A note on the number of edges guaranteeing a C_4 in Eulerian bipartite digraphs

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Abstract

Let G be an Eulerian bipartite digraph with vertex partition sizes m, n. We prove the following Turán-type result: If $e(G) > 2mn/3$ then G contains a directed cycle of length at most 4. The result is sharp. We also show that if $e(G) = 2mn/3$ and no directed cycle of length at most 4 exists, then G must be biregular. We apply this result in order to obtain an improved upper bound for the diameter of interchange graphs.

1 Introduction

All graphs considered here are finite, directed, and contain no parallel edges. For standard graph-theoretic terminology the reader is referred to [1]. In this paper we consider the most basic Turán-type problem in bipartite digraphs, namely, specifying conditions on the cardinality of the edge-set of the digraph that guarantee the existence of a directed simple cycle of length at most four. As usual in Turán type problems in directed graphs, one must impose constraints relating the indegree and outdegree of a vertex in order to avoid trivialities (if no such constraints exist then one may not have short directed cycles at all even if the graph is very dense, the extreme case being an acyclic orientation of a complete bipartite graph). The most interesting and natural constraint is the requirement that the digraph be Eulerian, namely, the indegree of a vertex must be equal to its outdegree.

Let $b(m,n)$ denote the maximum integer, such that there exists an Eulerian bipartite digraph with vertex partition sizes m,n having $b(m,n)$ edges and no directed cycle of length at most 4. A *biregular bipartite digraph* is an Eulerian bipartite digraph having the property that any two vertices in the same vertex class have the same indegree and outdegree.

The parameter $b(m,n)$ has been studied by Brualdi and Shen in [3], who proved $b(m,n) < \left(\sqrt{17}-1\right)mn/4$. They conjectured (the case $k=2$ of Conjecture 2 in [3]) that $b(m,n) \le 2mn/3$. In this paper we prove this conjecture, and together with a well-known construction obtain that it is sharp. Furthermore, we obtain that the extremal graphs must be biregular. Our main theorem is the following:

Theorem 1.1 *$b(m,n) \le 2mn/3$. Equality holds if and only if both m and n are divisible by 3. Any graph demonstrating an equality must be biregular.*

Brualdi and Shen have shown in [3] how an upper bound for $b(m,n)$ corresponds to an upper bound for the diameter of interchange graphs. These graphs are defined as follows: Let $R=(r_1,\dots,r_m)$ and $S=(s_1,\dots,s_n)$ be non-negative integral vectors with $\sum r_i = \sum s_j$. Let $\mathcal{A}(R,S)$ denote the set of all $\{0,1\}$-matrices with row sum vector R and column sum vector S, and assume that $\mathcal{A}(R,S) \neq \emptyset$. This set has been studied extensively (see [2] for a survey). The *interchange graph* $G(R,S)$ of $\mathcal{A}(R,S)$, defined by Brualdi in 1980, is the graph with all matrices in $\mathcal{A}(R,S)$ as its vertices, where two matrices are adjacent provided they differ in an interchange matrix. Brualdi conjectured that the diameter of $G(R,S)$, denoted $d(R,S)$, cannot exceed $mn/4$. Using a result of Walkup [4] that relates the distance between two vertices A and B in $G(R,S)$ to the maximum number of cycles in a cycle decomposition of an Eulerian bipartite digraph that corresponds to $A-B$, together with the upper bound for $b(m,n)$, it is shown in [3] that $d(R,S) \le (mn+b(m,n))/4$. Thus, the result in Theorem 1.1 also improves this upper bound for $d(R,S)$ giving

$$d(R,S) \le \frac{5}{12}mn.$$

It is worth mentioning that in Theorem 1.1, if v is any vertex with maximum normalized degree (by "normalized degree" we mean the ratio between its outdegree and the cardinality of the opposite vertex class), then there exists a directed cycle of length at most four that contains v. Thus, there is also a linear $O(mn)$ time algorithm for detecting such a cycle in these graphs; merely perform a breadth first search whose root is any vertex with maximum normalized degree.

2 Proof of the main result

Let $G=(V,E)$ be an Eulerian bipartite digraph. We may assume that G does not contain antiparallel edges, since otherwise G has a directed cycle of length 2 and we are done.

Let $V = A \cup B$ where A and B are the two (disjoint) vertex classes of G where $|A| = m$ and $|B| = n$. Let $0 \le \alpha \le 1$ satisfy $|E| = \alpha mn$. In order to prove the upper bound in Theorem 1.1 we need to show that if $\alpha > 2/3$ then G has a directed C_4.

For $v \in V$ let d_v denote the indegree and outdegree of v (it is the same by assumption). For $v \in A$, let $\rho_v = d_v/n$ and for $v \in B$, let $\rho_v = d_v/m$. Let $\rho = \max_{v \in V} \rho_v$. Notice that G is biregular if and only if $\rho_v = \rho = \alpha/2$ for each $v \in V$, or, more compactly, if and only if $\rho = \alpha/2$.

