A note on odd cycle-complete graph Ramsey numbers

Benny Sudakov *

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA
and Institute for Advanced Study, Princeton, NJ 08540, USA
bsudakov@math.princeton.edu

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Abstract

The Ramsey number $r(C_l, K_n)$ is the smallest positive integer m such that every graph of order m contains either cycle of length l or a set of n independent vertices. In this short note we slightly improve the best known upper bound on $r(C_l, K_n)$ for odd l.

1 Introduction

The Ramsey number $r(C_l, K_n)$ is the smallest positive integer m such that every graph of order m contains either cycle of length l or a set of n independent vertices. In this note we give an improved asymptotic bounds on $r(C_l, K_n)$ for odd $l > 5$.

Erdős et al. [5] proved that

$$r(C_l, K_n) \leq c(l)n^{1+1/k} \text{ where } k = \lceil l/2 \rceil - 1,$$

and $c(l)$ is a positive constant depending on l. A general lower bound for $r(C_l, K_n)$ was given by Spencer [8]. Later the asymptotics of $r(C_3, K_n)$ was determined up to a constant factor in [1] and [6]. For other values of l the result of Erdős et al. was slightly improved by Caro et al. [4]. In particular they showed that $r(C_{2k}, K_n) \leq c(k)(n/\ln n)^{k/(k-1)}$ for k fixed where n tends to infinity, and that $r(C_5, K_n) \leq cn^{3/2}/\sqrt{\ln n}$. In [4] the authors also

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suggested that one should be able to obtain a similar improvement for the cycle-complete graph Ramsey numbers for odd cycles of length greater than 5. Here we give such an improvement of the bound of Erdős et al. for $r(C_{2k+1}, K_n)$ for all remaining $k > 2$. Our main result is the following theorem.

Theorem 1.1 *For every fixed integer k and $n \to \infty$ the Ramsey numbers*

$$r(C_{2k+1}, K_n) \leq c(k)\frac{n^{1+1/k}}{\ln^{1/k} n}.$$

2 Proof of main result

In this section we prove Theorem 1.1. We will assume whenever this is needed that n is sufficiently large and make no attempt to optimize our absolute constants. First we need the following well known bound ([3], Lemma 15, Chapter 12) on the independence number of a graph containing few triangles (see also [2] for a more general result).

Proposition 2.1 *Let G be a graph on n vertices with average degree at most d and let h be the number of triangles in G. Then G contains an independent set of order at least*

$$0.1\frac{n}{d}\Big(\ln d - 1/2\ln(h/n)\Big).$$

From this proposition we can immediately deduce the following corollary.

Corollary 2.2 *Let G be a graph on n vertices with maximal degree d which does not contain a cycle of length $2k+1$. Then the independence number of G is at least*

$$\alpha(G) \geq 0.05\frac{n}{d}\Big(\ln d - \ln k\Big).$$

Proof. Since G has no cycle of length $2k+1$ it is easy to see that the neighborhood $N(v)$ of any vertex v contains no $2k$-vertex path. On the other hand it is well known that the graph with minimal degree $2k$ contains such a path. Therefore any induced subgraph of $G[N(v)]$ should contain vertex of degree smaller than $2k$. Delete from graph $G[N(v)]$ the vertex of minimal degree and repeat this procedure until the graph is empty. Note that at every step we remove at most $2k$ edges and in the end of the process we remove all the edges of $G[N(v)]$. Hence we obtain that $N(v)$ spans at most $2k|N(v)| \leq 2kd$ edges and the number of triangles in G, containing v is at most $2kd$. This implies that G contains at most $h = 2kdn/3$ triangles. Thus from Proposition 2.1 it follows that

$$\alpha(G) \geq 0.1\frac{n}{d}\Big(\ln d - 1/2\ln(h/n)\Big) \geq 0.1\frac{n}{d}\Big(\ln d - 1/2\ln(kd)\Big) = 0.05\frac{n}{d}\Big(\ln d - \ln k\Big). \quad \blacksquare$$

For the next statement we need to introduce some notations. Let G be a graph and v be an arbitrary vertex of G. Denote by $d(v, u)$ the length of the shortest path from v to u and let $N_i(v) = \{u|d(v, u) = i\}$ be the set of all vertices which are in distance exactly i from v. The following useful result about graphs without short cycles was proved by Erdős, Faudree, Rousseau and Schelp [5].

