On counting permutations by pairs of congruence classes of major index

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Abstract

For a fixed positive integer n, let S_n denote the symmetric group of $n!$ permutations on n symbols, and let $\mathrm{maj}(\sigma)$ denote the major index of a permutation σ. Fix positive integers $k < \ell \leq n$, and nonnegative integers i, j. Let $m_n(i\backslash k; j\backslash \ell)$ denote the cardinality of the set $\{\sigma \in S_n : \mathrm{maj}(\sigma) \equiv i \pmod k, \mathrm{maj}(\sigma^{-1}) \equiv j \pmod \ell\}$. In this paper we use combinatorial methods to investigate these numbers. Results of Gordon and Roselle imply that when k, ℓ are relatively prime,

$$m_n(i\backslash k; j\backslash \ell) = \frac{n!}{k \cdot \ell}.$$

We give a combinatorial proof of this in the case when ℓ divides $n-1$ and k divides n.

1 Introduction

Denote by S_n the symmetric group of all $n!$ permutations on the n symbols $1, \ldots, n$. First recall some combinatorial definitions pertaining to permutations. See, e.g., [St].

Definition 1.1 *Let $\sigma \in S_n$. For $1 \leq i \leq n-1$, i is said to be a descent of σ if $\sigma(i) > \sigma(i+1)$.*

Definition 1.2 *The major index of σ, denoted* $\mathrm{maj}(\sigma)$, *is the sum of the descents of σ.*

The values of the statistic maj range from 0 (for the identity) to $\binom{n}{2}$.

In [BS], the following result was discovered using certain representations of the symmetric group S_n, and then proved by means of a bijection as well.

Proposition 1.3 *([BS], Theorem 2.6) Fix an integer $0 \leq i \leq n-1$. Then*

$$(n-1)! = |\{\sigma \in S_n : \mathrm{maj}(\sigma) \equiv i \pmod n\}|.$$

This paper is similarly motivated by the following algebraic discovery. By considering characters of the symmetric group that are induced from irreducible characters of the cyclic subgroup of order n, and computing intertwining numbers, the following identity was discovered, for fixed nonnegative integers i and j ([Su]).

$$(n-2)! = |\{\sigma \in S_n : \mathrm{maj}(\sigma) \equiv i \pmod n, \mathrm{maj}(\sigma^{-1}) \equiv j \pmod{n-1}\}|, \qquad (A)$$

Let $m_n(i\backslash k; j\backslash l)$ be the number of permutations σ in the symmetric group S_n such that $\mathrm{maj}(\sigma) \equiv i \pmod k$ and $\mathrm{maj}(\sigma^{-1}) \equiv j \pmod l$. In this paper we give a combinatorial proof that if l divides $n-1$ and k divides n then $m_n(i\backslash k; j\backslash l) = n!/(kl)$. We also give a completely bijective proof of the special case (A) of the main result.

We are very grateful to the first referee for providing the following information. It turns out that our main result is an easy consequence of some results of Basil Gordon [Go]. In fact, Gordon's work implies a more general result: if k and l are relatively prime and at most n then $m_n(i\backslash k; j\backslash l) = n!/(kl)$. Gordon gives a formula determining a certain generating function, whose identification with the major index statistic is not made clear in his paper, but is made explicit in yet another paper [R]. Gordon evaluates his generating function at roots of unity, from which the result of this paper is easily derived by algebraic manipulation, as well as the more general result stated above. For the sake of completeness, we will present this proof in the next section. The methods of the present paper, however, are purely combinatorial.

As pointed out by the second referee, the numbers $m_n(i\backslash k; j\backslash l)$ also count the number of permutations in S_n whose major index is congruent to i modulo k, and whose inversion number is congruent to j modulo ℓ. This follows from a result of Foata and Schützenberger [FS] which states that major index and inversion number are equidistributed on descent sets. However in this paper our investigations focus exclusively on the major index.

The paper is organised as follows. In Section 2 the main technical lemmas are presented. In Section 3 we derive the main enumerative formulas of the paper, and in Section 4 we give some bijective proofs, including a bijective proof of the identity (A) above.

2 Preliminaries

This section contains the main lemmas that are needed for the rest of the paper.

