Bitableaux Bases for some Garsia-Haiman Modules and Other Related Modules

E. E. Allen *

Department of Mathematics
Wake Forest University
Winston-Salem, NC 27109
allene@wfu.edu

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Abstract

For certain subsets S and T of $\mathcal{A} = \{\cdots, (0,2), (0,1), (0,0), (1,0), (2,0), \cdots\}$ and factor spaces $\mathbb{C}_S[X,Y]$, $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ and $\mathbb{C}^-_{S,T}[X,Y,Z,W]$, bitableaux bases are constructed that are indexed by pairs of standard tableaux and sequences in the collections Υ_{ψ_S} and Υ_{ψ_T}. These bases give combinatorial interpretations to the appropriate Hilbert series of these spaces as well as the graded character of $\mathbb{C}_S[X,Y]$.

The factor space $\mathbb{C}_S[X,Y]$ is an analogue of the coinvariant ring of a polynomial ring in two sets of variables. $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ and $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ are analogues of coinvariant spaces in symmetric and skew-symmetric polynomial settings, respectively. The elements of the bitableaux bases are appropriately defined images in the polynomial spaces of bipermanents. The combinatorial interpretations of the respective Hilbert series and graded characters are given by statistics based on *cocharge tableaux*.

Additionally, it is shown that the Hilbert series and graded characters factor nicely. One of these factors gives the Hilbert series of a collection of Schur functions $s_{\lambda/\mu}$ where μ varies in an appropriately defined λ.

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1. Introduction.

Let $\mathcal{A}$ be the alphabet

$$\mathcal{A} \;=\; \Big\{(f,g) : f \text{ and } g \text{ are nonnegative integers such that } fg = 0\Big\}. \tag{1.1}$$

Specifically,

$$\mathcal{A} \;=\; \Big\{\cdots,(0,3),(0,2),(0,1),(0,0),(1,0),(2,0),(3,0),\cdots\Big\}.$$

The elements of $\mathcal{A}$ are the coordinates (i,k), of the cells of the hookshape in the first quadrant of the plane as shown:

$$\begin{array}{llllll}
\vdots & & & & & \\
\hline
\multicolumn{1}{|l|}{(3,0)} & & & & & \\
\hline
\multicolumn{1}{|l|}{(2,0)} & & & & & \\
\hline
\multicolumn{1}{|l|}{(1,0)} & & & & & \\
\hline
\multicolumn{1}{|l|}{(0,0)} & \multicolumn{1}{l|}{(0,1)} & \multicolumn{1}{l|}{(0,2)} & \multicolumn{1}{l|}{(0,3)} & \multicolumn{1}{l|}{(0,4)} & \multicolumn{1}{l|}{(0,5)} \quad \cdots. \\
\hline
\end{array}$$

We will say that

$$(a_1,b_1) <_{\mathcal{A}} (a_2,b_2)$$

if and only if $a_1 - b_1 < a_2 - b_2$.

Let $\mathbb{C}[X,Y,Z,W]$ denote the polynomial ring with complex coefficients in $X = \{x_1,x_2,\cdots,x_n\}$, $Y = \{y_1,y_2,\cdots,y_n\}$, $Z = \{z_1,z_2,\cdots,z_n\}$ and $W = \{w_1,w_2,\cdots,w_n\}$. Given a subset

$$S \;=\; \Big\{(a_1,b_1),(a_2,b_2),\cdots,(a_n,b_n)\Big\} \subset \mathcal{A}$$

of the alphabet $\mathcal{A}$ listed in increasing order with respect to $<_{\mathcal{A}}$, we define M_S to be the $n \times n$ matrix

$$M_S \;=\; (x_i^{a_k} y_i^{b_k})_{1\le i,k\le n}$$

and $\Delta_S(X,Y)$ to be the determinant of M_S. Let ∂_{x_i} denote the partial differential operator with respect to x_i. With $P(X,Y) \in \mathbb{C}[X,Y]$, we will set

$$P(\partial_X,\partial_Y) \;=\; P(\partial_{x_1},\partial_{x_2},\cdots,\partial_{x_n},\partial_{y_1},\partial_{y_2},\cdots,\partial_{y_n}).$$

Setting $\mathcal{I}_S(X,Y)$ to be the ideal

$$\mathcal{I}_S(X,Y) \;=\; \Big\{P(X,Y) \in \mathbb{C}[X,Y] : P(\partial_X,\partial_Y)\Delta_S(X,Y) \;=\; 0\Big\}, \tag{1.2}$$

we define $\mathbb{C}_S[X,Y]$ to be the polynomial quotient ring

$$\mathbb{C}_S[X,Y] \;=\; \mathbb{C}[X,Y]/\mathcal{I}_S(X,Y). \tag{1.3}$$

The rings $\mathbb{C}_S[X,Y]$ are called *Garsia-Haiman modules.* A. Garsia and M. Haiman introduced modules of this type to study the q,t-Kostka coefficients (see [10]).

