Sumsets of finite Beatty sequences

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Dedicated to Aviezri Fraenkel, with respect and gratitude.

Abstract

An investigation of the size of $S+S$ for a finite Beatty sequence $S=(s_i)=(\lfloor i\alpha+\gamma\rfloor)$, where $\lfloor\ \rfloor$ denotes "floor", α, γ are real with $\alpha\geq 1$, and $0\leq i\leq k-1$ and $k\geq 3$. For $\alpha>2$, it is shown that $|S+S|$ depends on the number of "centres" of the Sturmian word $\Delta S=(s_i-s_{i-1})$, and hence that $3(k-1)\leq|S+S|\leq 4k-6$ if S is not an arithmetic progression. A formula is obtained for the number of centres of certain finite periodic Sturmian words, and this leads to further information about $|S+S|$ in terms of finite nearest integer continued fractions.

1 Introduction

For the purposes of this paper, an *infinite sequence* is a two-way infinite sequence, that is, a sequence indexed by the set Z of all integers. An *infinite Beatty sequence* is a strictly increasing sequence of integers

$$s=(s_i)=(s_i)_{i\in Z}$$

such that for all integers i

$$s_i=\lfloor i\alpha+\gamma\rfloor\,, \tag{1}$$

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where $\lfloor\ \rfloor$ denotes "floor" or "integer part" and α, γ are fixed real numbers with $\alpha \geq 1$. Let s be such a sequence. It easily shown that for all integers i, j with $j \geq 0$ we have

$$s_{i+j} - s_i \in \{\lfloor j\alpha\rfloor,\ \lfloor j\alpha\rfloor + 1\}\,, \tag{2}$$

and in particular that for all i

$$s_{i+1} - s_i \in \{\lfloor \alpha\rfloor,\ \lfloor \alpha\rfloor + 1\}\,. \tag{3}$$

The *difference sequence* of s is the sequence

$$\Delta s = (\Delta_i)_{i\in Z}\,, \tag{4}$$

where for all i

$$\Delta_i = s_i - s_{i-1}\,. \tag{5}$$

From (3) we see that we can view Δs as a binary sequence in two symbols a and b by denoting one of $\lfloor\alpha\rfloor$, $\lfloor\alpha\rfloor + 1$ by a and the other by b. Both symbols must occur except in the special case when α is an integer. In this case $\Delta s = (\Delta_i)$ is a constant sequence and the sequence s is an infinite arithmetic progression with common difference $\alpha = \lfloor\alpha\rfloor$, or, equivalently, a residue class modulo α. Thus the concept of Beatty sequence extends that of arithmetic progression. In all cases, we shall call α the *modulus* of the sequence s given by (1).

For $k \geq 1$ a *finite Beatty sequence* S with cardinality

$$|S| = k$$

is a finite nonempty set of integers

$$S = \{s_0,\, s_1,\, \ldots,\, s_{k-1}\} \tag{6}$$

such that s_i satisfies (1) for all i in

$$I = I_k = \{0,\, 1,\, \ldots,\, k-1\}\,, \tag{7}$$

where α, γ are fixed real numbers with $\alpha \geq 1$. We shall call α a *modulus* of S. The set S and its properties are determined by the infinite Beatty sequence $s = (s_i)$. However the sequence s and its modulus are not uniquely determined by the set S.

Consider now any set S of integers such that $|S| = k \geq 1$. Let

$$S = \{s_0,\, s_1,\, \ldots,\, s_{k-1}\},$$

where

$$s_0 < s_1 < \ldots < s_{k-1}\,. \tag{8}$$

The *sumset* of S is the set

$$S+S=\{t+u:\ t\in S,\, u\in S\}\,.$$

For $k=1$, we have $|S+S|=1$, for $k=2$, $|S+S|=3$, and for $k\geq 3$ it is easily shown that

$$|S+S|\geq 2k-1, \tag{9}$$

with equality if and only if S is an arithmetic progression. As we shall see further below, arithmetic progressions play a special role in results on sets with small sumset. Since finite Beatty sequences can be regarded as a generalisation of finite arithmetic progressions, their sumsets are of special interest. In this paper, I shall give some results on the size of $S+S$ when S is a finite Beatty sequence and $|S|\geq 3$.

The results obtained will depend on the notion of a "centre" of a *binary word*. For $k\geq 1$, by a k-letter *binary word* x in two letters a and b we mean a finite sequence $(x_i)_{i\in I}$ where the index set I consists of k consecutive integers and $x_i\in\{a,\, b\}$ for all i in I.

Consider such a word x indexed by $I=\{0,\, 1,\, \ldots,\, k-1\}$, and write

$$x=x_0x_1\ldots x_{i-1}x_i\ldots x_{k-1}\,.$$

Let $i\in I$. We say that x has a *centre at* x_i if $x_{i-j}=x_{i+j}$ for all $j\geq 0$ such that $i\pm j$ both belong to I. For $i\geq 1$, we say that x has a *centre between* x_{i-1} *and* x_i if $x_{i-1-j}=x_{i+j}$ for all $j\geq 0$ such that $i-1-j$ and $i+j$ both belong to I. We can think of a centre as a position about which the word has as much mirror symmetry as possible. We note that x always has a centre at the first letter x_0 and the last letter x_{k-1}. The number of centres of x is at most $2k-1$, with equality if and only if

$$x=\underbrace{aa\ldots a}_{k\ a\text{'s}}=a^k \quad \text{or} \quad x=b^k.$$

For given α and γ with $\alpha\geq 1$ and integral $k\geq 2$, we shall consider a finite Beatty sequence S as in (6) indexed by $I=I_k$ as in (7) with s_i satisfying (1) for all i in I. We shall assume that α is non-integral and $\alpha>2$. In this situation it turns out that the size of $S+S$ is determined by the combinatorial nature of the *difference sequence*

$$\Delta S=(\Delta_1,\, \Delta_2,\, \ldots,\, \Delta_{k-1})=(s_1-s_0,\, s_2-s_1,\, \ldots,\, s_{k-1}-s_{k-2})\,, \tag{10}$$

when viewed as finite binary word.

The material in the remaining sections is arranged as follows.

