of our assertions are either given or referenced in [Co], [CS2], [ATLAS].

A Steiner system $S(5,6,12)$ is a collection of special six point subsets (hexads) of a twelve point set $\omega$ such that every five point subset of $\omega$ is contained in exactly one special hexad. It turns out that under the symmetric group $S_{12}$ of permutations of $\omega$ there is just one orbit of systems of hexads with this property. In other words, up to permutation of (the names of) its points, there is just one $S(5,6,12)$ on $\omega$. The $S_{12}$-stabilizer of a Steiner system is the group $M_{12}$ of order 95040.

A standard Steiner system is exhibited in [CS2]. This system is described with reference to an arrangement of points of $\omega$ into a $3 \times 4$ rectangle, called the Minimog. The hexads of the Steiner system are described by 132 convenient patterns of points in the Minimog. Although we shall not need to know all of these patterns, we will need their column distributions. We define the column distribution of a subset of $\omega$ to be the multiset of sizes of its intersections with the four Minimog columns. For example, the the column distribution of a hexad is one of the following four possibilities: $3^2 0^2, 1^3 3^1, 2^2 1^2, 2^3 0^1$. The hexads do include all patterns with distribution $3^2 0^2$ (pairs of Minimog columns), but only certain representatives of the patterns with other possible column distributions.

If we attach the numbers of $\Omega = \{0, 1, 2, \ldots, 10, 11\}$ to points of $\omega$ we obtain a "labelling" of the Steiner system. We can then describe the hexads as subsets of $\Omega$. A particularly fruitful choice, called the shuffle labelling, has the convenient property that all six point subsets of $\Omega$ that sum to 21 are hexads, there are eleven such subsets. Under the shuffle labelling, all 132 hexads of the Steiner system have point sums between 21 and 45.

For the remainder of this paper, we shall work only with the shuffle labelled $S(5,6,12)$. We will reserve the term hexad for sets of six numbers of $\Omega$ that give a shuffle labelled hexad. We shall refer to other six element subsets of $\Omega$ as non-special hexads, if we are sure that they are not hexads, or as 6-sets, if their status with respect to the Steiner system is undetermined. In [CS2], the shuffle labelled hexads are identified by numbering the Minimog as follows:

| | | | |
|---|---|---|---|
| 6 | 3 | 0 | 9 |
| 5 | 2 | 7 | 10 |
| 4 | 1 | 8 | 11 |

We could apply any permutation of $M_{12}$ without changing the Steiner system, hence there are 95039 other ways to number the Minimog to produce the same labelled Steiner system. We shall refer to such a numbering as a choice of (Minimog) co-ordinates.

Since $M_{12}$ is transitive on hexads, we can choose co-ordinates so that any given hexad forms the left pair of columns (the left brick) of the Minimog. Moreover, the $M_{12}$-stabilizer of a hexad is transitive on the hexad, so we can choose Minimog co-ordinates that arrange the points of a specified hexad in any desired order within the left brick.

If $N$ is a non-special hexad, and $i$ is a point of $N$, we can extend $N \setminus i$ to a (special) hexad (by the Steiner property). We write the extending element as $N(i)$, and we shall refer to $N(i)$ as the correction of $N$ at $i$. We note that the six corrections of a non-special hexad, since they must be distinct, form its complement in $\Omega$.

The positions of the hexad game are described by 6-sets (giving the locations of the 6 coins). The rules of the hexad game are plainly invariant under a useful duality: for any 6-set $\{x_1, x_2, \dots x_6\}$ we define the dual 6-set as the complement of $\{11-x_1, 11-x_2, \dots 11-x_6\}$. This duality is nothing more than the game played on the complementary board with coins moving to the right. It preserves sums of 6-sets and is well known to preserve the hexads of the Steiner system. It is surprising that although the hexad game has no other symmetries, its $P$-positions give a highly symmetric set: our next theorem shows that they are preserved by all of the 95040 symmetries of $M_{12}$.