Let $\gamma \in S_n$ be the n-cycle which takes i to $i+1$ modulo n, for all i. We will sometimes write γ_n for clarity. The circular class of σ is the set of permutations $[\sigma] = \{\sigma\gamma^i, 0 \le i \le n-1\}$. The following observation is due to Klaychko [K]. For our purposes it is more convenient to state the result in terms of the inverse permutation. This formulation also admits an easy proof, which we give below for the sake of completeness.

Lemma 2.1 *([K], [Ga], Lemma 4.1) Let $\sigma \in S_n$. Then the function $\tau \mapsto \mathrm{maj}(\tau^{-1})$ takes on all n possible values modulo n in the circular class of σ. More precisely, we have that* $\mathrm{maj}(\sigma\gamma^i)^{-1} \equiv \mathrm{maj}(\sigma^{-1}) + i \pmod n, 0 \le i \le n-1$.

Proof. Let $\tau = a_1 \dots a_n$ (written as a word). Then $\tau\gamma = a_2 \dots a_n a_1$. Note that i is a descent of τ^{-1} if and only if i appears to the right of $i+1$ when τ is written as a word in $\{1, 2, \dots, n\}$.

By looking at occurrences of i to the right of $i+1$, it is easy to see that $\mathrm{maj}(\tau\gamma)^{-1} - \mathrm{maj}(\tau^{-1}) = a_1 - (a_1 - 1) = +1$, if $a_1 \ne 1, a_1 \ne n$. If $a_1 = 1$, then clearly $\mathrm{maj}(\tau\gamma)^{-1} - \mathrm{maj}(\tau^{-1}) = +1$, while if $a_1 = n$, then the difference is $-(n-1)$.

Hence in all cases the difference is $+1$ modulo n. $\square$

The basic idea of the next lemma is also to be found in [Gu]. We are again grateful to the first referee for bringing this reference to our attention.

Lemma 2.2 *Let* $\sigma \in S_{n-1}$, *and let* σ_i *denote the permutation in* S_n *obtained by inserting* n *in position* i *of* σ, $1 \leq i \leq n$. *Then for each* k *between 1 and* n, *the values of the major index on the set* $\{\sigma_1, \ldots, \sigma_k\}$ *form a consecutive segment of integers* $[m+1, m+k]$, *and the value of* $\operatorname{maj}(\sigma_{k+1})$ *is either* m *or* $m+k+1$ *according as* k *is a descent of* σ *or not, respectively. Note that* $\operatorname{maj}(\sigma_n) = \operatorname{maj}(\sigma)$.

In particular, on the set $\{\sigma_1, \ldots, \sigma_n\}$, *the function* maj *takes on each of the* n *values in the interval* $[\operatorname{maj}(\sigma), \operatorname{maj}(\sigma) + n - 1]$.

Proof. Let $\sigma' = a_1 \ldots a_{n-1}$, with descents in positions $i_1, \ldots i_d$. Let σ be the permutation in S_n obtained by appending n to σ'. Hence $\operatorname{maj}(\sigma) = \operatorname{maj}(\sigma')$. We shall show that the value of $\operatorname{maj}(\sigma')$ increases successively by 1 as n is inserted into σ' in the following order:

1. first in the positions immediately following a descent, starting with the right-most descent and moving to the left;

2. then in the remaining positions, beginning with position 1, from left to right.

For instance, if $\sigma' = 14253$, then the resulting permutations, beginning with σ and then in the order specified above, are

$$142536, 142563, 146253, 614253, 164253, 142653,$$

with respective major indices 6,7,8,9,10,11.

Let σ_k denote the permutation in S_n obtained from σ' by inserting n in position k. Thus $\sigma_k = a_1 \ldots a_{k-1}\ n\ a_k a_{k+1} \ldots a_{n-1}$ for $k = 2, \ldots, n-1$, and $\sigma_1 = n\ a_1 \ldots a_{n-1}$, $\sigma_n = a_1, \ldots a_{n-1}\ n$. Let Δ_k denote the difference $\operatorname{maj}(\sigma_k) - \operatorname{maj}(\sigma')$.

The following facts are easily verified:

1. If n is inserted immediately after a descent of σ', i.e., if $k = i_j + 1,\ 1 \leq j \leq d$, then n contributes a descent in position $i_j + 1$, but the i_jth element ceases to be a descent. Also the $(d - j)$ descents to the right of n are shifted further to the right by one. Thus
$$\Delta_k = (d-j) + (i_j + 1) - i_j = d - j + 1,$$
and hence the difference Δ_k ranges from 1 through d.