After giving some further background on sets with small sumset in Section 2, I shall consider finite Beatty sequences and their sumsets in Section 3 and derive the basic result (Proposition 1) that for a finite Beatty sequence S with $\alpha>2$ and $|S|=k\geq 3$ we have

$$|S+S|=4k-4-C\,,$$

where C is the number of centres of the binary word ΔS given by (10). In Section 4, I shall consider the number of centres of a binary word and hence show, in particular, that if S as above is not an arithmetic progression then

$$3k - 3 \leq |S + S| \leq 4k - 6\,. \tag{11}$$

In Section 5, I shall give some further auxiliary results, first on *rational* Beatty sequences (those whose modulus α is rational), then on infinite periodic Sturmian sequences and their connection with the nearest integer algorithm. This will lead, in Section 6, to Proposition 3, which gives a precise formula for the number of centres of certain finite periodic Sturmian words. Application of Proposition 3 to ΔS when S is a finite Beatty sequence will then yield information about $|S + S|$ in terms of nearest integer continued fractions.

Finally, in Section 7, I shall briefly mention related results in Z^2, and suggest some possible directions of further investigation.

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$$\{\alpha i + \gamma\} = \alpha i + \gamma - \lfloor \alpha i + \gamma \rfloor\,,$$

for fixed irrational α and integral i in a specified interval. I would also like to thank the referees for their helpful comments.

2 Sets of integers with small sumset

Starting from the inequality (9), Freiman studied the structure of finite sets S of k integers for which $|S + S|$ is not too far above the minimum value $2k - 1$ and showed that they are closely related to arithmetic progressions. His precise results for the cases

$$2k - 1 \leq |S + S| \leq 3k - 3$$

are given in the following theorem, and he also obtained detailed results on the case $|S + S| = 3k - 2$.

Theorem A *(Freiman).* Let S be a finite set of k integers.

(i) Suppose $|S+S| = 2k-1+\ell$, where $0 \le \ell \le k-3$. Then there is an arithmetic progression L such that $S \subseteq L$ and $|L| = k+\ell$.

(ii) Suppose $|S+S| = 3k-3$ and $k \ge 7$. Then

either (a) there is an arithmetic progression L such that

$$S \subseteq L \quad \text{and} \quad |L| = 2k-1$$

or (b) S is a union of two arithmetic progressions with the same common difference.

Proof. See Freiman [4], Theorems 1.9 and 1.11.

Freiman also obtained a widely applicable fundamental result (now known as Freiman's Main Theorem) which gives information about the structure of S as above when $|S| = k$ and $|S+S| \le \sigma k$, where σ is a fixed real number such that $\sigma \ge 2$. Freiman's proof appeared in different versions in [4] and [5]. Intensive investigation in recent years has led to different formulations and extensions of the theorem, a new proof by Rusza [9] and significant modification by Bilu of Freiman's proof. In [1] Bilu presents his proof in the context of an exposition of the Main Theorem with full references. Nathanson [7] gives a self-contained presentation of Rusza's proof in Chapter 8 and provides extensive background to the Main Theorem.

Before formulating an appropriate special case of the Main Theorem, we need some definitions.

For given sets $A \subseteq Z^n$, $B \subseteq Z^m$, a mapping $\varphi:\ A \to B$ is an *isomorphism* if it is a bijection of A onto B such that for all x, y, z, w in A

$$x+y = z+w \Leftrightarrow \varphi(x)+\varphi(y) = \varphi(z)+\varphi(w)\,.$$

We call A and B *isomorphic* if such an isomorphism exists, and note that if A and B are isomorphic then

$$|A+A| = |B+B|\,.$$

(In the language of Bilu [1], an isomorphism as above is an F_2-isomorphism.)

An *arithmetic progression* in Z^n is a set P of the form

$$P = \{v_0 + l_1 v_1 :\ l_1 = 0, 1, \ldots, L_1 - 1\}\,,$$

where L_1 is a positive integer, v_0, v_1 are in Z^n, and v_1 is non-zero. We note that the mapping φ from $\{0,\, 1,\, \ldots,\, L_1 - 1\}$ to P given by

$$\varphi(l_1) = v_0 + l_1 v_1$$

is an isomorphism.

A *generalised arithmetic progression* of *rank at most* 2 in Z^n is of the form

$$P = \{v_0 + l_1 v_1 + l_2 v_2 :\ l_1 = 0, 1, \ldots, L_1 - 1;\ l_2 = 0, 1, \ldots, L_2 - 1\}\,,$$

where L_1, L_2 are positive integers and v_0, v_1, v_2 are in Z^n. If, further, the mapping φ from $\{0, 1, \ldots, L_1 - 1\} \times \{0, 1, \ldots, L_2 - 1\}$ to P given by

$$\varphi(l_1, l_2) = v_0 + l_1 v_1 + l_2 v_2$$

is an isomorphism, we shall say that P is *proper.* We note that in this case $|P| = L_1 L_2$ and P is a union of arithmetic progressions. (In the language of Bilu [1], P is an F_2-progression.)

In this paper we shall be mainly concerned with finite sets of integers S such that $3k \leq |S + S| \leq 4k$ (compare (11) above). Hence the following very special case of the Main Theorem is relevant.

Theorem B. (*Special Case of Main Theorem*) Let σ be a real number such that $3 \leq \sigma < 4$, k an integer such that

$$k > \frac{6}{4 - \sigma},$$

and S a set of integers such that $|S| = k$. Suppose that

$$|S + S| \leq \sigma k.$$

Then there is a set P of integers such that P is a proper generalised arithmetic progression of rank at most 2, $S \subseteq P$, and $|P| \leq ck$, where $c = c(\sigma)$ is a positive constant depending only on σ.

Proof. See Theorem 1.2 of Bilu [1] and its proof. The above result is obtained by taking $s = 2$, K a subset of the Abelian group Z, and $3 \leq \sigma < 4$ in that theorem.

3 Finite Beatty sequences and their sumsets

3.1 Infinite Beatty sequences

We shall look first at infinite Beatty sequences and their difference sequences. For this purpose, we need some vocabulary associated with a binary sequence $x = (x_i)_{i \in Z}$ in two symbols a and b. For $j \geq 1$ a j-letter *word of* x is simply a binary word in a and b of the form

$$w = x_i x_{i+1} \ldots x_{i+j-1}\,.$$

A j-letter *block* in x is a maximal word of x with all letters in it identical, that is, of the form

$$w = a^j \quad \text{or} \quad w = b^j$$

for some $j \geq 1$. A symbol, b, say, is *isolated* in x if it occurs in x ($x_i = b$ for some i) but its square does not (there is no i such that $x_i = x_{i+1} = b$).

A binary sequence $x = (x_i)_{i \in Z}$ in a and b is said to be *Sturmian* (or "two-distance" or "almost constant") if it satisfies the following *Sturmian condition:* If v and w are two words in x with the same number of letters then the number of a's in v differs from the number of a's in w by at most one. In the following proposition we gather together well known basic results on Sturmian sequences.