Fix $v^* \in V$ satisfying $\rho_{v^*} = \rho$. Without loss of generality, assume $v^* \in A$ (since otherwise we can interchange the roles of m and n, as we did not impose any cardinality constraints upon them). It clearly suffices to prove the following:

Lemma 2.1 *If no directed C_4 contains v^* as a vertex then $\alpha \le 2/3$.*

Proof: We assume that no directed C_4 contains v^* as a vertex. Let

$$A^+ = \{w \in A \;:\; (v^*, x) \in E \implies (x, w) \notin E\}$$

$$A^- = \{w \in A \;:\; (x, v^*) \in E \implies (w, x) \notin E\}.$$

Since no directed C_4 contains v^* as a vertex, we must have that every $w \in A$ appears in at least one of A^- or A^+ (it may appear in both; in particular, v^* appears in both A^- and A^+ as there are no antiparallel edges). Hence, $A^- \cup A^+ = A$. Thus, at least one of them has cardinality at least $m/2$. Assume, without loss of generality, that $|A^+| \ge m/2$ (otherwise we can reverse the directions of all edges and the result remains intact). Order the vertices of A such that $A = \{v_1, \ldots, v_m\}$ and $v_1 = v^*$, $v_i \in A^+$ for $i = 1, \ldots, |A^+|$, and $v_i \in A^-$ for $i = |A^+|+1, \ldots, m$. Order the vertices of B such that $B = \{u_1, \ldots, u_n\}$ where $(v^*, u_i) \in E$ for $i = 1, \ldots, \rho n$, $(u_i, v^*) \in E$ for $i = \rho n + 1, \ldots, 2\rho n$. Consider the adjacency matrix of G, denoted by M, where M has m rows and n columns, and $M(i,j) = 1$ if $(v_i, u_j) \in E$, $M(i,j) = -1$ if $(u_j, v_i) \in E$ and otherwise $M(i,j) = 0$. Notice that by our ordering of the vertices, the upper left block of M does not contain -1. Namely $M(i,j) \ne -1$ for $i = 1, \ldots, |A^+|$ and $j = 1, \ldots, \rho n$. Denote this upper left block by M_1. Also note that, similarly, $M(i,j) \ne 1$ for $i = |A^+| + 1, \ldots, m$ and $j = \rho n + 1, \ldots, 2\rho n$. Denote this block M_2. Denote by M_3 the block consisting of the rows $i = |A^+| + 1, \ldots, m$ and the columns $j = 1, \ldots, \rho n$. Denote by M_4 the block consisting of the rows $i = |A^+| + 1, \ldots, m$ and the columns $j = 2\rho n + 1, \ldots, n$. Denote by M_5 the block consisting of the rows $i = 1, \ldots, |A^+|$ and the columns $j = 2\rho n + 1, \ldots, n$. Define $\beta = |A^+|/m$. Figure 1 visualizes these terms.

Let $c(s,k)$ denote the number of entries of M equal to k in the block M_s for $s = 1, 2, 3, 4, 5$ and $k = -1, 0, 1$. For normalization purposes, define $f(s,k) = c(s,k)/mn$. Consider the following equalities:

$$f(3,-1) + f(3,0) + f(3,1) = \rho(1-\beta). \tag{1}$$

$$f(1,-1) = 0 \qquad f(1,1) = f(3,-1) - f(3,1) \qquad f(1,0) = \rho\beta - f(3,-1) + f(3,1). \tag{2}$$

$$f(2,1) = 0 \qquad f(2,-1) + f(2,0) = \rho(1-\beta). \tag{3}$$

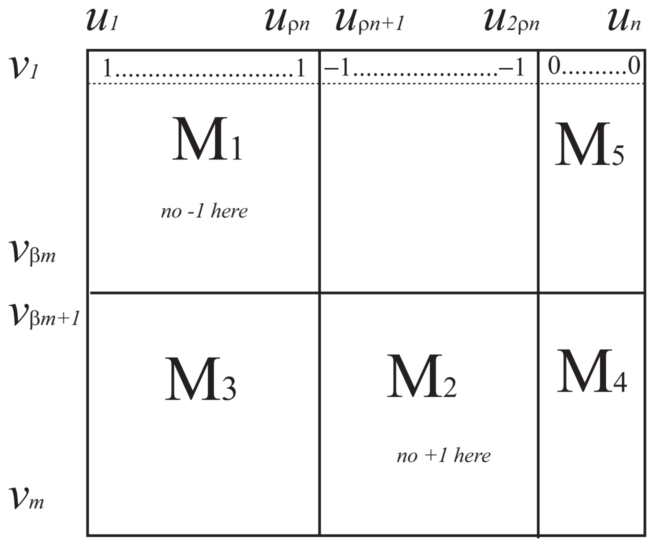

Figure 1: The adjacency matrix M and its blocks

Equality (1) follows from the fact that M_3 contains $\rho(1-\beta)mn$ cells. The equalities in (2) follow from the fact that M_1 does not contain -1 entries, has $\rho\beta mn$ cells, and the fact that $M_1 \cup M_3$ has the same number of $+1$ entries as -1 entries (since the graph G is Eulerian). The equalities in (3) follow from the fact that M_2 does not contain $+1$ entries, and has $\rho(1-\beta)mn$ cells.