2. If $1 \leq k \leq i_1$, then the d descents to the right are shifted over by 1, and thus
$$\Delta_k = d + k,$$
and hence Δ_k ranges from $d+1$ through $d + i_1$.

3. If n is inserted in position k between two descents, but not immediately following a descent, i.e., if $1 + i_j < k \leq i_{j+1},\ j \leq d-1$, then
$$\Delta_k = (d - j) + k,$$
and hence Δ_k ranges from $(d - j + 2) + i_j$ through $d - j + i_{j+1}$.

4. Finally when $i_d + 2 \leq k \leq n-1$,

$$\Delta_k = k,$$

and hence Δ_k ranges from $i_d + 2$ through $n-1$.

This establishes the claim. It also shows that as n is inserted into σ' from left to right, the difference in major index goes up (from $\mathrm{maj}(\sigma')$) first by $(d+1)$, then up by one at each step, except when it is inserted immediately after the *jth* descent, in which case it goes down to $(d-j+1)$. Since when n is in position n, $\mathrm{maj}(\sigma')$ is unchanged, this establishes the statement of the lemma. □

Remark 2.3 *Note that in Lemma 2.2, it need not be true that the values of* maj *on an arbitrary set* $\{\sigma_j, \ldots, \sigma_{j+r}\}$, $j > 1$, *form a consecutive set of integers.*

Lemma 2.4 *Let* $\sigma \in S_{n-1}$, *and let* σ_i *denote the permutation in* S_n *obtained by inserting* n *in position* i *of* σ, *for* $1 \leq i \leq n$. *Then* $\mathrm{maj}(\sigma_i^{-1}) \equiv \mathrm{maj}(\sigma^{-1}) \pmod{n-1}$.

Proof. Consider the effect of inserting n on the set of descents of σ^{-1}. If n is inserted to the right of $n-1$, there is no change; if n is inserted to the left of $n-1$, then $n-1$ becomes a descent of σ_i^{-1}. In either case, the major index of the inverse permutation is unchanged modulo $n-1$. □

Finally we shall need the following result, which generalises Proposition 1.3. It is perhaps known, although we do not know of a precise reference. There is an easy generating function proof which we include for the sake of completeness. In Section 4 we will give a constructive proof of the equivalent statement for inverse permutations.

Proposition 2.5 *Let* $1 \leq k \leq n$. *Then*

$$\frac{n!}{k} = |\{\sigma \in S_n : \mathrm{maj}(\sigma) \equiv j \pmod{k}\}|.$$

Proof. Recall the well-known formula due to MacMahon (see [St], p. 216, Corollary 4.5.9)

$$\sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = \prod_{i=1}^{n-1} (1 + q + \ldots + q^i) \qquad (B)$$

Note that Lemma 2.2 gives an immediate inductive proof of formula (B).

Now fix integers $1 \leq k \leq n$ and $0 \leq j \leq k-1$. To show that the number of permutations in S_n with major index congruent to $j \pmod{k}$ is $n!/k$, it suffices to show that, modulo the polynomial $(1-q^k)$, the left-hand side of (B) equals $(n!/k) \cdot (1 + q + \ldots q^{k-1})$.

Since $1 + q + \ldots + q^i = (1-q^{i+1})/(1-q)$, it follows from the generating function that for fixed $k \leq n$, the sum on the left-hand side vanishes at all kth roots of unity not equal to 1. Hence, modulo $(1-q^k)$, there is a constant c such that

$$\sum_{\sigma \in S_n} q^{\mathrm{maj}(\sigma)} = c(1 + q + \ldots + q^{k-1}).$$

Putting $q=1$ yields $c=n!/k$, as required. $\square$

We end this section with the result pointed out by the first referee, mentioned in the introduction, which is proved along the same lines as Proposition 2.5 by using crucial results of Gordon [Go] and Roselle[R].

Theorem 2.6 *Let $k,\ell \le n$, and assume k, ℓ, are relatively prime. Then*

$$m_n(i\backslash k; j\backslash \ell) = \frac{n!}{k\ell}.$$

Proof. Let $H_n(q_1,q_2) = \sum_{\sigma\in S_n} q_1^{\mathrm{maj}(\sigma)} q_2^{\mathrm{maj}(\sigma^{-1})}$. Let ω_k denote a primitive kth root of unity. It follows from results of Gordon [Go] and Roselle [R] that if $1 < g \neq h \le n$, then $H_n(\omega_g, \omega_h) = 0$.