Proposition C. (i) Suppose s is an infinite Beatty sequence with non-integral modulus α. Then the difference sequence Δs given by (4) and (5) is non-constant and Sturmian in two symbols a and b denoting $\lfloor \alpha \rfloor$ and $\lfloor \alpha \rfloor + 1$ (in some order).

(ii) Let $x = (x_i)_{i \in Z}$ be Sturmian in two symbols a and b and suppose x is non-constant. Then at least one of a and b is isolated in x and the only case when both are isolated is the Sturmian sequence

$$x = \ldots ababa \ldots = (ab)^{\infty} . \tag{12}$$

(iii) Let $x = (x_i)_{i \in Z}$ be Sturmian in two symbols a and b. Suppose x is non-constant and not of the form (12), and let b be the isolated symbol in x. Then there is a unique integer $\nu \geq 1$ (called the *a-width* of x) such that every block of a's in x is

$$\textit{either} \quad a^{\nu} \quad \textit{or} \quad a^{\nu+1}$$

and a^{ν} occurs as a block in x.

(iv) For x, a, b, ν as in (iii), replacement of maximal subwords of the form ba^{ν} by y and $ba^{\nu+1}$ by z yields a sequence (y_i) which is Sturmian in y and z. This process is called *left derivation*. Similarly, *right derivation*, replacing $a^{\nu}b$ by y and $a^{\nu+1}b$ by z, yields either (y_i) or (y_{i+1}).

Proof. Part (i) follows easily from (2), and parts (ii) to (iv) from the Sturmian condition.

For full discussion of infinite Sturmian sequences and their derived sequences, see for example, Lunnon and Pleasants [6] and the references given there.

3.2 Finite Beatty sequences

The above vocabulary extends in the obvious way to finite binary sequences, and Proposition C provides information about finite Beatty sequences and finite Sturmian sequences. If

$$w = x_1 x_2 \ldots x_j$$

is a j-letter binary word in a and b in which both letters appear, then we can write

$$w = y_1 y_2 \cdots y_t$$

where $y_1, y_2, \ldots, y_t$ are distinct blocks in w and $t \geq 2$.

We call y_1 the *first* block in w, y_t the *last* block in w, and $y_2, \ldots, y_{t-1}$ (if $t \geq 3$) the *internal* blocks in w. If w is determined by an infinite Sturmian sequence x in which b is isolated and ν is as in Proposition C (iii), then the first and last blocks of w can each be any of

$$b, \, a, \, a^2, \, \ldots, \, a^{\nu+1},$$

but the only possible internal blocks are

$$b, \, a^{\nu}, \, a^{\nu+1} .$$

Let S be a finite Beatty sequence such that $|S| = k \geq 2$. Then S is of the form (6), where s_i is given by (1) for all i in the index set $I = I_k$ as in (7). It follows from (2) that S satisfies the *difference condition*: we have

$$|(s_{i+j} - s_i) - (s_{u+j} - s_u)| \leq 1$$

for all i, j, u such that $j \geq 0$ and i, u, $i+j$, $u+j$ all belong to I.

It is easily seen that the difference condition is equivalent to the following *sum condition*: for all i, t, u, v in I

$$u + v = i + t \Rightarrow |(s_u + s_v) - (s_i + s_t)| \leq 1\,. \tag{13}$$

Boshernitzan and Fraenkel [2] have shown that the sum condition (in a slightly different form) characterises finite Beatty sequences. The following theorem gathers these results together.

Theorem D. For $k \geq 2$, let $S = \{s_0, s_1, \ldots, s_{k-1}\}$ be a finite set of integers indexed by $I = I_k$ as in (7) such that (8) holds. Then the following three conditions are equivalent.

(i) There exist real numbers α, γ with $\alpha \geq 1$ such that (1) holds for all i in I, that is, S is a finite Beatty sequence.

(ii) The sequence $(s_i)_{i\in I}$ satisfies the difference condition stated above.

(iii) The sequence $(s_i)_{i\in I}$ satisfies the sum condition stated above.

3.3 The mid-points of a finite Beatty sequence

We now consider a finite Beatty sequence S as in (6) such that $|S| = k$, where (1) holds for all i in $I = I_k$ as in (7) and $k \geq 3$. Let

$$M = \left\{\tfrac{1}{2}(t+u) : \ t \in S,\ u \in S\right\}$$

be the set of all *mid-points* of S. Then

$$|M| = |S + S|$$

and it is easy to think geometrically in terms of M since all the mid-points belong to $\frac{1}{2}Z$ and to the closed interval $[s_0, s_{k-1}]$.

The *basic* mid-points of S are the $2k-1$ distinct elements of the set

$$B = S \cup \left\{\tfrac{1}{2}(s_{i-1} + s_i) : \ i = 1, 2, \ldots, k-1\right\}\,.$$

The *family* of mid-points associated with the basic mid-point

$$m = s_i \ \left(= \tfrac{1}{2}(s_i + s_i)\right)$$

is

$$\mathcal{F}(s_i) = \left\{\tfrac{1}{2}(s_{i-j}+s_{i+j}) : j \geq 0,\, i-j,\, i+j \in I\right\} \tag{14}$$
$$= \left\{\tfrac{1}{2}(s_u+s_v) : u+v=2i,\, u,v \in I\right\} .$$

Similarly, the family associated with $m = \frac{1}{2}(s_{i-1}+s_i)$ is

$$\mathcal{F}\left(\tfrac{1}{2}(s_{i-1}+s_i)\right) = \left\{\tfrac{1}{2}(s_{i-1-j}+s_{i+j}) : j \geq 0,\, i-1-j,\, i+j \in I\right\} .$$

Trivially we have

$$|\mathcal{F}(s_0)| = |\mathcal{F}(s_{k-1})| = 1.$$

For all other basic mid-points m in B, it follows from the sum condition (13) that

$$1 \leq |\mathcal{F}(m)| \leq 2,$$

and we now determine when $|\mathcal{F}(m)| = 1$.

For $m = s_i$ with $1 \leq i \leq k-2$, we see from (14) that $|\mathcal{F}(m)| = 1$ if and only if

$$s_{i-j} + s_{i+j} = 2s_i \tag{15}$$

for all $j \geq 1$ such that $i-j$ and $i+j$ both belong to I. In terms of the difference sequence ΔS as in (10),

$$s_{i+j} - s_i = \Delta_{i+1} + \Delta_{i+2} + \cdots + \Delta_{i+j},$$
$$s_i - s_{i-j} = \Delta_i + \Delta_{i-1} + \cdots + \Delta_{i-j+1}.$$

It follows that (15) holds for all j as above if and only if

$$\Delta_{i-j+1} = \Delta_{i+j}$$

for all such j, that is, if and only if the binary sequence ΔS has a centre between the letters corresponding to Δ_i and Δ_{i+1}.