The action of $\sigma \in S_n$ (where S_n denotes the symmetric group on n letters) on the polynomial

$$P(X,Y,Z,W) \in \mathbb{C}[X,Y,Z,W]$$

is defined by setting

$$\begin{aligned} &\sigma_{X,Y,Z,W}\ P(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_n,z_1,\cdots,z_n,w_1,\cdots,w_n) \\ &\quad = P(x_{\sigma_1},x_{\sigma_2},\cdots,x_{\sigma_n},y_{\sigma_1},y_{\sigma_2},\cdots,y_{\sigma_n},z_{\sigma_1},\cdots,z_{\sigma_n},w_{\sigma_1},\cdots,w_{\sigma_n}), \end{aligned} \tag{1.4}$$

$$\begin{aligned} &\sigma_{X,Y}\ P(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_n,z_1,\cdots,z_n,w_1,\cdots,w_n) \\ &\quad = P(x_{\sigma_1},x_{\sigma_2},\cdots,x_{\sigma_n},y_{\sigma_1},y_{\sigma_2},\cdots,y_{\sigma_n},z_1,\cdots,z_n,w_1,\cdots,w_n) \end{aligned} \tag{1.5}$$

and

$$\begin{aligned} &\sigma_{Z,W}\ P(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_n,z_1,\cdots,z_n,w_1,\cdots,w_n) \\ &\quad = P(x_1,x_2,\cdots,x_n,y_1,y_2,\cdots,y_n,z_{\sigma_1},\cdots,z_{\sigma_n},w_{\sigma_1},\cdots,w_{\sigma_n}). \end{aligned}$$

Note that the subscripts ${}_{X,Y,Z,W}$, ${}_{X,Y}$ and ${}_{Z,W}$ denote the sets of variables on which σ acts.

For subsets S and T of $\mathcal{A}$, let

$$\mathbb{C}^{S_n}[X,Y,Z,W] = \big\{P \in \mathbb{C}[X,Y,Z,W] : \sigma_{X,Y,Z,W}\ P \;=\; P\ \forall\ \sigma \in S_n\big\}, \tag{1.6}$$

$$\begin{aligned} &\mathcal{I}^+_{S,T}(X,Y,Z,W) \\ &= \big\{P \in \mathbb{C}^{S_n}[X,Y,Z,W] : P(\partial_X,\partial_Y,\partial_Z,\partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W) \;=\; 0\big\} \end{aligned} \tag{1.7}$$

and

$$\mathbb{C}^+_{S,T}[X,Y,Z,W] \;=\; \mathbb{C}^{S_n}[X,Y,Z,W]/\mathcal{I}^+_{S,T}(X,Y,Z,W). \tag{1.8}$$

Analogously, with $sgn(\sigma)$ denoting the sign of the permutation σ, let

$$\mathbb{C}^-[X,Y,Z,W] = \Big\{P \in \mathbb{C}[X,Y,Z,W] : \sigma_{X,Y,Z,W}\ P = sgn(\sigma)\ P\ \forall\ \sigma \in S_n\Big\}, \tag{1.9}$$

$$\begin{aligned} &\mathcal{I}^-_{S,T}(X,Y,Z,W) \\ &= \big\{P \in \mathbb{C}^-[X,Y,Z,W] : P(\partial_X,\partial_Y,\partial_Z,\partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W) = 0\big\} \end{aligned} \tag{1.10}$$

and

$$\mathbb{C}^-_{S,T}[X,Y,Z,W] \;=\; \mathbb{C}^-[X,Y,Z,W]/\mathcal{I}^-_{S,T}(X,Y,Z,W). \tag{1.11}$$

It should be noted that $\mathbb{C}^{S_n}[X,Y,Z,W]$ and $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ are rings. $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ is not closed under multiplication and hence is not a ring. Thus we will be considering $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ simply as a module.