## 3. The $P$-positions of the hexad game.

We now give our new, non-enumerative proof of:

**Theorem.** *The $P$-positions of the hexad game are the (special) hexads of the shuffle numbered Steiner system $S(5, 6, 12)$.*

We note that the terminal positions of the game all have sum 21 and are therefore hexads. Moreover, it is impossible to move from one hexad to another in the game (because a pair of hexads contain at most four common points). Hence to show that the hexads are the $P$-positions, it suffices to show that every non-special hexad that is a legitimate position of the game admits a correction that is a decrease. (Note that every correction does result in a legitimate position of the hexad game, since the special hexads all have point sums of at least 21.) Accordingly, we can complete the proof of our theorem with part (e) of the following Lemma.

**Lemma.** *Suppose that $N$ is a non-special hexad for which every correction is an increase. Then:*

**(a).** *$N$ contains the point $0$ but does not contain the point $11$.*

**(b).** *$N$ meets $\{0, 1, 2, 3, 4\}$ in at least three points.*

**(c).** *$N$ meets $\{0, 1, 2, 3, 4\}$ in more than three points.*

**(d).** *If $N$ meets $\{0, 1, 2, 3, 4\}$ in four points then $10 \notin N$.*

**(e).** *The sum of the points of $N$ is at most 21.*

Proof. Let $S = \{0, 1, 2, 3, 4\}$ be the set of "small values" in $\Omega$, we note that $H = S \cup \{11\}$ is a hexad (since its points sum to 21).

(a) If 11 was in $N$, we would have $11 > N(11)$. Similarly, if 0 was not in $N$, it would be the correction of some $i \in N$ with $i > 0 = N(i)$.

(b) If $|N \cap S| \leq 2$ then at least three points of $S$ are corrections of points of $N$. At least one of these corrections must be from a point $i$ of $N \setminus S$. We would then have $i > N(i)$.

(c) If $|N \cap S| = 3$, we choose Minimog co-ordinates with left brick columns $N \cap S$ and $H \setminus N$. The column distribution of $N$ must be $3, 0, 1, 2$ ($3^2 0^2$ cannot occur since $N$ is non-special). Let $\{i, j\}$ be the two points of $N$ in a single column of the right brick. The corrections of $N$ at $i$ and $j$ are both in $H \setminus N$ (to give corrected hexads with column-distribution $3^1, 1^3$). Hence, these corrections map to a column with just one large value and two very small values. Accordingly, at least one of the two corrections has to be a decrease.

(d) Otherwise, the correction of 10 could only be 11, and the corresponding corrected hexad would meet $H$ in exactly five points (4 points in $S$ and 11), in contradiction to the Steiner property.

(e) Suppose that $N$ has a point sum in excess of 21. Then $N$ can not contain $S$ (since $21 = 0 + 1 + 2 + 3 + 4 + 11$). Hence by (c), we have $|N \cap S| = 4$; let $s \epsilon S$ be the missed point. A similar argument shows that the dual of $N$ meets $S$ in 4 points. Therefore $N$ has exactly one large entry of the form $7 + \delta$ with $0 \leq \delta \leq 2$. The sixth entry of $N$ must have a moderate size $5 + \epsilon$ with $0 \leq \epsilon \leq 1$.

Now, the sum of the points of $N$ is $(0 + 1 + 2 + 3 + 4 - s) + (5 + \delta) + (7 + \epsilon) = 21 + (1 + \delta + \epsilon) - s$. Hence, $s < 1 + \delta + \epsilon \leq 4$. We deduce that $1 + \delta + \epsilon \in N \cap S$, and therefore the point decrease $1 + \delta + \epsilon \mapsto s$ must be a correction of $N$ to a hexad with sum 21: a contradiction. ∎

## 4. Nim sums and Welter values of hexads.

We have observed that the hexad game is closely related to Welter's game (see [We1], [We2], [ONAG], [WW], [KF]). Hence it is natural to ask what are the Welter values of the hexads? The following lemma shows that it is also very natural to ask what are the nim sums of the hexads?