Now apply this result with $g=k$, $h=\ell$. Let

$$R_n(q_1,q_2) = \sum_{0\le i\le k-1, 0\le j\le \ell-1} m_n(i\backslash k; j\backslash \ell) q_1^i q_2^j.$$

By the result mentioned above, this vanishes for $q_1=\omega_k, q_2=\omega_\ell$. But $\omega_k^i\omega_\ell^j = \omega_{k\ell}^{i\ell+jk}$. It is easy to verify that because k and ℓ are relatively prime, the exponent $i\ell+jk$ runs over all $k\ell$ distinct residue classes modulo $k\ell$, for $0\le i\le k-1$ and $0\le \ell\le j-1$. Hence $R_n(\omega_k,\omega_\ell)$ is a linear combination of all $k\ell$ roots of unity which vanishes. This implies that the polynomial

$$P_n(q) = \sum_{0\le i\le k-1, 0\le j\le \ell-1} m_n(i\backslash k; j\backslash \ell) q^{\alpha(i,j)},$$

where $\alpha(i,j)$ is some fixed permutation of the $k\ell$ integers $\{0,\ldots,k\ell-1\}$, vanishes at a primitive $k\ell$-th root of unity. Hence there is a constant c such that

$$P_n(q) = c(1+q+\ldots+q^{k\ell-1}).$$

It follows immediately that $m_n(i\backslash k; j\backslash \ell) = c$. Putting $q=1$ gives $c=\frac{n!}{k\ell}$. $\square$

3 Enumerative Results

Recall that $m_n(i\backslash k; j\backslash \ell)$ denotes the number of permutations $\sigma\in S_n$ with $\mathrm{maj}(\sigma)\equiv i \pmod k$ and $\mathrm{maj}(\sigma^{-1})\equiv j \pmod \ell$. We give a combinatorial proof of:

Theorem 3.1 *Let ℓ be a divisor of $n-1$, $\ell\neq 1$, and let k be a divisor of n, $k\neq 1$. Fix $0\le i\le k-1$, $0\le j\le \ell-1$. Then*

$$m_n(i\backslash k; j\backslash \ell) = \frac{n!}{k\cdot\ell}.$$

Proof. Let $\sigma \in S_{n-1}$, and construct σ_i, $i = 1, \ldots, n$ in S_n as in Lemma 2.2, by inserting n in position i. Since $\ell | n-1$, we have by Lemma 2.4 that for all i,

$$\mathrm{maj}(\sigma^{-1}) \equiv \mathrm{maj}(\sigma_i^{-1}) \pmod{\ell}.$$

By Lemma 2.2, since the set $\{\mathrm{maj}(\sigma_i) : i = 1, \ldots, n\}$ consists of n consecutive integers, each congruence class modulo k appears exactly $\frac{n}{k}$ times. Hence we have

$$m_n(i \backslash k; j \backslash \ell) = \frac{n}{k} \cdot |\{\sigma \in S_{n-1} : \mathrm{maj}(\sigma^{-1}) \equiv j \pmod{\ell}\}|$$

and the result now follows from Proposition 2.5. □

By examining Lemma 2.2 more closely, we obtain the following recurrence on n for these numbers in the case when k and ℓ are divisors of $n-1$.

Proposition 3.2 *Let ℓ, k be divisors of $n-1$, $\ell \neq 1$, $k \neq 1$. Then*

$$m_n(i \backslash k; j \backslash \ell) = (n-2)! \frac{(n-1)^2}{k \cdot \ell} + m_{n-1}(i \backslash k; j \backslash \ell).$$

Proof. Let $\sigma \in S_{n-1}$, and construct σ_i, $i = 1, \ldots, n$ in S_n as in Lemma 2.2, by inserting n in position i. Since $\ell | n-1$, we have by Lemma 2.4 that for all i,

$$\mathrm{maj}(\sigma^{-1}) \equiv \mathrm{maj}(\sigma_i^{-1}) \pmod{\ell}.$$

Now let $k | n-1$. By Lemma 2.2, the major indices of the first $n-1$ elements σ_i, $i = 1, \ldots, n-1$, form a segment of $n-1$ consecutive integers, and hence the residue class i modulo k appears exactly $\frac{n-1}{k}$ times among them. Also note that $\mathrm{maj}(\sigma) = \mathrm{maj}(\sigma_n)$.