In this paper, we construct bases for $\mathbb{C}_S[X,Y]$, $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ and $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ (for certain general classes of S and T that we shall call *dense*) that are indexed by pairs of standard tableaux and sequences in the collections Υ_{ψ_S} and Υ_{ψ_T}. In Section Two, we introduce tableaux, bitableaux, bipermanents and bideterminants. In Section Three, we define *dense Garsia-Haiman Modules.* The bases for $\mathbb{C}_S[X,Y]$, $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ and $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ are constructed in Sections Three, Four and Five, respectively, using bipermanents and bideterminants. Specifically, with $\mathcal{ST}_n$ denoting the collection of standard tableaux with n cells, $sh(Q)$ denoting the shape of the tableaux Q, $\mathcal{CO}_S$ denoting a collection of *cocharge tableaux* and $[Q,C]_{per}$, $[U,V]^+_{per}$ and $[U,V]^-_{per}$ denoting certain images of bipermanents in the factor spaces $\mathbb{C}_S[X,Y]$, $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ and $\mathbb{C}^-_{S,T}[X,Y,Z,W]$, respectively, we will prove the following theorems:

Theorem 1.1.

If S is dense then the collection

$$\mathcal{B}_{\psi_S} \;=\; \Big\{[Q,C]_{per} : Q \in \mathcal{ST}_n,\; C \in \mathcal{CO}_S \;\textit{and}\; sh(Q) = sh(C)\Big\}$$

is a basis for $\mathbb{C}_S[X,Y]$ with coefficients from $\mathbb{C}$.

Theorem 1.2.

If S and T are dense then

$$\mathcal{BSQ}_{S,T} \;=\; \Big\{[U,V]^+_{per} : U \in \mathcal{CO}_S, V \in \mathcal{CO}_T \;\textit{and}\; sh(U) = sh(V)\Big\}$$

is a basis for $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$.

Theorem 1.3.

If S and T are dense then the collection

$$\mathcal{BAQ}_{S,T} \;=\; \Big\{[U,V]^-_{per} :\; U \in \mathcal{CO}_S,\; V \in \mathcal{CO}_T \;\textit{and}\; sh(U) = sh(V)\Big\}$$

is a basis for $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$.

Note that these theorems are Theorem 3.8, Theorem 4.4 and Theorem 5.7, respectively.

If R_{u_1,u_2,u_3,u_4} is a homogeneous subspace of dimension u_1 in X, u_2 in Y, u_3 in Z and u_4 in W, then we define the Hilbert series $\mathcal{H}(R)$ to be

$$\mathcal{H}(R) \;=\; \sum_{u_1,u_2,u_3,u_4} dim(R_{u_1,u_2,u_3,u_4})\; t^{u_1}\; q^{u_2}\; r^{u_3}\; s^{u_4}.$$

These bases imply combinatorial interpretations for the Hilbert series of $\mathbb{C}_S[X,Y]$, $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ and $\mathbb{C}^-_{S,T}[X,Y,Z,W]$.

Theorem 1.4.

If S is dense then the Hilbert series $\mathcal{H}(\mathbb{C}_S[X,Y])$ is given by

$$\mathcal{H}(\mathbb{C}_S[X,Y]) \;=\; \sum_{\lambda\vdash n} h_\lambda \sum_{M\in\mathcal{ST}_\lambda}\; \sum_{\rho\in\Upsilon_{\psi_S}} t^{|C_{\rho,1}(M)|} q^{|C_{\rho,2}(M)|}$$

where $\mathcal{ST}_\lambda$ denotes the collection of standard tableaux of shape λ, h_λ denotes the number of standard tableaux of shape λ, Υ_{ψ_S} is a collection of sequences defined in equation (3.4) and $|C_{\rho,1}(M)|$ and $|C_{\rho,2}(M)|$ denote the sums of the first and second coordinates, respectively, of the entries of $C_\rho(M)$.

Theorem 1.5.

If S and T are dense then the Hilbert series $\mathcal{H}(\mathbb{C}^+_{S,T}[X,Y,Z,W])$ for $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ is given by

$$\begin{aligned}&\mathcal{H}(\mathbb{C}^+_{S,T}[X,Y,Z,W])\\ &= \sum_{\lambda\vdash n}\sum_{\rho\in\Upsilon_{\psi_S}}\sum_{\rho'\in\Upsilon_{\psi_T}}\sum_{(M,N)\in\mathcal{ST}_\lambda\times\mathcal{ST}_\lambda} t^{|C_{\rho,1}(M)|} q^{|C_{\rho,2}(M)|} r^{|C_{\rho',1}(N)|} s^{|C_{\rho',2}(N)|}\end{aligned}$$

where $\mathcal{ST}_\lambda$ denotes the collection of standard tableaux of shape λ and Υ_{ψ_S} and Υ_{ψ_T} are collections of sequences defined in equation (3.4).

Theorem 1.6.