**Lemma N.** *If $X$ is a set of six distinct non-negative integers with Welter value $w$ and nim sum $n$, then $w + 1 \equiv n \pmod 8$.*

Proof. Following [ONAG] we set the mating function $(x|y)$ of a pair of distinct non-negative integers as $2^{i+1} - 1$ where $2^i$ is the highest power of 2 dividing $x - y$.

Write $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ so that $(x_1|x_2)$ is at least as large as the mating function of any other pair, and $(x_3|x_4)$ is at least as large as the mating function of any other pair from $X \setminus \{x_1, x_2\}$. We recall from [ONAG] that $w = n \oplus (x_1|x_2) \oplus (x_3|x_4) \oplus (x_5|x_6)$ (where we overload the operator $\oplus$ to denote a nim sum).

The pigeonhole principle applies (using the four residue classes mod 4) to show that $4|(x_1 - x_2)$. Hence $(x_1|x_2) \equiv 7 \pmod 8$. A similar argument shows that $(x_3|x_4) \equiv 3 \pmod 4$.

We note that if $n$ is odd, then $x_5$ and $x_6$ have opposite parity so that $(x_5|x_6) = 1$. We also observe that one of $x_5$ and $x_6$ must share the parity of $x_3$ and $x_4$. Another appeal to the pigeonhole principle will now show that $x_3 \equiv x_4 \pmod 4$ so that $(x_3|x_4) \equiv 7 \pmod 8$. Hence $w = n \oplus 1 \pmod 8$, and since $n$ is odd $w = n - 1 \pmod 8$.

Alternatively, if $n$ is even, $(x_3|x_4) \geq (x_5|x_6) \geq 3$. In the case where $(x_5|x_6) \geq 7$, we have $n \equiv 0 \pmod 4$. That is, the last two binary digits of $n$ must be 00. Therefore,

$n-1 \equiv n \oplus 7 \pmod 8 = n \oplus 7 \oplus 7 \oplus 7 \pmod 8 \equiv n \oplus (x_1|x_2) \oplus (x_3|x_4) \oplus (x_5|x_6) \pmod 8 = w \pmod 8$. In the case where $(x_5|x_6) = 3$ but $(x_3|x_4) \geq 7$ we have $n \equiv 2 \pmod 4$. That is, the last two binary digits of $n$ must be 10. Therefore, $n-1 \equiv n \oplus 3 \pmod 8 = n \oplus 7 \oplus 7 \oplus 3 \pmod 8 \equiv n \oplus (x_1|x_2) \oplus (x_3|x_4) \oplus (x_5|x_6) \pmod 8 = w \pmod 8$. Finally, if $(x_3|x_4) = 3$, the pigeonhole argument shows that $x_3$, $x_4$, $x_5$, and $x_6$ must represent a complete set of residues modulo 4. It follows that $n \equiv 0 \pmod 4$ and, as above, $n-1 \equiv n\oplus 7 \pmod 8 = n\oplus 7\oplus 3\oplus 3 \pmod 8 \equiv n\oplus(x_1|x_2)\oplus(x_3|x_4)\oplus(x_5|x_6) \pmod 8 = w \pmod 8$. ∎

For any integer $i$ with $0 \leq i \leq 15$, we write $H_w(i)$ for the number of hexads with Welter value $i$, and $H_n(i)$ for the number of hexads with nim sum $i$. Our following theorems show that these tallies of nim sums and Welter values do obey a surprising regularity. We note that one of the previously observed mysterious features of the shuffle numbering is a regularity in the tallies of sums of hexads.