Similarly, for $m = \frac{1}{2}(s_{i-1}+s_i)$ with $1 \leq i \leq k-1$, $|\mathcal{F}(m)| = 1$ if and only if ΔS has a centre at the letter corresponding to Δ_i. Thus we now conclude that

$$\sum_{m \in B} |\mathcal{F}(m)| = 2(2k-1) - 2 - C = 4(k-1) - C, \tag{16}$$

where C is the number of centres of the binary word ΔS.

Since every mid-point $\frac{1}{2}(s_u+s_v)$ belongs to one of the families $\mathcal{F}(m)$, we have

$$|M| \leq \sum_{m \in B} |\mathcal{F}(m)| . \tag{17}$$

The following lemma gives a condition for equality to hold here.

Lemma 1. Let S be a finite Beatty sequence of the form (6), where s_i is given by (1) for all i in $I = I_k$ as in (7) and $|S| = k \geq 3$. Suppose the modulus α satisfies $\alpha > 2$. Then for all i, t, u, v in I we have

$$s_u + s_v = s_i + s_t \quad \Rightarrow \quad u + v = i + t\,.$$

Proof. Suppose $s_u + s_v = s_i + s_t$. We have

$$\lfloor (u+v)\alpha + 2\gamma \rfloor = s_u + s_v + \varepsilon_1,$$
$$\lfloor (i+t)\alpha + 2\gamma \rfloor = s_i + s_t + \varepsilon_2,$$

where $\varepsilon_1 \in \{0, 1\}$, $\varepsilon_2 \in \{0, 1\}$. Hence it follows that

$$|\lfloor (u+v)\alpha + 2\gamma \rfloor - \lfloor (i+t)\alpha + 2\gamma \rfloor| \leq 1\,.$$

However by (3) the Beatty sequence $(t_i) = (\lfloor i\alpha + 2\gamma \rfloor)$ has

$$|t_\nu - t_s| \geq \lfloor \alpha \rfloor \geq 2$$

whenever $\nu \neq s$. Hence we must have $u + v = i + t$.

The following corollaries are immediate consequences:

Corollary 1 to Lemma 1. Under the assumptions of the lemma, the families of mid-points $\mathcal{F}(m)$ with m in B (the set of basic mid-points) are pairwise disjoint.

Corollary 2 to Lemma 1. Under the assumptions of the lemma, the mapping φ defined by

$$\varphi(s_i) = (i,\, s_i)$$

is an isomorphism (in the sense of Section 2) of S onto $\varphi(S) = Z^2 \cap B$, where B is the plane parallelogram

$$B:\ 0 \leq x \leq k-1,\ \alpha x + \gamma - 1 < y \leq \alpha x + \gamma\,.$$

By Corollary 1 we see that under the conditions of the lemma equality holds in (17). By combining this with (16) and the preceding discussion, we obtain the following proposition.

Proposition 1. Let S be a finite Beatty sequence of the form (6), where s_i is given by (1) for all i in $I = I_k$ as in (7) and $|S| \geq k \geq 3$. Suppose that the modulus α satisfies $\alpha > 2$. Then

$$|S + S| = 4(k-1) - C,$$

where C is the number of centres of ΔS as in (10) when ΔS is viewed as a binary sequence.

Since ΔS always has a centre at each of the letters corresponding to Δ_1 and Δ_{k-1} for $k \geq 3$, we always have $C \geq 2$ and so

$$|S + S| \leq 4k - 6\,. \tag{18}$$

We note that the restriction $\alpha > 2$ is not very serious. If $1 < \alpha < 2$ then

$$s_{k-1} = \lfloor \alpha(k-1) + \gamma \rfloor \leq 2k - 2 + s_0\,,$$

so that

$$S \subseteq L = \{s_0,\, s_0 + 1,\, \ldots,\, s_0 + 2k - 2\}\,,$$

and L is an arithmetic progression with $|L| = 2k - 1$, that is, S satisfies the conclusion of Theorem A (ii) (a).

We note also that if S is an arithmetic progression with any modulus and $|S| = k \geq 3$, then ΔS has $C = 2k - 3$ centres and the conclusion of Proposition 1 holds.

4 Centres of binary words

For any finite binary word x in two distinct symbols a and b, let

$$C(x) = \text{number of centres of } x\,.$$

We note that $C(x)$ is unchanged by reversing x or interchanging a and b.

Consider now a fixed binary word in a and b,

$$w = x_1 x_2 \ldots x_j\,,$$

say, in which both a and b appear. Write

$$w = y_1 y_2 \ldots y_t, \qquad (t \geq 2) \tag{19}$$

where each y_i is a (maximal) block (of a's or of b's) as in Section 3. No centre occurs between y_i and y_{i+1} for any i, since this position is between two distinct letters (a, b or b, a). We note the following simple observations about $C(w)$.

(i) If w has a centre occurring at some position in a subword y of w, then y itself has a centre in this position.

(ii) Each internal block of w contributes at most one centre to $C(w)$.

(iii) If the first block y_1 of w is of the form

$$y_1 = a^r \qquad (r \geq 1)$$

then w has exactly r centres at positions in y_1 (in fact in the first r possible positions).

(iv) For w as in (19) above and $1 \leq i < t$ we have

$$C(w) \leq C(y_1 \ldots y_i) + C(y_{i+1} \ldots y_t)\,.$$

Combining (ii) and (iii), we obtain the first part of the following proposition. The second part is easily checked.

Proposition 2. Let w be a finite binary word in a and b in which both a and b appear.

(i) The number of centres of w is less than or equal to the number of letters in w.

(ii) Suppose, further, that w is Sturmian with b isolated. Then equality occurs in (i) if and only if *either* both letters are isolated (and so appear alternately)

$$\text{or} \qquad w = a^r b a^s \,, \qquad r \geq 0,\ s \geq 0,\ r+s \geq 2\,.$$

If S is a finite Beatty sequence with $|S| = k \geq 3$ and S is not an arithmetic progression, then Proposition 2 applies to ΔS and so the number C of centers of ΔS satisfies $C \leq k-1$. Thus by Proposition 1 and (18) we obtain:

Corollary to Proposition 2. Let S be a finite Beatty sequence with $|S| = k \geq 3$. Suppose that S has modulus $\alpha > 2$ and S is not an arithmetic progression. Then

$$3(k-1) \leq |S+S| \leq 4k-6\,,$$

with equality on the left if and only if ΔS satisfies one of the conditions in (ii) of Proposition 2.