If S and T are dense the Hilbert series $\mathcal{H}(\mathbb{C}^-_{S,T}[X,Y,Z,W])$ for $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ is given by

$$\begin{aligned}&\mathcal{H}(\mathbb{C}^-_{S,T}[X,Y,Z,W])\\ &= \sum_{\lambda\vdash n}\sum_{\rho\in\Upsilon_{\psi_S}}\sum_{\rho'\in\Upsilon_{\psi_T}}\sum_{(M,N)\in\mathcal{ST}_\lambda\times\mathcal{ST}_\lambda} t^{|C_{\rho,1}(M)|} q^{|C_{\rho,2}(M)|} r^{|C_{\rho',1}(N^t)|} s^{|C_{\rho',2}(N^t)|}\end{aligned}$$

where $\mathcal{ST}_\lambda$ denotes the collection of standard tableaux of shape λ, Υ_{ψ_S} and Υ_{ψ_T} are collections of sequences defined in equation (3.4) and N^t denotes the transpose of the standard tableau N.

Note that the above three theorems are Corollary 3.10, Corollary 4.5 and Corollary 5.8, respectively. Furthermore, since the action of S_n on a basis for $\mathbb{C}_S[X,Y]$ can be described in terms of irreducible representations of S_n, we will be able to compute the graded character of the spaces $\mathbb{C}_S[X,Y]$ (see Corollary 3.12).

Theorem 1.7.

With S dense, the graded character $char_{q,t}(\mathbb{C}_S[X,Y])$ of $\mathbb{C}_S[X,Y]$ is given by:

$$char_{q,t}(\mathbb{C}_S[X,Y]) \;=\; \sum_{\lambda\vdash n}\chi^\lambda \sum_{\rho\in\Upsilon_{\psi_S}}\;\sum_{M\in\mathcal{ST}_\lambda} t^{|C_{\rho,1}(M)|}q^{|C_{\rho,2}(M)|}$$

where $\mathcal{ST}_\lambda$ denotes the collection of standard tableaux of shape λ, Υ_{ψ_S} is a collection of sequences defined in equation (3.4), and $|C_{\rho,1}(M)|$ and $|C_{\rho,2}(M)|$ denote the sum of the first and second coordinates, respectively, of the entries of $C_\rho(N)$ and χ^λ denotes the irreducible S_n character corresponding to shape λ.

Note that in Theorem 1.4, Theorem 1.5, Theorem 1.6 and Theorem 1.7, due to the construction of the cocharge statistic, we are able to factor the resulting Hilbert series as well as the graded character (see the theorems in the respective chapters). Additionally, one of the factors of these polynomials turns out to be the Hilbert series of a collection of skew Schur functions $s_{\lambda/\mu}$ as μ varies in a partition λ that corresponds to to a dense set S (see Corollary 3.13).

It should be noted that $\mathbb{C}_S[X,Y]$, $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ and $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ are generalizations of some well-studied modules. For example, if

$$S \;=\; \Big\{(0,0),(1,0),\cdots,(n-1,0)\Big\} \tag{1.12}$$

then $\Delta_S(X,Y)$ is the Vandermonde determinant in the variables $\{x_1,x_2,\cdots,x_n\}$ and $\mathcal{I}_S(X,Y)$ is the ideal generated by the elementary symmetric functions

$$e_k \;=\; \sum_{1\le i_1<i_2<\cdots<i_k\le n} x_{i_1}x_{i_2}\cdots x_{i_k}$$

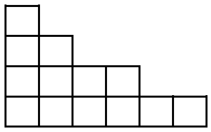
for $1\le k\le n$ and the monomials $\{y_1,y_2,\cdots,y_n\}$. In this case, $\mathbb{C}_S[X,Y]$ is the ring of coinvariants in the variables $X=\{x_1,x_2,\cdots,x_n\}$ associated with the symmetric group S_n. Bases for $\mathbb{C}_S[X,Y]$ are given in [13] (in which it is shown that the collection $\{x_1^{\epsilon_1}x_2^{\epsilon_2}\cdots x_n^{\epsilon_n} : 0\le \epsilon_i\le i-1\}$ is a basis), in [8] (in which a basis is constructed using the descent monomials) and [14] and [15] (in which it is shown that the Schubert Polynomials form a basis). $\mathbf{C}^+_{S,S}$ has been shown to have a basis closely related to the descent monomials (see [2] or [16]). A. Garsia computed the Hilbert series of $\mathbf{C}^-_{S,S}$ in [9]. It should be noted that all of the above results are with the collection S as given in (1.12). The results of this paper are related to a much larger class of collections. Additionally, the construction of a basis for $\mathbf{C}^-_{S,T}$ (for general classes of dense sets S

and T) corresponds to constructing a basis in a noncommutative *letter place algebra*. Specifically, these factor spaces $\mathbf{C}^{-}_{S,T}$ are analogues of coinvariant rings in an exterior algebra setting. The proofs of these theorems include algorithms for expanding elements of these modules in terms of the respective bases.