It is useful to consider analogous tallies of nim sums and Welter values of duads. Accordingly, we write $D_n(i)$ (respectively $D_w(i)$) for the number of duads in $\Omega$ with nim sum $i$ (respectively Welter value $i$). It is easy to determine the values of $D_n(i)$. For example, if we classify integers between 0 and 15 as $S$ (small, 0 to 3), $M$ (moderate, 4 to 7), $L$ (large, 8 to 11), or $E$ (enormous, 12 to 15), we have the following classification of nim sums. (The table just reflects the obvious group homomorphism from $(\{0, 1, \ldots, 15\}, \oplus)$ to the elementary abelian group on $\{S, M, L, E\}$.)

| $\oplus$ | $S$ | $M$ | $L$ | E |
|---|---|---|---|---|
| $S$ | $S$ | $M$ | $L$ | E |
| $M$ | $M$ | $S$ | $E$ | L |
| $L$ | $L$ | $E$ | $S$ | M |
| $E$ | $E$ | $L$ | $M$ | S |

The top left $3 \times 3$ square in the table corresponds to nim sums of ordered pairs of $\Omega$. Hence, lifting back to $\{0, 1, \ldots, 15\}$ and restricting to nim sums of unordered duads we find that $D_n(i)$ is 0, 6, or 4 according as $i = 0$, $1 \leq i \leq 3$, or $4 \leq i \leq 15$. Moreover, since the Welter value of a duad is one less that the nim sum [ONAG] we deduce that that $D_w(i)$ is 6, 4, or 0 according as $0 \leq i \leq 2$, $3 \leq i \leq 14$, or $i = 15$.

**Theorem N.** $H_n(i) = 2D_n(i)$.

Proof. There is a 2 to 1 map from the 132 hexads to the $66 = C(12, 2)$ duads of $\Omega$ such that:
• Complementary hexads map to the same duad.
• If $h_1$ and $h_2$ are hexads with $|h_1 \cap h_2| = 3$, then the corresponding duads $d_1$ and $d_2$ have $|d_1 \cap d_2| = 1$, and $h_1 \oplus h_2$ is a hexad that corresponds to $d_1 \oplus d_2$.
• The eleven hexads that sum to 21 correspond to a chain of duads of the form $\{a_0, a_1\}$, $\{a_1, a_2\}$, …, $\{a_{10}, a_{11}\}$. Moreover, by adjusting the hexad-duad correspondence appropriately, we can achieve any desired bijection between $\Omega$ and $\{a_0, a_1, \ldots, a_{11}\}$.
We shall choose the hexad-duad correspondence to extend the following assignment:

$\{1,2,3,4,5,6\} \mapsto \{\overline{0},\overline{7}\}$
$\{0,1,2,3,7,8\} \mapsto \{\overline{7},\overline{8}\}$
$\{0,1,2,4,5,9\} \mapsto \{\overline{8},\overline{3}\}$
$\{0,1,3,4,6,7\} \mapsto \{\overline{3},\overline{4}\}$
$\{0,1,2,3,5,10\} \mapsto \{\overline{4},\overline{11}\}$
$\{0,1,2,4,6,8\} \mapsto \{\overline{11},\overline{2}\}$
$\{0,2,3,4,5,7\} \mapsto \{\overline{2},\overline{5}\}$
$\{0,1,2,3,6,9\} \mapsto \{\overline{5},\overline{10}\}$
$\{0,1,3,4,5,8\} \mapsto \{\overline{10},\overline{1}\}$
$\{0,1,2,5,6,7\} \mapsto \{\overline{1},\overline{6}\}$
$\{0,1,2,3,4,11\} \mapsto \{\overline{6},\overline{9}\}$

(We have used the permutation $(\overline{1},\overline{7},\overline{5},\overline{11},\overline{9})(\overline{2},\overline{8},\overline{10},\overline{6})$ to relabel the right hand duads given in [CS2]). Our choice of correspondence has the property that the nim sum of the two numbers in any of the given duads matches the nim sum of both of its corresponding hexads. (Note that complementary pairs of hexads have a common nim sum since $\bigoplus_{i\in\Omega} i = 0$.)