5 Infinite periodic Sturmian sequences

Periodic Sturmian sequences arise as difference sequences of rational Beatty sequences. In this section, we will start with auxiliary results on rational Beatty sequences and the nearest integer algorithm. We will then describe the connection between left and right derivation of an infinite periodic Sturmian sequence and the nearest integer algorithm.

5.1 Rational Beatty sequences

A finite or infinite Beatty sequence S is called a *rational Beatty sequence* if it has a rational modulus $\alpha = P/Q$ where P, Q are relatively prime positive integers. We first gather together the basic properties in the infinite case.

Lemma 2. Let $s = (s_i)$ be a strictly increasing sequence of integers indexed by Z.

(i) Suppose s is a rational Beatty sequence with modulus $\alpha = P/Q$, where P, Q are relatively prime positive integers, $P > Q \geq 2$. Let c_0, R_1, ε_1 be the unique integers such that

$$P = c_0 Q + \varepsilon_1 R_1$$
$$\varepsilon_1 = \pm 1,\ R_1 \geq 1,\ -\tfrac{1}{2}Q < \varepsilon_1 R_1 \leq \tfrac{1}{2}Q\,.$$

Then Δs as in (4) and (5) is periodic with least period Q, and Sturmian in a and b with b isolated, where

$$a = c_0, \qquad b = c_0 + \varepsilon_1 .$$

The number of b's in each Q-letter word of Δs is exactly R_1.

(ii) Conversely, suppose Δs is periodic with least period $Q \geq 2$ and Sturmian in two symbols a, b representing two positive integers which differ by exactly one. Then s is a rational Beatty sequence with modulus P/Q, where $P = \Delta_1 + \Delta_2 + \ldots + \Delta_Q$.

Proof. Part (i) follows easily from Proposition C, the observation that s_{i+Q} equals $s_i + P$ for all i, and the uniqueness of the expression for P in the form

$$P = (Q - R_1)c_0 + R_1(c_0 + \varepsilon_1) ,$$

with $0 < R_1 \leq Q - R_1$ which follows from the choice of c_0, R_0, ε_1 above. Part (ii) is well known. It follows, for example, from the results of Lunnon and Pleasants [6] and is proved in Pitman and Wolff [8].

We note that c_0 as in Lemma 2 is the nearest integer to P/Q and is the initial partial quotient in the nearest integer continued fraction expansion of P/Q. (This expansion is discussed in Section 5.2 below.)

The following result is easily checked (using, for example, the ideas of Lemma 1 of Simpson [10] and the fact that translation by an integer is an isomorphism of Z onto itself.)

Lemma 3. Let P, Q be relatively prime positive integers such that

$$P > Q \geq 2.$$

Let

$$S = \left\{ \lfloor i\tfrac{P}{Q} + \gamma \rfloor \;:\; 0 \leq i \leq k-1 \right\}$$

be a finite rational Beatty sequence with $|S| = k \geq 2$ and let $s = (s_i)$ be an infinite rational Beatty sequence with modulus P/Q.

(i) The Beatty sequence S is isomorphic to one of the Q finite Beatty sequences of the form

$$\left\{ \lfloor i\tfrac{P}{Q} + \tfrac{T}{Q} \rfloor \;:\; 0 \leq i \leq k-1 \right\}$$

with $T = 0, 1, \ldots, Q-1$.

(ii) Each of the Q sequences in (i) is isomorphic to one of the Q sequences

$$\{s_j,\, s_{j+1},\, \ldots,\, s_{j+k-1}\}$$

with $j = 0, 1, \ldots, Q-1$, that is, it is isomorphic to a sequence obtained by using an appropriate shift of s.

It easily seen (for example by using Corollary 2 to Lemma 1) that if $k = tQ$ $(t \geq 1)$ then S as in Lemma 3 is the union of Q arithmetic progressions

$$\{u_i + jP : \ j = 0, 1, \ldots, t-1\}$$

such that for $i = 0, 1, \ldots, Q-1$

$$u_{i+1} - u_i \equiv d \pmod{P},$$

where d is an integer relatively prime to P and $d \geq 2$. We can think of S as satisfying a weaker definition of generalised arithmetic progression of rank at most 2 than that given in Section 2.

5.2 The nearest integer algorithm

In order to cover multiples of $\frac{1}{2}$, we define the *nearest integer* $N = N(x)$ to a real number x to be the unique integer N such that

$$N - \tfrac{1}{2} < x \leq N + \tfrac{1}{2} \, .$$

The *nearest integer algorithm* is the analogue of the Euclidean algorithm when division with least remainder (in absolute value) replaces ordinary division. We start with two relatively prime positive integers R_0, R_1 such that

$$R_0 \geq 2R_1, \quad R_1 \geq 1,$$

and produce unique integers $n \geq 1$, c_1, c_2, $\ldots$, c_n, $\varepsilon_2, \ldots, \varepsilon_n$, $R_2, \ldots, R_n$ such that R_i and R_{i+1} are always relatively prime, as follows.

At Step 1, let

$$c_1 = \text{nearest integer to } R_0/R_1 \, .$$

If $R_1 = 1$, we have $R_0 = c_1$, $n = 1$ and the process stops. Otherwise, let ε_2, R_2 be the unique integers such that

$$R_0 = c_1 R_1 + \varepsilon_2 R_2, \qquad \varepsilon_2 = \pm 1, \quad R_2 \geq 1 \, .$$

If Steps 1, 2, $\ldots$, $j-1$ have occurred, so that $R_j \geq 1$, then at Step j, let

$$c_j = \text{nearest integer to } R_{j-1}/R_j \, .$$

If $R_j = 1$, we have $R_{j-1} = c_j$, $n = j$, and the process stops. Otherwise, let ε_{j+1}, R_{j+1} be the unique integers such that

$$R_{j-1} = c_j R_j + \varepsilon_{j+1} R_{j+1}, \quad \varepsilon_{j+1} = \pm 1, \quad R_{j+1} \geq 1 \, .$$

Since $R_0 > R_1 > \ldots > R_{j-1} > R_j$ we must reach n such that $R_n = 1$, $R_{n-1} = c_n$ and the process stops. By the definition of c_j it follows that

$$1 \leq R_j \leq \tfrac{1}{2} R_{j-1} \quad \text{for } j = 1, 2, \ldots, n,$$

in fact with strict $<$ in the second inequality except when $j = n$ and $R_{j-1} = 2$.