M. Haiman recently announced a proof showing that the dimension of $\mathbb{C}_S[X,Y]$ for classes of S related to partitions have dimension $n!$ (see [12]). This particular paper deals with a different type of class for S (specifically dense sets), construction of appropriate bases for $\mathbb{C}_S[X,Y]$ and its relation to the rings $\mathbb{C}^{+}_{S,T}[X,Y,Z,W]$ and $\mathbb{C}^{-}_{S,T}[X,Y,Z,W]$.

2. Bitableaux, Bipermanents and Bideterminants

General references for much of the material in this section (specifically, letter place algebras, bitableaux, bideterminants and bipermanents) can be found in [7] or [11]. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ be a partition of n. In other words, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ and $n = \lambda_1 + \lambda_2 + \cdots + \lambda_k$. This is commonly denoted by $\lambda \vdash n$. We will use the French notation for depicting Ferrers diagrams and tableaux. A Ferrers diagram of shape λ has λ_1 cells in the first row, and continuing north, has λ_2 cells in the second row, etc. For example,



is a Ferrers diagram of shape $(6,4,2,1)$. A *tableau* of shape λ is a Ferrers diagram of shape λ where each cell contains an entry from some alphabet. A tableau Q of shape λ is said to be *injective* if the alphabet is $\{1,2,\cdots,n\}$ and each of the letters appear exactly once as entries in the cells of Q. (Note that if Q has shape $\lambda \vdash n$ then Q has exactly n cells.) We will say that a tableau Q is *standard* if Q is injective, $sh(Q) = \lambda$ where $\lambda \vdash n$ and the entries strictly increase from west to east (left to right) and from south to north (bottom to top). We will say that a tableau Q is *column-strict* if the entries of Q increase weakly from west to east but increase strictly from south to north. A tableau Q is said to be *row-strict* if the entries increase strictly in the rows from west to east and increase weakly in the columns from south to north. We will denote the collections of all column-strict tableaux and row-strict tableaux with entries from the alphabet $\mathcal{A}$ and exactly n cells by $\mathcal{CS}_n$ and $\mathcal{RS}_n$, respectively. The set of standard tableaux with entries $\{1,2,\cdots,n\}$ will be denoted by $\mathcal{ST}_n$. We will denote the shape of a tableau Q (i.e., the shape of the underlying Ferrers diagram) as $sh(Q)$.

The column sequence $cs(Q)$ of a tableau Q is a listing of the entries of Q from south to north (bottom to top) in each column starting with the column farthest west (left) and continuing east (right). Analogously, the row sequence $rs(Q)$ of a tableau Q is a listing of the entries of Q from west to east in each column starting with the row farthest south and continuing north.

If Q is a tableau of shape $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ and R is a tableau of shape $\mu = (\mu_1, \mu_2, \cdots, \mu_j)$, we will say that Q is *longer* than R if and only if λ is lexicographically larger than μ. Similarly, we will say that Q is *higher* than R if and only if the conjugate partition λ' is lexicographically larger than the conjugate partition μ'. The transpose Q^t of a tableau Q of shape λ is the tableau of shape λ' obtained by reflecting Q along its diagonal.

Example

Let

$$Q = \begin{array}{llll} 6 & 8 & & \\ 3 & 5 & 7 & \\ 1 & 2 & 4 & 9 \end{array} .$$

Then

$$Q^t = \begin{array}{lll} 9 & & \\ 4 & 7 & \\ 2 & 5 & 8 \\ 1 & 3 & 6 \end{array},$$

$$rs(Q) = 1, 2, 4, 9, 3, 5, 7, 6, 8$$

and

$$cs(Q) = 1, 3, 6, 2, 5, 8, 4, 7, 9. \square$$

Let $\mathcal{LP}$ be the algebra of polynomials over $\mathbb{C}$ in the indeterminants $(a_i|b_k)$ where a_i and b_k are elements from some alphabets $\mathcal{AL}$ and $\mathcal{BL}$ respectively. $\mathcal{LP}$ is called the *letter place algebra.* Note that the letter place algebra $\mathcal{LP}$ is commutative. Specifically, we have that

$$\begin{aligned} &(a_1|b_1)\ (a_2|b_2)\ (a_3|b_3) \cdots (a_n|b_n) \\ &= (a_{\sigma(1)}|b_{\sigma(1)})\ (a_{\sigma(2)}|b_{\sigma(2)})\ (a_{\sigma(3)}|b_{\sigma(3)}) \cdots (a_{\sigma(n)}|b_{\sigma(n)}) \end{aligned}$$

for all $\sigma \in S_n$ (the symmetric group). Note that this implies

$$(a_{\sigma(1)}|b_1)\ (a_{\sigma(2)}|b_2) \cdots (a_{\sigma(n)}|b_n) = (a_1|b_{\sigma^{-1}(1)})\ (a_2|b_{\sigma^{-1}(2)}) \cdots (a_n|b_{\sigma^{-1}(n)}). \quad (2.1)$$