Hence we have

$$m_n(i \backslash k; j \backslash \ell) = \frac{n-1}{k} |\{\sigma \in S_{n-1} : \mathrm{maj}(\sigma^{-1}) \equiv j \pmod{\ell}\}|$$
$$+ |\{\sigma \in S_{n-1} : \mathrm{maj}(\sigma^{-1}) \equiv j \pmod{\ell},\ \mathrm{maj}(\sigma) \equiv i \pmod{k}\}|.$$

Collecting terms and using Proposition 2.5, we obtain

$$m_n(i \backslash k; j \backslash \ell) = \frac{n-1}{k} \frac{(n-1)!}{\ell} + m_{n-1}(i \backslash k; j \backslash \ell),$$

as required. □

We note that while the above arguments are not symmetric in k and ℓ, the numbers $m_n(i \backslash k; j \backslash \ell)$ satisfy

$$m_n(i \backslash k; j \backslash \ell) = m_n(j \backslash \ell; i \backslash k). \qquad (C)$$

This follows by applying the involution $\tau \mapsto \tau^{-1}$.

For arbitrary choices of k, ℓ, these numbers usually depend on the values of i and j. For example for $n = 4$, we have $m_4(0 \backslash 2; 0 \backslash 2) = 8 = m_4(1 \backslash 2; 1 \backslash 2)$, and $m_4(1 \backslash 2; 0 \backslash 2) = 4 = m_4(0 \backslash 2; 1 \backslash 2)$. When $k = \ell = 3$, we have $m_4(0 \backslash 3; 0 \backslash 3) = 4$, $m_4(0 \backslash 3; 1 \backslash 3) = 2 = m_4(0 \backslash 3; 2 \backslash 3)$; and $m_4(1 \backslash 3; 1 \backslash 3) = 3 = m_4(1 \backslash 3; 2 \backslash 3)$. The other values follow by symmetry from (C).

Note that in view of Proposition 2.5, we know that, for fixed ℓ, the sum over $i = 0, 1, \ldots, k-1$ of the numbers $m_n(i \backslash k; j \backslash \ell)$ is $\frac{n!}{\ell}$.

4 Some bijections

In this section we present bijective proofs for some of the results derived in Sections 3 and 2. Recall that this paper was originally motivated by the algebraic discovery of the formula (A). We now give a bijective proof of (A), which is the special case $k = n, \ell = n-1$ of Theorem 3.1, namely

$$m_n(i\backslash n; j\backslash n-1) = (n-2)!$$

Proposition 4.1 *(Bijection for the case $k = n, \ell = n-1$ of Theorem 3.1.) Fix integers $0 \le i \le n-1, 0 \le j \le n-2$. Then the number of permutations σ in S_n such that $\mathrm{maj}(\sigma) \equiv i \pmod{n}$ and $\mathrm{maj}(\sigma^{-1}) \equiv j \pmod{n-1}$, equals $(n-2)!$*

Proof. First note that $(n-2)!$ counts the number of permutations in S_{n-1} having $n-1$ as a fixed point. Let A_{n-1} be this set of permutations, and let B_n be the subset of S_n with major indices as prescribed in the statement of the theorem. Given $\sigma \in A_{n-1}$, by Lemma 2.1 there is a unique circular rearrangement σ' in S_{n-1} whose inverse has major index congruent to $j \pmod{n-1}$. Lemma 2.2 then shows that, for each $i = 0, 1, \ldots, n-1$, there is a unique position in σ' in which to insert n, in order to obtain a permutation $\sigma'' \in S_n$ such that $\mathrm{maj}(\sigma'') \equiv i \pmod{n}$. By Lemma 2.4, the passage from σ' to σ'' does not change the major index of the inverses modulo $n-1$, and thus $\mathrm{maj}({\sigma''}^{-1}) = \mathrm{maj}({\sigma'}^{-1}) \equiv j \pmod{n-1}$. Hence $\sigma \mapsto \sigma''$ gives a well-defined map from A_{n-1} to B_n. To see that this is a bijection, given $\sigma'' \in B_n$, erase the n to obtain $\sigma' \in S_{n-1}$, and let σ be the unique circular rearrangement of σ' such that $\sigma(n-1) = n-1$. Then $\sigma \in A_{n-1}$, and clearly the map is a bijection. $\square$