Proposition 2.3 *Let G be a graph which has no cycles of length $2k+1$. Then for any $1 \le i \le k$ the induced subgraph $G[N_i(v)]$ contains an independent set of order at least $|N_i(v)|/(2k-1)$.*

We are now ready to complete the proof of our main result.

Proof of Theorem 1.1. Let G be a graph on $m = 80(kn)^{1+1/k}/\ln^{1/k} n$ vertices without C_{2k+1} and let $d = 2(kn)^{1/k}\ln^{1-1/k} n$. We start with $G' = G$ and $I = \emptyset$ and as long as G' has a vertex of degree at least d we do the following iterative procedure. Pick a vertex $v \in G'$ with degree at least d. If $N_k(v)$ in G' has size at least $2kn$, then by Proposition 2.3 it contains an independent set of size greater than n and we are done. Otherwise, since $|N_1(v)|/|N_0(v)| = |N_1(v)| \ge d$ there exist an index $1 \le i \le k-1$ such that

$$\frac{|N_{i+1}(v)|}{|N_i(v)|} \le \left(\frac{2kn}{d}\right)^{1/(k-1)} = \frac{(kn)^{1/k}}{\ln^{1/k} n} = x.$$

Pick the smallest i with this property. By Proposition 2.3 $N_i(v)$ contains an independent set I' of size at least $|N_i(v)|/(2k-1)$. Set $I = I \cup I'$ and remove all vertices in $N_{i-1}(v)$, $N_i(v)$ and $N_{i+1}(v)$ from G'. Note that the number of vertices which we have removed is at most

$$\begin{aligned} |N_{i-1}(v)| + |N_i(v)| + |N_{i+1}(v)| &\le \left(\frac{1}{x} + 1 + x\right)|N_i(v)| \qquad (1) \\ &\le \frac{2(kn)^{1/k}}{\ln^{1/k} n}|N_i(v)| \le \frac{4k(kn)^{1/k}}{\ln^{1/k} n}|I'|, \end{aligned}$$

and they contain all the neighbors of the vertices in I'. Therefore during the whole process I stays always independent. In addition, by (1) the ratio between the total number of vertices which we remove and the order of I is at most $4k(kn)^{1/k}/\ln^{1/k} n$.

Let G' be a graph obtained in the end of this process. Either we done or by definition its maximal degree is less than d. If it has at least $m/2$ vertices, then by Corollary 2.2 it contains an independent set of size $0.05(m/2d)(\ln d - \ln k) > n$. Here we needed that $m = 80(kn)^{1+1/k}/\ln^{1/k} n$. On the other hand if we remove more than $m/2$ vertices during our process, then we constructed an independent set I in G of order

$$|I| \ge \frac{m/2}{4k(kn)^{1/k}/\ln^{1/k} n} = \frac{40(kn)^{1+1/k}/\ln^{1/k} n}{4k(kn)^{1/k}/\ln^{1/k} n} > n.$$

This completes the proof of the theorem. ∎

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Note added in proof. When this paper was written we learned that independently of our work Y. Li and W. Zang [7] obtained a similar result.

References

[1] M. Ajtai, J. Komlós and E. Szemerédi, A note on Ramsey numbers, *J. Combinatorial Theory A* 29 (1980), 354-360.

[2] N. Alon, M. Krivelevich and B. Sudakov, Coloring graphs with sparse neighborhoods, *J. Combinatorial Theory Ser. B* 77 (1999), 73-82.

[3] B. Bollobás, **Random Graphs**, Academic Press, London, 1985.

[4] Y. Caro, Y. Li, C. Rousseau and Y. Zhang, Asymptotic bounds for some bipartite graph-complete graph Ramsey numbers, *Discrete Mathematics* 220 (2000), 51-56.

[5] P. Erdős, R. Faudree, C. Rousseau and R. Schelp, On cycle-complete graph Ramsey numbers, *J. Graph Theory* 2 (1978), 53-64.

[6] J. Kim, The Ramsey number $R(3,t)$ has order of magnitude $t^2/\log t$, *Random Structures and Algorithms* 7 (1995), 173-207.

[7] Y. Li and W. Zang, The independence number of graphs with a forbidden cycle and Ramsey numbers, preprint.

[8] J. Spencer, Asymptotic lower bounds for Ramsey functions, *Discrete Mathematics* 20 (1977), 69-76.