We now show that

$$\begin{array}{lll} \text{(a)} & 4\rho^2 - 3\rho + \alpha & \leq 2f(3,-1) - f(2,0), \\ \text{(b)} & 2\rho^2 - \rho & \leq f(2,0) - 2f(3,-1) - f(3,0). \end{array}$$

By the definition of ρ, each column of M contains at least $(1-2\rho)m$ entries equal to 0. Thus $(f(4,0)+f(5,0))mn \geq (1-2\rho)^2 mn$ as M_4 and M_5 together occupy $(1-2\rho)n$ columns of M. Since M has exactly $(1-\alpha)mn$ entries equal to 0, we have

$$(1-\alpha)mn \geq mn\sum_{i=1}^{5} f(i,0) \geq (f(1,0)+f(2,0)+f(3,0))mn + (1-2\rho)^2 mn;$$

that is,

$$1-\alpha \geq f(1,0)+f(2,0)+f(3,0)+(1-2\rho)^2. \tag{4}$$

By equality (2) we know that $f(1,0) = \rho\beta - f(3,-1) + f(3,1)$ and by equality (1) we have $f(3,-1)+f(3,0)+f(3,1) = \rho(1-\beta)$. Using these equalities and inequality (4) we have

$$\begin{aligned} 1-\alpha &\geq \rho\beta - f(3,-1) + f(3,1) + f(2,0) + f(3,0) + (1-2\rho)^2 \\ &= \rho\beta - 2f(3,-1) + f(2,0) + \rho(1-\beta) + (1-2\rho)^2 \\ &= -2f(3,-1) + f(2,0) + 4\rho^2 - 3\rho + 1, \end{aligned}$$

giving inequality (a).

To prove inequality (b), let M' be the submatrix of M consisting of rows $\beta m+1,\ldots,m$ and all columns of M. Since each column of M contains at most ρm entries equal to 1, we have

$$(f(4,1)+f(5,1))mn \le \rho m(1-2\rho)n.$$

Since G is bipartite Eulerian, the number of -1's in M' equals the number of 1's in M'. Thus,

$$\begin{aligned}(f(2,-1)+f(3,-1)+f(4,-1))mn &= (f(2,1)+f(3,1)+f(4,1))mn\\ &= (f(3,1)+f(4,1))mn\\ &\le f(3,1)mn+(f(4,1)+f(5,1))mn\\ &\le f(3,1)mn+\rho(1-2\rho)mn,\end{aligned}$$

which implies,

$$f(2,-1)+f(3,-1) \le f(3,1)+\rho(1-2\rho).$$

Since $f(3,1)+f(3,-1)+f(3,0)=\rho(1-\beta)=f(2,0)+f(2,-1)$, we have

$$\begin{aligned}\rho(2\rho-1) &\le f(3,1)-f(2,-1)-f(3,-1)\\ &= f(2,0)-2f(3,-1)-f(3,0),\end{aligned}$$

proving inequality (b).

Adding inequalities (a) and (b) we have $6\rho^2-4\rho+\alpha \le -f(3,0) \le 0$. Thus

$$\alpha \le -6\rho^2+4\rho = -6\left(\rho-\frac{1}{3}\right)^2+\frac{2}{3} \le \frac{2}{3}. \qquad \square$$

Proof of Theorem 1.1: The last inequality shows that $b(m,n) \le 2mn/3$. Now, suppose G is an Eulerian bipartite digraph with edge density exactly $2/3$ and no directed cycle of length at most 4. The last inequality shows that in this case we must have $\rho = 1/3 = \alpha/2$. Hence, G must be biregular and the cardinality of each vertex class of G must be divisible by 3. For any pair m and n both divisible by 3 it is easy to construct a biregular Eulerian bipartite digraph with edge density $2/3$ and no directed C_4 nor antiparallel edges. We use a construction from [3]. Let $|M_0|=|M_1|=|M_2|=m/3$ and let $|N_0|=|N_1|=|N_2|=n/3$. Construct a bipartite graph with vertex classes $M=M_0\cup M_1\cup M_2$ and $N=N_0\cup N_1\cup N_2$. Create all possible directed edges from M_i to N_i, $i=0,1,2$ and from N_i to M_{i+1} $i=0,1,2$ (modulo 3). Clearly this graph has no antiparallel edges and no directed C_4. It is biregular and has $2mn/3$ edges. This completes the proof of Theorem 1.1. $\square$

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