Example 4.1.1 *Let $n = 6, i = 2, j = 3$. Take $\sigma = 21345 \in A_5$. Note that $\mathrm{maj}(\sigma^{-1}) = 1$. The unique circular rearrangement whose inverse has major index equal to $3 (\equiv 3 \pmod{5})$ is $\sigma' = 34521$. Now $\mathrm{maj}(\sigma') = 7$, since the descents are in positions 3 and 4. Now use the proof of Lemma 2.2. To obtain a permutation with major index 8 $(\equiv 2 \pmod{6})$, insert 6 into position 5, immediately after the right-most descent. This gives $\sigma'' = 345261 \in B_6$.*

The remainder of this section is devoted to giving a constructive proof of Proposition 2.5. A bijection for the case $k = n$ was given in [BS], using Lemma 2.1. We do not know of a bijection for arbitrary k, but a bijection for the case $k = n-1$ is given in the proof which follows.

Proposition 4.2 *(Bijection for the case $k = n-1$ of Proposition 2.5.) Fix an integer $0 \le j \le n-2$. The number of permutations in S_n with major index congruent to $j \pmod{n-1}$ is $n(n-2)! = n!/(n-1)$.*

Proof. Let B_n denote the set $\{\sigma \in S_n : \mathrm{maj}(\sigma^{-1}) \equiv j \pmod{n-1}\}$. It suffices to show that this set has cardinality $n(n-2)!$. Let C_n denote the set of permutations $\tau \in S_n$ such that, when n is erased, $n-1$ is a fixed point of the resulting permutation τ' in S_{n-1}. Observe that C_n has cardinality $n(n-2)!$, since the number of permutations in S_{n-1} which fix $n-1$ is $(n-2)!$, and there are n positions in which n can be inserted.

We describe a bijection between C_n and B_n. If $\tau \in C_n$, let τ' be the permutation in S_{n-1} obtained by erasing n. By definition of C_n, $\tau'(n-1) = n-1$. By Lemma 2.1, there is a unique circular rearrangement $\tau'' \in S_{n-1}$ of τ' such that the major index of the inverse of τ'' is congruent to $j \pmod{n-1}$. Now construct $\tilde{\tau} \in S_n$ by inserting n into τ'' in the same position that it occupied in τ, i.e., $\tilde{\tau}^{-1}(n) = \tau^{-1}(n)$. By Lemma 2.4, $\mathrm{maj}(\tilde{\tau}^{-1}) = \mathrm{maj}(\tau''^{-1}) \equiv j \pmod{n-1}$. Hence we have a map $\tau \mapsto \tilde{\tau} \in B_n$. It is easy to see that this construction can be reversed exactly as in the proof of Proposition 4.1, and hence we have the desired bijection. $\square$

Example 4.2.1 *Let* $n = 5, j = 2$. *Take* $\tau = 32154$. *Then* τ *belongs to the set* C_5 *of the preceding proof. Erasing 5 yields* $\tau' = 3214$, *whose inverse major index is 3. The third cyclic rearrangement* $\tau'' = 4321$ *then has inverse major index* $6 \equiv 2 \pmod 4$, *and* $\tau \mapsto \tilde{\tau} = 43251$.

Now we examine Klyachko's Lemma 2.1 more closely. We obtain the following result, which specialises, in the case $k = n$, to Proposition 1.3.

Lemma 4.3 *Fix integers* $1 \le k \le n$, $0 \le j \le k-1$ *and* $1 \le a \le n-k+1$.

1. *Then*

$$(n-1)! = |\{\sigma \in S_n : \mathrm{maj}(\sigma^{-1}) \equiv j \pmod{k}, \sigma^{-1}(n) \in [n-a-k+2, n-a+1]\}|.$$

2. *Let* $n = qk + r$, $0 \le r \le k-1$. *Fix an integer* s *between 1 and* q. *Then*

$$s(n-1)! = |\{\sigma \in S_n : \mathrm{maj}(\sigma^{-1}) \equiv j \pmod{k}, \sigma^{-1}(n) \in [n-sk+1, n]\}|.$$

Proof. Let A_n denote the set of permutations in S_n which fix n, and let B_n denote the subset of S_n subject to the conditions in the statement of part (1). Let $\tau \in A_n$. Consider the circular class of τ consisting of the set $\{\tau, \tau\gamma, \ldots, \tau\gamma^{n-1}\}$. The proof of Lemma 2.1 shows that because $\tau(n) = n$, we have the exact equality $\mathrm{maj}(\tau\gamma^i) = \mathrm{maj}(\tau) + i$, for $0 \le i \le n-1$. In particular, for any $1 \le k \le n$, the first k circular rearrangements $\tau\gamma^i, 0 \le i \le k-1$, have the property that the major indices of their inverses form a complete residue system modulo k. More generally, this observation holds for any k consecutive circular rearrangements $\tau\gamma^i, a \le i \le a+k-1$, where a is any fixed integer $1 \le a \le n-k+1$.