If the process stops at Step n, where $n > 1$, we obtain the *nearest integer continued fraction expansion* of R_0/R_1,

$$\begin{aligned}\frac{R_0}{R_1} =& c_1 + \frac{\varepsilon_2}{c_2+} \cdots \frac{\varepsilon_{n-1}}{c_{n-1}+} \frac{\varepsilon_n}{c_n} \\ =& [c_1;\, \varepsilon_2 c_2,\, \varepsilon_3 c_3,\, \ldots,\, \varepsilon_n c_n]\,.\end{aligned}$$

(We note that $c_i \geq 2$ for all i, $c_i \geq 3$ if $\varepsilon_{i+1} = -1$, and $\varepsilon_n = 1$ if $c_n = 2$.)

We note that at Step j we have

$$R_{j-1} = (R_j - R_{j+1})c_j + R_{j+1}(c_j + \varepsilon_{j+1})\,,$$

and $\ell = R_j - R_{j+1}$, $m = R_{j+1}$, $c = c_j$, $\varepsilon = \varepsilon_{j+1}$ are the unique solutions in integers of

$$R_{j-1} = \ell c + m(c + \varepsilon) \tag{20}$$

such that $\varepsilon = \pm 1$, $\ell + m = R_j$, $m \leq \ell$.

5.3 Derivation of a periodic infinite Sturmian sequence

Consider an infinite sequence $x = (x_i)$ which is Sturmian in two symbols a and b with b isolated and is also periodic with least period $Q \geq 2$. By a *period* of x we shall mean a Q-letter word of x. Let ν be the a-width of x (as in Proposition C (iii)) and R_1 the number of b's per period of x. If $R_1 = 1$, then x is of the form

$$x = (ba^{Q-1})^\infty = (a^{Q-1}b)^\infty\,, \tag{21}$$

and we exclude this case from now on.

We can view x as made up of consecutive words of the forms ba^ν, $ba^{\nu+1}$, which we will call *left-basic*, with both left-basic words appearing, and similarly in terms of right-basic words $a^\nu b$, $a^{\nu+1}b$. By a *basic* word we shall mean a word which is either left-basic or right-basic. By Proposition C (iv), we know that left derivation, that is, replacement of ba^ν by one symbol and $ba^{\nu+1}$ by another, yields a sequence x' which is a Sturmian in the two new symbols, at least one of which must be isolated in x'. (Exactly one is isolated unless x' has period 2.) We call a left-basic word *isolated in* x if the corresponding symbol is isolated in x', and similarly for right-basic words. The following lemma gives the connection between repeated derivation of x and the nearest integer algorithm for Q/R_1.

Lemma 4. Let $x = (x_i)$ be an infinite binary sequence which is Sturmian and periodic, with least period Q and exactly R_1 isolated symbols in each period, where $R_1 \geq 2$.

Write $Q = R_0$ and carry out the nearest integer algorithm for R_0/R_1 as set out above, finishing with $R_{n-1} = c_n$, $R_n = 1$, where $n \geq 2$.

Let $x^{(0)}$ be the given sequence x, and, for $i = 1, 2, \ldots, n$, let $x^{(i)}$ be obtained from $x^{(i-1)}$ by either left or right derivation. Then the sequences $x^{(0)}, \ldots, x^{(i)}, \ldots, x^{(n)}$ have the following properties.

(i) For given i such that $1 \leq i \leq n-1$, let a and b be the two symbols in the sequence $x^{(i-1)}$, with b isolated. The sequence $x^{(i-1)}$ is periodic with least period R_{i-1} and has exactly R_i isolated symbols b in each period. A period of $x^{(i-1)}$ beginning with b is made up of R_i consecutive left-basic words, of which R_{i+1} are isolated. The left-basic words of $x^{(i-1)}$ are

$$ba^{c-1+\varepsilon}, \quad ba^{c-1},$$

the first being isolated, where $c = c_i$, $\varepsilon = \varepsilon_{i+1}$. The a-width of $x^{(i-1)}$ is

$$\min\{c-1+\varepsilon,\, c-1\} = c_i - 1 - \tfrac{1}{2}(1-\varepsilon_{i+1})\,.$$

(ii) Finally, the sequence $x^{(n-1)}$ is of the form

$$x^{(n-1)} = (zy^{c_n-1})^{\infty}$$

and $x^{(n)}$ is a constant sequence.

(iii) Corresponding results in terms of the right-basic words also hold.

Proof. Part (i) is easily proved by induction on i, using Section 5.2 and, in particular, the observation above regarding the equation (20). Part (ii) follows since $x^{(n-1)}$ has period $R_{n-1} = c_n$ and exactly one isolated symbol as $R_n = 1$. Part (iii) follows, for example, by reversing the sequence x.

5.4 Centres of infinite periodic binary sequences

Consider now an infinite periodic binary sequence $x = (x_i)$ with least period $Q \geq 2$. The sequence has a *centre* in a given position (at x_i or between x_{i-1} and x_i, for some i) if the whole sequence has mirror symmetry about that position. We regard a period of x, $w = x_{j+1}x_{j+2}\ldots x_{j+Q}$, say, as including exactly one of the two adjacent positions (before x_{j+1} and after x_{j+Q}). It is easily shown that the sequence has *either* no centres *or* exactly two centres per period.

If, further, the sequence x as above is Sturmian, it is easily seen that the derivation process does not change the number of centres per period; since the sequences (21), and hence those in Lemma 4 (ii), have exactly two centres per period, it follows that every infinite periodic Sturmian sequence has exactly two centres per period. This was proved geometrically by Lunnon and Pleasants [6] (see Theorem 4).

6 Centres of finite periodic Sturmian words

We can now use the above results on derivation to obtain a formula for the number of centres of certain finite periodic Sturmian words and hence to evaluate $|S+S|$ for certain finite Beatty sequences.

6.1 Finite periodic Sturmian words

We would like to use derivation to investigate the number of centres in a finite word w of an infinite periodic Sturmian sequence x. However we run into difficulties because repeated derivation of a given word may not be feasible, since we may reach a word which does not start or finish with an isolated symbol or one which does not consist only of complete basic words. For this reason, we confine attention to the case when w consists of t consecutive v's, where v is a suitable period of x.

The following lemma shows the effect of derivation on $C(w)$ for w as above, where, as in Section 4, $C(w)$ denotes the number of centres in w.