All duads are generated when we close our given set under the operation of forming the symmetric difference of an intersecting pair. Moreover, since the nim sum of a symmetric difference of sets is just the nim sum of the nim sums of the sets, we deduce that every duad must have the same nim sum as its corresponding hexads. Hence there is a 2 to 1 map from the multiset of nim sums of hexads to the multiset of nim sums of duads. ∎

We shall apply the particular "hexad-duad correspondence" of Theorem N repeatedly. The proof of Theorem N gives us:

**Corollary N.** *The hexad-duad correspondence preserves nim sums.*

Let $\overline{\Omega} = \{\overline{0},\overline{1},\ldots,\overline{11}\}$. The twenty-four points of $\Omega \cup \overline{\Omega}$ support a Steiner system $S(5,8,24)$ (see [ATLAS], [CS2]) whose 759 special octads can be generated out of symmetric differences of octads formed as the union of a hexad in $\Omega$ and its corresponding duad in $\overline{\Omega}$. Later, we shall use standard properties of $S(5,8,24)$ to give a short proof of our Lemma W. These properties, including the exhibition of octads and dodecads inside a $4\times 6$ array called the MOG, are explained in [CS2], [ATLAS]. It is possible to avoid the use of $S(5,8,24)$ and give an alternative easy but tedious enumerative proof of Lemma W by determining the complete hexad-duad correspondence.

We note that our particular choice of $S(5,8,24)$ is implicitly labelled, since it is constructed from our particular hexad-duad correspondence. We shall make use of this labelling to refer to appropriate subsets of $\Omega \cup \overline{\Omega}$ as octads or dodecads. Of course, as in the analogous situation of the Minimog and $S(5,6,12)$, there are many ways to label the MOG array to specify the same labelled $S(5,8,24)$, we shall call each of these a choice of MOG co-ordinates. The following choice of MOG co-ordinates is convenient:

| | | | | | |
|---|---|---|---|---|---|
| $\overline{4}$ | $4$ | $\overline{8}$ | $\overline{0}$ | $8$ | $0$ |
| $5$ | $\overline{5}$ | $2$ | $10$ | $\overline{3}$ | $\overline{11}$ |
| $6$ | $\overline{6}$ | $3$ | $11$ | $\overline{1}$ | $\overline{9}$ |
| $7$ | $\overline{7}$ | $9$ | $1$ | $\overline{10}$ | $\overline{2}$ |

(That this is a choice of MOG co-ordinates is verified by checking that our given hexad-duad pairings do give octads and that $\Omega$ gives a dodecad with respect to this labelling.)

From Corollary N, we see that the set of octads from our $S(5,8,24)$ that extend hexads do all have nim sums of zero. Since we can generate the entire $S(5,8,24)$ from this collection by using the symmetric difference operation, we deduce that every labelled octad nim sums to zero. (In computing this nim sum, we can interpret the value of an element $\overline{i}$ in $\overline{\Omega}$ as the negative integer $-1 \oplus i$ as in [ONAG].)

We shall need to pay special attention to those "mated duads" of $\Omega$ and $\overline{\Omega}$ that have a nim sum of exactly 8. The mated duads in $\Omega$ are $\{0,8\}$, $\{1,9\}$, $\{2,10\}$, $\{3,11\}$. The mated duads are visible as adjacent entries in the two right hand octads of our MOG labelling.

We shall make use of mated duads to determine the "uneven hexads" whose nim sums and Welter values differ in the 8-place of their binary expansions. The standard formula that we used in Lemma N for the Welter value (in terms of the nim sum and mating functions) shows that a hexad is uneven if and only if it contains an odd number of mated duads. Each duad $\{\overline{i},\overline{j}\}$ corresponds to a pair of complementary hexads, and we write $X_{\overline{i},\overline{j}}$ for the number of uneven hexads in this pair. Our next goal is to find a convenient formula for $X_{\overline{i},\overline{j}}$, and in order to do this we first show that a large group of symmetries preserves this quantity.

We classify elements of $\Omega$ and $\overline{\Omega}$ as moderate (resp. extreme) if they do (resp. do not) have a value in the set $\{4,5,6,7\}$.

**Lemma.** *Let $G$ be the group of permutations of $\Omega \cup \overline{\Omega}$ that preserves the Steiner system $S(5,8,24)$, the sets $\Omega$ and $\overline{\Omega}$, the moderate values, and the set of mated duads. Then $G$ has four orbits on the duads of $\overline{\Omega}$: an orbit of mated duads, an orbit of non-mated extreme duads, an orbit of moderate duads, and an orbit of moderate-extreme pairings.*

Proof. The group $G$ must have at least the indicated orbits, since for example, by its definition, $G$ can never map a moderate point to an extreme one.