Let I be an injective tableau of shape $\lambda = (\lambda_1, \cdots, \lambda_k)$. Let R_i $(1 \le i \le k)$ denote the collection of integers in the i^{th} row of I. Similarly, let D_i $(1 \le i \le j)$ denote the collection of integers in the i^{th} column of I. Set

$$R(I) = S_{R_1} \times S_{R_2} \times \cdots \times S_{R_k} \quad (2.2)$$

and

$$D(I) = S_{D_1} \times S_{D_2} \times \cdots \times S_{D_j}, \quad (2.3)$$

where S_{R_i} and S_{D_i} denote the symmetric group on the collections of elements R_i and D_i respectively. Define, in the group algebra $\mathbb{C}[S_n]$,

$$P(I) \;=\; \sum_{\sigma \in R(I)} \sigma \tag{2.4}$$

and

$$N(I) \;=\; \sum_{\sigma \in D(I)} sgn(\sigma)\; \sigma. \tag{2.5}$$

Similarly, let $[i_1, i_2, \cdots, i_k]$ and $[i_1, i_2, \cdots, i_k]'$ denote the formal sums in the group algebra $\mathbb{C}[S_n]$ of S_n,

$$[i_1, i_2, \cdots, i_k] \;=\; \sum_{\sigma \in S_{\{i_1, i_2, \cdots, i_k\}}} \sigma$$

and

$$[i_1, i_2, \cdots, i_k]' \;=\; \sum_{\sigma \in S_{\{i_1, i_2, \cdots, i_k\}}} sgn(\sigma)\; \sigma.$$

Now, given two tableaux, U and V, of the same shape λ and an injective tableau I (also of shape λ), let u_i and v_i be the entries in U and V that correspond to the cell containing i in I, respectively. The *bideterminant* $(U, V)_{det}$ is defined to be

$$\begin{aligned} (U, V)_{det} &= N(I)\; (u_1|v_1) \cdots (u_n|v_n) \\ &= \sum_{\sigma \in D(I)} sgn(\sigma)\; \sigma\; (u_1|v_1) \cdots (u_n|v_n) \\ &= \sum_{\sigma \in D(I)} sgn(\sigma)\; (u_{\sigma(1)}|v_1) \cdots (u_{\sigma(n)}|v_n) \end{aligned}$$

and the *bipermanent* $(U, V)_{per}$ is defined as

$$\begin{aligned} (U, V)_{per} &= P(I)\; (u_1|v_1) \cdots (u_n|v_n) \\ &= \sum_{\sigma \in R(I)} (u_{\sigma(1)}|v_1) \cdots (u_{\sigma(n)}|v_n). \end{aligned}$$

The *content* $con(U, V)$ of a bideterminant $(U, V)_{det}$ (or a bipermanent $(U, V)_{per}$) is

$$con(U, V) \;=\; ((\alpha_1, \alpha_2, \cdots, \alpha_k), (\beta_1, \beta_2, \cdots, \beta_j))$$

where α_i denotes the number of entries of a_i in U and β_h denotes the number of entries of b_h in V. We will say that the bitableaux

$$(U, V) <_{s'c} (M, Q),$$

where U and V have shape λ and M and Q have shape μ when

1. $\lambda' <_L \mu'$ (where $>_L$ denotes the lexicographic ordering); or
2. if $\lambda = \mu$ then $cs(U)\ cs(V) >_L cs(M)\ cs(Q)$.

The following theorem may be found in [7] or [11]. The proof of Theorem 2.1 included here (which was pointed out to this author by A. Garsia) is different than the proof found in either [7] or [11]. This particular proof is included since it provides an algorithm that will be useful later in this development.

Theorem 2.1.

The collections

$$\mathcal{CSD} \ = \ \Big\{(M,Q)_{det} : M,Q \in \mathcal{CS}, sh(M) = sh(Q)\Big\} \tag{2.6}$$

and

$$\mathcal{CSP} \ = \ \Big\{(M,Q)_{per} : M,Q \in \mathcal{CS}, sh(M) = sh(Q)\Big\},$$

where $\mathcal{CS}$ denotes the collection of column-strict tableaux, linearly span the letter place algebra $\mathcal{LP}$ with coefficients from $\mathbb{C}$.