Lemma 5. Let $x = (x_i)$ be an infinite periodic Sturmian sequence in a and b, with b isolated, a-width ν and period $Q \geq 3$. Let v be a period of x which either begins or ends with an isolated basic word. Let

$$w = v^t \qquad (t \geq 1)\,,$$

and let w' be the word obtained from w by left derivation if v begins with b or right derivation if v ends with b. Then

$$C(w) \geq C(w') + \nu + 1,$$

with equality if, further, v ends or begins with $a^{\nu+1}$.

Proof. We note first that our hypothesis on v ensures that v (and hence also w) begins or ends with b, so that the derivation is possible. We assume that v starts with b (the other case being exactly similar). Let y and z be the two symbols in x', with y replacing ba^ν and z replacing $ba^{\nu+1}$. Since v is a period of x, it is easily seen that exactly one of the following two cases occur.

Case (i) In this case

$$\begin{aligned} w &= b|\ldots\ldots b|a^{\nu+1}\,, \\ w' &= \ \ |\ldots\ldots|z\,. \end{aligned}$$

The first b contributes one centre to $C(w)$ and disappears in w'. The last $a^{\nu+1}$ contributes $\nu+1$ centres to $C(w)$ and is replaced by z, which contributes one centre to $C(w')$. Each internal block of w contributes a centre to $C(w)$ if and only if the corresponding letter or position contributes a centre to $C(w')$. Thus

$$C(w) - C(w') = (1-0) + ((\nu+1)-1) = \nu + 1\,.$$

Case (ii) In this case, for some $r \geq 1$,

$$\begin{aligned} w &= b|\ldots\ldots a^{\nu+1}|(ba^\nu)^r \\ w' &= \ \ |\ldots\ldots\ldots z|y^r\,. \end{aligned}$$

This time the last $(ba^{\nu})^r$ contributes $r+\nu$ centres to $C(w)$ and is replaced by y^r, which contributes r centres to $C(w')$. For each centre in w' preceding y^r, there is a centre at a corresponding position in w, but there may also be other centres in w (caused by the possibility of "matching" the last a^{ν} with an a^{ν} occurring in a block $a^{\nu+1}$). Thus this time

$$C(w) - C(w') \geq (1-0) + (r+\nu-r) = \nu + 1\,.$$

Since the only case when v ends with $a^{\nu+1}$ is Case (i), when equality occurs, this completes the proof of the lemma.

Derivable periods. Let $x = (x_i)$ be an infinite periodic Sturmian sequence such that derived sequences $x^{(i)}$ exist and are non-constant for

$$1 \leq i < n$$

and $x^{(n)}$ is constant, where $n \geq 2$. Let v be a period of x beginning or ending with an isolated symbol. Let $v' = \delta_1(v)$, where δ_1 is the unique operation of either left or right derivation such that $\delta_1(v)$ exists (and so $\delta_1(v)$ is a period of x'). We call v *derivable* if there is a sequence $\delta_1, \ldots, \delta_n$ with each δ_i either a left or a right derivation such that $v^{(i)} = \delta_i(v^{(i-1)})$ exists and is non-constant for $1 \leq i \leq n-1$ while $v^{(n)}$ exists and $v^{(n)} = u$, say, where $x^{(n)} = u^{\infty}$. By working backwards from u, we can see that for any sequence $\delta_1, \ldots, \delta_n$ of derivations there is a derivable period v of x such that the $v^{(i)}$ are obtained by these operations.

Remark. Suppose x and v are as in Lemma 5, and let the isolated basic words of x be $ba^{c-1+\varepsilon}$, $a^{c-1+\varepsilon}b$, where $\varepsilon = \pm 1$. We note that if v begins with b, then v will end with $a^{\nu+1}$ provided that *either* $\varepsilon = -1$ and v begins with the isolated word *or* $\varepsilon = 1$ and v ends with the isolated word, and a corresponding result holds if v ends with b. It follows that the operations $\delta_n, \delta_{n-1}, \ldots, \delta_1$ and hence v can be chosen so that each $v^{(i)}$ begins or ends with

$$(a_i)^{1+\nu_i}\,,$$

where a_i is the non-isolated symbol of $x^{(i)}$ and ν_i is the a_i-width of $x^{(i)}$.

These observations together with Lemma 4 and 5 lead to the main combinatorial result of this paper which is as follows:

Proposition 3. Let $x = (x_i)$ be an infinite binary sequence which is Sturmian and periodic with least period Q and has exactly R_1 isolated symbols per period, where $R_1 \geq 2$. Let the nearest integer continued fraction expansion of Q/R_1 be

$$Q/R_1 = [c_1;\, \varepsilon_2 c_2, \varepsilon_3 c_3,\, \ldots,\, \varepsilon_n c_n]\,,$$

where $n \geq 2$.

(i) Let

$$w = v^t\,,$$

where $t \geq 1$ and v is a derivable period of x. Then $C(w)$, the number of centres of w, satisfies

$$C(w) \geq 2t + C',$$

where

$$C' = c_n + \sum_{i=1}^{n-1} \left(c_i + \tfrac{1}{2}\varepsilon_{i+1}\right) - \tfrac{1}{2}(n+3)\,. \tag{22}$$

(ii) There exists a derivable period v_0 of x such that for all $t \geq 1$

$$C(v_0^t) = 2t + C'\,.$$

(iii) We have

$$C' = C(v_0) - 2,$$
$$2 \leq C' \leq Q - 3\,.$$

Proof. (i) Let $v^{(1)}$, $v^{(2)}$, ..., $v^{(n)}$ be the successive derived words obtained from v as above. Note that, for each i , $w^{(i)}$ exists and equals $(v^{(i)})^t$. By Lemma 4 (i) and Lemma 5, for $1 \leq i \leq n-1$, we have

$$C(w^{(i-1)}) \geq C(w^{(i)}) + c_i + \tfrac{1}{2}(\varepsilon_{i+1} - 1)\,,$$

and hence

$$C(w) \geq \sum_{i=1}^{n-1} \left(c_i + \tfrac{1}{2}\varepsilon_{i+1}\right) - \tfrac{1}{2}(n-1) + C(w^{(n-1)})\,.$$

By Lemma 4 (ii)

$$w^{(n-1)} = \left(zy^{c_n - 1}\right)^t \quad \text{or} \quad \left(y^{c_n-1}z\right)^t\,,$$

and it is easily checked that

$$C(w^{(n-1)}) = c_n + 2(t-1)\,.$$

The required inequality now follows.

(ii) The remark preceding the proposition shows that there exists a period v_0 such that if $v = v_0$ then equality holds in each application of Lemma 5 and hence $C(w)$ equals $2t + C'$, as required.