The group $M_{24}$ of permutations that preserve $S(5,8,24)$ contains elements:
$(\overline{5},\overline{6},\overline{7})(5,6,7)(2,3,9)(10,11,1)(\overline{3},\overline{1},\overline{10})(\overline{11},\overline{9},\overline{2})$,
$(\overline{6},\overline{7})(6,7)(3,9)(11,1)(\overline{1},\overline{2})(\overline{9},\overline{10})(8,0)(\overline{3},\overline{11})$,
$(\overline{6},\overline{4},\overline{5})(7,5,4)(8,3,9)(0,11,1)(\overline{8},\overline{1},\overline{10})(\overline{0},\overline{9},\overline{2})$,
$(\overline{4},\overline{5})(4,5)(3,9)(11,1)(\overline{1},\overline{2})(\overline{9},\overline{10})(\overline{8},\overline{0})(2,10)$
that preserve the indicated objects and therefore generate a subgroup $H$ of $G$. The first pair of these elements are standard MOG automorphisms, illustrated, for example, in Figure 11.30 of [CS2]. The second pair are the same automorphisms with respect to the following MOG co-ordinates:

| | | |
|---|---|---|
| $6\ \overline{7}$<br>$\overline{6}\ 7$ | $\overline{3}\ \overline{11}$<br>$8\ \ 0$ | $2\ 10$<br>$\overline{8}\ \overline{0}$ |
| $\overline{4}\ 5$<br>$\overline{5}\ 4$ | $3\ 11$<br>$9\ \ 1$ | $\overline{1}\ \ \overline{9}$<br>$\overline{10}\ \overline{2}$ |

The moderate points and the extreme points are the two orbits of $H$ on $\overline{\Omega}$. Moreover, the $H$-stabilizer of a moderate point is visibly transitive on the extreme points and on the other moderate points in $\overline{\Omega}$. Similarly, the $H$-stabilizer of an extreme point has three orbits on the extreme points of $\overline{\Omega}$: the point itself, the other point in its mated duad, and the remaining six extreme points. The $H$-orbits on duads are therefore just the four stated orbits of $G$, moreover since $G$ can not have any fewer orbits, the result follows. ∎

**Lemma W.** *Suppose that $\{\overline{i}, \overline{j}\}$ is a duad of $\overline{\Omega}$, then $X_{\overline{i},\overline{j}}$ is the number of extreme values in $\{i, j\}$.*

Proof. The extreme values, and the values of $X_{\overline{i},\overline{j}}$ are preserved under the action of $G$, so we need only check the validity of the claim on duads giving orbit representatives of $G$. For example, we can use the convenient duads $\{\overline{0}, \overline{7}\}$, $\{\overline{0}, \overline{8}\}$, $\{\overline{8}, \overline{3}\}$, and $\{\overline{7}, \overline{4}\}$ that respectively correspond to the hexads $\{1, 2, 3, 4, 5, 6\}$, $\{0, 4, 5, 6, 7, 8\}$, $\{0, 1, 2, 4, 5, 9\}$, and $\{0, 1, 5, 6, 8, 9\}$. ∎

**Theorem W.** *$H_w(i) = 2D_w(i)$.*

Proof. We first note that every duad with nim sum 8 is a mated (extreme) pair. So by Corollary N and Lemma W, every hexad with nim sum 8 is uneven and therefore has Welter value 7 (using Lemma N). There are no other hexads with nim sum divisible by 8. We deduce that $H_w(7) = H_n(8) = 2D_n(8) = 2D_w(7)$ and $H_w(15) = 0 = 2D_w(15)$.