Hence for every $\tau \in A_n$, there is a unique $i, a \le i \le a+k-1$ such $\sigma = \tau\gamma^i$ has $\mathrm{maj}(\sigma^{-1}) \equiv j \pmod{k}$. Since n is in position $n-i$ in $\tau\gamma^i$, clearly $n-a-k+2 \le \sigma^{-1}(n) \le n-a+1$. Thus $\tau \mapsto \sigma$ gives a well-defined map from A_n to B_n. Conversely given $\sigma \in B_n$, with $\sigma^{-1}(n) = n-i+1,\ a \le i \le a+k-1$, let $\tau \in S_n$ be defined by $\tau\gamma^i = \sigma$. Then clearly $\tau(n) = n$, and $\tau \in A_n$. This shows that our map is a bijection, and (1) is proved.

For (2), again we start with the set A_n of the $(n-1)!$ permutations in S_n which fix n. Let $\tau \in A_n$. Then as in the preceding proof, for $i = 0, 1, \ldots, sk-1$, the first sk circular rearrangements $\tau\gamma^i$ have n in position $(n-i)$, and $\mathrm{maj}((\tau\gamma^i)^{-1}) = \mathrm{maj}(\tau) + i$. In particular, for each $J = 1, \ldots, s$, the major index of the inverse permutations in the subset $\{\tau\gamma^{(J-1)k+i} : 0 \le i \le k-1\}$ is a complete residue system modulo k. Hence the first sk rearrangements contain exactly s permutations with inverse major index congruent to $j \pmod{k}$. This establishes (2). $\square$

We are now ready to give a constructive proof of an equivalent restatement of Proposition 2.5, by looking at the circular classes of permutations $\tau \in S_n$ which fix n. Note that the statement of Proposition 4.4 (or Proposition 2.5) is invariant with respect to taking inverses, i.e., it says that $\frac{n!}{k}$ is also the number of permutations in S_n with constant major index modulo k. Our constructive proof, however, works only for the inverse permutations.

Proposition 4.4 *(Constructive proof)*

$$\frac{n!}{k} = |\{\sigma \in S_n : \mathrm{maj}(\sigma^{-1}) \equiv j \pmod{k}\}|.$$

Proof. We proceed inductively. We assume $k \leq n-1$, since the case $k = n$ was dealt with in Proposition 1.3. It is easy to verify directly that the statement holds for $n = 3$. Assume we have constructed the permutations in S_{n-1} with inverse major index congruent to $j \pmod{k}$. Note that this means we can identify these permutations in the subset A_n of S_n. Let $\tau \in A_n$. We show how to pick out the permutations in the circular class of τ with inverse major index congruent to $j \pmod{k}$. Let $n = qk + r$. Taking $s = q$ in Lemma 4.3 (2), the proof shows how to pick out the q permutations in the first qk circular rearrangements $\tau\gamma^i, 0 \leq i \leq qk-1$. Now consider the remaining r (recall $r < k$) rearrangements $\tau\gamma^i, qk \leq i \leq qk+r-1$. These will contain a (necessarily unique) permutation with inverse major index congruent to $j \pmod{k}$, iff $\mathrm{maj}(\tau^{-1}) \equiv j - i \pmod{k}$, for $qk \leq i \leq qk+r-1$, i.e., iff $\mathrm{maj}(\tau^{-1}) \equiv j - t \pmod{k}$, for $t = 0, \ldots, r-1$. By induction hypothesis for each $t = 0, \ldots, r-1$, there are exactly $(n-1)!/k$ such permutations in A_n. Hence there are $r(n-1)!/k$ permutations in A_n whose circular class is such that, among the last r rearrangements, there is a permutation with inverse maj congruent to $j \pmod{k}$.

We have thus accounted for exactly $q(n-1)! + r(n-1)!/k = n!/k$ permutations $\sigma \in S_n$ with $\mathrm{maj}(\sigma^{-1}) \equiv j \pmod{k}$. $\square$

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