Proof

Note that

$$(u_1|v_1)(u_2|v_2)(u_3|v_3)\cdots(u_n|v_n) \ = \ (U,V)_{det}$$

where

$$U \ = \ u_1\ u_2\ u_3\ \cdots\ u_n,$$
$$V \ = \ v_1\ v_2\ v_3\ \cdots\ v_n$$

and

$$I \ = \ 1\ \ 2\ \ 3\ \ \cdots\ n.$$

Thus $\mathcal{LP}$ is spanned by the collection

$$\Big\{(U,V)_{det} : sh(U) = sh(V)\Big\}$$

where U and V are not necessarily column-strict.

Suppose that U and V are two tableaux of shape λ, suppose I is an injective tableau of shape λ and let u_i and v_i be the entries in U and V that correspond to the cell containing i in I respectively. Recall that if

$$\Big\{i_{j,1}, i_{j,2}, \cdots, i_{j,k_j}\Big\}$$

is the j^{th} column of I, (where I has h columns and thus $1 \le j \le h$) then

$$N(I) \ = \ [i_{1,1}, i_{1,2}, \cdots, i_{1,k_1}]'[i_{2,1}, i_{2,2}, \cdots, i_{2,k_2}]' \cdots [i_{h,1}, i_{h,2}, \cdots, i_{h,k_h}]'$$

and

$$(U,V)_{det} \;=\; N(I)\ (u_1|v_1)(u_2|v_2)(u_3|v_3)\cdots(u_n|v_n).$$

For a given content $((\alpha_1,\alpha_2,\cdots,\alpha_k),(\beta_1,\beta_2,\cdots,\beta_j))$, let (U,V) be a bitableau such that

$$con(U,V) \;=\; ((\alpha_1,\alpha_2,\cdots,\alpha_k),(\beta_1,\beta_2,\cdots,\beta_j))$$

and $sh(U) = sh(V) = 1^n$. Specifically, without loss of generality we may assume that

$$U \;=\; \begin{array}{c} u_n \\ u_{n-1} \\ \vdots \\ u_2 \\ u_1 \end{array}$$

and

$$V \;=\; \begin{array}{c} v_n \\ v_{n-1} \\ \vdots \\ v_2 \\ v_1 \end{array}$$

where $u_i \leq u_{i+1}$ and $v_i \leq v_{i+1}$ for $1 \leq i \leq n-1$. If $u_i = u_{i+1}$ or $v_i = v_{i+1}$ for any i, then $(U,V)_{det} = 0$ and $(U,V)_{det}$ is in the linear span of $\mathcal{CSD}$. If $u_i < u_{i+1}$ and $v_i < v_{i+1}$ for all i, then $(U,V)_{det} \in \mathcal{CSD}$.

Let (U,V) be the largest bitableau with respect to $>_{s'c}$ such that $(U,V)_{det}$ cannot be written as a linear combination of

$$\mathcal{CSD}_n \;=\; \Big\{(M,Q)_{det} : M \in \mathcal{CS}_n, Q \in \mathcal{CS}_n, sh(U) = sh(V)\Big\}. \tag{2.7}$$

Without loss of generality, we may assume that even though at least one of U and V is not column-strict, the entries of the columns of both U and V increase strictly from south to north (or bottom to top). Suppose U is not column-strict. Thus, let us suppose that (j,m) is the smallest lexicographic pair of integers such that $u_{j,m} > u_{j+1,m}$. Note that if $u_{j,m} \leq u_{j+1,m}$ for all j and m then U is column-strict. Essentially, in columns j and $j+1$ of U we have

$$\begin{array}{ccc} u_{j,k_j} & & \\ \vdots & & u_{j+1,k_{j+1}} \\ \vdots & & \vdots \\ u_{j,m} & > & u_{j+1,m} \\ \vdots & & \vdots \\ u_{j,1} & & u_{j+1,1} \end{array}$$

where $u_{j,m} > u_{j+1,m}$. Particularly, we have

$$u_{j+1,1} < u_{j+1,2} < u_{j+1,3} < \cdots < u_{j+1,m} < u_{j,m} < u_{j,m+1} < \cdots < u_{j,k_j}.$$

Recall that a right transversal of a subgroup H of a group G is a subset K of G consisting of exactly one element from each right coset of H in G (see, for example, [17]). Let K_1 be the right transversal of

$$H_1 = S_{\{i_{j+1,1}, i_{j+1,2}, \cdots i_{j+1,m}\}} \times S_{\{i_{j,m}, i_{j,m+1}, \cdots, i_{j,k_j}\}}$$

in the group

$$G_1 = S_{\{i_{j+1,1}, i_{j+1,2}, \cdots, i_{j+1,m}, i_{j,m}, i_{j,m+1}, \cdots, i_{j,k_j}\}}.$$