Now consider the case where $i \not\equiv 0 \pmod 8$. Lemma N shows that $H_w(i-1) = H_n(i) - U(i) + U(i \oplus 8)$, where $U(i)$ is the number of uneven hexads with nim sum $i$. Now $U(i)$ is found by summing $X_{\overline{k},\overline{l}}$ over duads that nim sum to $i$ (by Corollary N). Hence, from Lemma W we deduce that $U(i)$ is the number of extreme values $x \in \Omega$ for which $i \oplus x \in \Omega$. However, $i$ and $x$ have this relationship if and only if $i \oplus 8$ and $x \oplus 8$ have a similar relationship. Therefore, $U(i) = U(i \oplus 8)$. We conclude that $H_w(i-1) = H_n(i) = 2D_n(i) = 2D_w(i-1)$. ∎

## 5. Comments and Questions.

We have viewed the hexad game as 6-coin Welter's game played on a strip of 12 squares with the restriction that the sum of the occupied squares remain greater than or equal to 21. The restriction can be characterized in a rather more general way as follows.

In the misere version of a game, the last player to move is the loser rather than the winner as in normal play. We will make a slight modification to this definition, which does not lead to any change in the evaluation or play of any non-terminal positions. For any impartial game $G$, we write $M(G)$ (the Misere version of $G$) for the impartial game consisting of all non-terminal positions of $G$ and the same moves between positions as in $G$. Although, this definition of the $M(G)$ leads to a standard Misere evaluation of all non-terminal $G$ positions, the omission of terminal positions does cause some changes. For example, $M(M(G))$ is not the same game as $G$. We shall write $M^2(G)$ for $M(M(G))$, and more generally, we shall set $M^n(G) := M(M^{n-1}(G))$.

If $T(G)$ is the game tree of $G$ then $T(M(G))$ is gotten from $T(G)$ by pruning all the leaves (terminal positions) and their corresponding edges. Thus the newly formed leaves will now be the winning terminal positions of $T(M(G))$ (the positions that can only lead in the original to a now losing terminal position of $G$).

Let $W$ stand for the 6-coin Welter's game played on the above strip with no restriction on the sum of the occupied squares, hence, this sum must be greater than or equal to 15. It is not hard to see that $M(W)$ has the restriction that the sum of coin positions is at least 16. In this way, it follows that $M^6(W)$ is precisely our hexad game. We ask whether any general results about Misere games can be used to recover any of our properties of the hexad game? We believe that such an approach would give new insight into the arithmetic and combinatorial properties of $S(5, 6, 12)$ and its shuffle numbering. In [CS2], Conway comments that it will not be easy to explain these properties since they extend to deep results in finite group theory and lattice theory.

**References.**


[ATLAS] Conway, J. H., Curtis, R. T., Norton, S. P., Parker, R. A., Wilson, R. A., Atlas of finite groups. Oxford University Press, Oxford, 1985.

[Co] Conway, J. H., Hexacode and tetracode—MOG and MINIMOG. Computational group theory (Durham, 1982), 359–365, Academic Press, London-New York, 1984.

[CS] Conway, John H., Sloane, N. J. A., Lexicographic codes: error-correcting codes from game theory. IEEE Trans. Inform. Theory 32 (1986), no. 3, 337–348.

[CS2] Conway, J. H., Sloane, N. J. A., Sphere packings, lattices and groups. Second edition., Grundlehren der Mathematischen Wissenschaften, 290. Springer-Verlag, New York, 1993.

[KF] Kahane, Joseph, Fraenkel, Aviezri S., $k$-Welter—a generalization of Welter's game, J. Combin. Theory Ser. A 46 (1987), no. 1, 1–20.

[ONAG] Conway, J. H., On numbers and games. London Mathematical Society Monographs, No. 6. Academic Press, London-New York, 1976.

[We1] Welter, C. P., The theory of a class of games on a sequence of squares, in terms of the advancing operation in a special group., Indagationes Math. 16, (1954). 194–200.

[We2] Welter, C. P., The advancing operation in a special abelian group., Indagationes Math. 14, (1952). 304–314.

[WW] Berlekamp, Elwyn R., Conway, John H., Guy, Richard K., Winning ways for your mathematical plays., Academic Press, London-New York, 1982.