(iii) Taking $t = 1$ in (ii) we obtain $C(v_0) = 2 + C'$. Since $n \geq 2$, we have

$$C' \geq c_2 + c_1 + \tfrac{1}{2}(\varepsilon_2 - 1) - 2 \geq 2\,.$$

Since $R_1 \geq 2$, v_0 does not satisfy (ii) of Proposition 2 and so $C(v_0) \leq Q - 1$, giving $C' \leq Q - 3$.

6.2 Sumsets of finite Beatty sequences

The rational case

The result below follows immediately from Proposition 3 by using Proposition 1 and Lemmas 2 and 3.

Corollary 1 to Proposition 3. Let P, Q be relatively prime positive integers such that $P > 2Q$, $Q \geq 5$ and P/Q has nearest integer continued fraction expansion

$$P/Q = [c_0;\, \varepsilon_1 c_1,\, \varepsilon_2 c_2,\, \ldots,\, \varepsilon_n c_n]\,,$$

where $n \geq 2$. Then there exists an integer T such that $0 \leq T \leq Q-1$ and for every $t \geq 1$, if $k = tQ+1$ and S is the rational Beatty sequence

$$S = \left\{ \lfloor i\tfrac{P}{Q} + \tfrac{T}{Q} \rfloor \;:\; i = 0, 1, \ldots, k-1 \right\}\ , \tag{23}$$

then

$$|S+S| = 2(k-1)\left(2 - \tfrac{1}{Q}\right) - C'\ ,$$

where C' is given by (22).

From the results of Section 5.2 with $R_0 = Q$, we have

$$Q \geq \left(c_1 - \tfrac{1}{2}\right)\left(c_2 - \tfrac{1}{2}\right)\cdots\left(c_{n-1} - \tfrac{1}{2}\right)c_n\ ,$$

and hence $C'/(k-1) = C'/(tQ)$ is small if n is large or $c_1, \ldots, c_n$ are large, even if $t = 1$.

We note that a $2Q$-letter word of a periodic sequence with least period Q includes every period of the sequence. The following corollary can be obtained by using this observation.

Corollary 2 to Proposition 3. For P, Q as in Corollary 2 above, suppose that

$$k = tQ + 1 + R = (t-1)Q + 1 + (Q+R)\,,$$

where $0 \leq R \leq Q-1$ and $t \geq 2$. Then for every T such that $0 \leq T \leq Q-1$ the Beatty sequence S given by (23) satisfies

$$|S+S| \geq 2(k-1)\left(2 - \tfrac{1}{Q}\right) - 2(Q+R)\left(1 - \tfrac{1}{Q}\right) - C'\,,$$

where C' is given by (22).

We note that if S is a rational Beatty sequence with modulus P/Q such that $Q = 2, 3$ or 4 then each period of ΔS is of a form covered by Proposition 2, and so the number C of centres of ΔS is easily determined and $|S+S|$ can be obtained directly from Proposition 1.

The irrational case

For irrational $\alpha > 2$, let P, Q be relatively prime positive integers such that $Q \geq 5$ and

$$\alpha = \tfrac{P}{Q} + \tfrac{\varepsilon}{Q^2}\,, \qquad 0 < \varepsilon < 1\,. \tag{24}$$

Then it is easily seen that for every given integer T such that $0 \leq T \leq Q-1$ we have

$$\lfloor i\alpha + \tfrac{T}{Q} \rfloor = \lfloor i\tfrac{P}{Q} + \tfrac{T}{Q} \rfloor \quad \text{for } i = 0, 1, \ldots, Q\,.$$

Thus we can use Corollary 1 above with $t = 1$ to obtain information about the sumset of a finite Beatty sequence with irrational modulus. (Unfortunately we cannot simply use the nearest integer continued fraction expansion of α since its convergents do not in general satisfy (24) with $\varepsilon \geq 0$.)

Suppose, for example, that α has an ordinary (regular) continued fraction expansion

$$\alpha = (c_0;\, c_1,\, c_2,\, \ldots)$$

where all $c_i \geq 2$, let p_{2m}/q_{2m} be an even convergent, and write $P = p_{2m}$, $Q = q_{2m}$. Then (24) holds and we have $\varepsilon_i = 1$ for all i and

$$\tfrac{P}{Q} = [c_0;\, c_1,\, c_2,\, \ldots,\, c_{2m}]\,.$$

Hence there exists T such that $0 \leq T \leq Q-1$ and the finite Beatty sequence S with $|S| = k = Q+1$ given by

$$S = \left\{ \lfloor i\alpha + \tfrac{T}{Q} \rfloor \;:\; 0 \leq i \leq Q \right\}$$

satisfies

$$|S+S| = 4Q - 2 - \sum_{i=1}^{2m} c_i = 4k - 6 - \sum_{i=1}^{2m} c_i\,.$$

7 Concluding Comments

7.1 Sumsets in Z^2

By Corollary 2 to Lemma 1 we know that if $S = \{s_0,\, s_1,\, \ldots,\, s_{k-1}\}$ is a finite Beatty sequence with non-integral modulus $\alpha > 2$ and $k \geq 3$ then S is isomorphic to the set of points

$$T = \varphi(S) = \{(i, s_i) :\; i = 0, 1, \ldots, k-1\}$$

in Z^2. Further, if $\alpha = P/Q$ with P and Q relatively prime positive integers then the points of T form arithmetic progressions in Z^2, each progression consisting of consecutive lattice points on one of Q adjacent lattice lines.

For subsets of Z^2, results along the lines of those in Section 2 have been obtained by Freiman and significantly extended recently by Stanchescu. References and details are given in Stanchescu [11]. For a finite subset T of Z^2, this work includes bounds on $|T+T|$ which ensure that T is covered by Q parallel lattice lines and also includes detailed study of the case when T is covered by 2 or 3 parallel lines. When interpreted geometrically in Z^2, the above results on sumsets of finite rational Beatty sequences provide a fund of examples illustrating this work of Freiman and Stanchescu.

7.2 Further investigation

We note that Beatty sequences have close connections with special expansions of positive integers (see, for example, Fraenkel, Levitt and Simshoni [3]). It seems likely that further investigation of the sumset of a finite Beatty sequence and the number of centres of its difference sequence may require some such expansion, possibly related to the nearest integer continued fraction.

In connection with sets of integers with small sumsets, it would be desirable to consider generalisations of Beatty sequences, for example, sequences (s_i) such that

$$|(s_{i+j} - s_i) - (s_{u+j} - s_u)| \leq d$$

for a given positive integer $d \geq 2$. I hope to study such sequences (s_i) further.

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