We may assume that $\epsilon \in K_1$, where ϵ is the identity element of the group G_1. Now,

$$\begin{aligned}
& N(I)\ [i_{j+1,1}, i_{j+1,2}, \cdots, i_{j+1,m}, i_{j,m}, \cdots, i_{j,k_j}]'\ (u_1|v_1)(u_2|v_2)\cdots(u_n|v_n) \\
&= N(I)\ [i_{j+1,1}, i_{j+1,2}, \cdots i_{j+1,m}]'\ [i_{j,m}, \cdots, i_{j,k_j}]' \sum_{\alpha \in K_1} sgn(\alpha)\ \alpha\ (u_1|v_1)(u_2|v_2)\cdots(u_n|v_n) \\
&= m!\ (k_j - m + 1)!\ N(I) \sum_{\alpha \in K_1} sgn(\alpha)\ \alpha\ (u_1|v_1)(u_2|v_2)\cdots(u_n|v_n) \\
&= m!\ (k_j - m + 1)!\ (U, V)_{det} \\
&+ m!\ (k_j - m + 1)!\ N(I) \sum_{\substack{\alpha \in K_1 \\ \alpha \neq \epsilon}} sgn(\alpha)\ \alpha\ (u_1|v_1)(u_2|v_2)\cdots(u_n|v_n),
\end{aligned} \tag{2.8}$$

For $\alpha \in K_1$ and $\alpha \neq \epsilon$, we have that

$$N(I)\ \alpha\ (u_1|v_1)(u_2|v_2)\cdots(u_n|v_n) = (P_\alpha, V)_{det}$$

where P_α is a tableau of shape λ and $cs(P_\alpha) < cs(U)$.

Now let K_2 be the left transversal of

$$\begin{aligned}
H_2 &= S_{\{i_{j,1}, i_{j,2}, \cdots, i_{j,m-1}\}} \times S_{\{i_{j,m}, i_{j,m+1}, \cdots, i_{j,k_j}\}} \\
&\times S_{\{i_{j+1,1}, i_{j+1,2}, \cdots, i_{j+1,m}\}} \times S_{\{i_{j+1,m+1}, \cdots, i_{j+1,k_{j+1}}\}}
\end{aligned}$$

in the group

$$G_2 = S_{\{i_{j,1}, i_{j,2}, \cdots, i_{j,k_j}\}} \times S_{\{i_{j+1,1}, i_{j+1,2}, \cdots, i_{j+1,k_{j+1}}\}}.$$

Specifically,

$$\begin{aligned}
& [i_{j,1}, i_{j,2}, \cdots, i_{j,k_j}]'\ [i_{j+1,1}, i_{j+1,2}, \cdots, i_{j+1,k_{j+1}}]' \\
&= \sum_{\beta \in K_2} sgn(\beta)\ \beta\ [i_{j,1}, i_{j,2}, \cdots, i_{j,m-1}]'[i_{j,m}, \cdots, i_{j,k_j}]' \\
& [i_{j+1,1}, \cdots, i_{j+1,m}]'[i_{j+1,m+1}, \cdots, i_{j+1,k_{j+1}}]'.
\end{aligned}$$

Therefore,

$$
\begin{aligned}
& N(I)\ [i_{j+1,1}, i_{j+1,2}, \cdots i_{j+1,m}, i_{j,m}, \cdots, i_{j,k_j}]'\ (u_1|v_1)(u_2|v_2)\cdots(u_n|v_n) \\
= & [i_{1,1}, i_{1,2}, \cdots, i_{1,k_1}]' \cdots [i_{j-1,1}, i_{j-1,2}, \cdots, i_{j-1,k_{j-1}}]' \\
& [i_{j,1}, i_{j,2}, \cdots, i_{j,k_j}]'\ [i_{j+1,1}, i_{j+1,2}, \cdots, i_{j+1,k_{j+1}}]' \\
& [i_{j+2,1}, i_{j+2,2}, \cdots, i_{j+2,k_{j+2}}]'\ \cdots [i_{h,1}, i_{h,2}, \cdots, i_{h,k_h}]' \\
& [i_{j+1,1}, i_{j+1,2}, \cdots, i_{j+1,m}, i_{j,m}, \cdots, i_{j,k_j}]'\ (u_1|v_1)(u_2|v_2)\cdots(u_n|v_n) \\
= & [i_{1,1}, i_{1,2}, \cdots, i_{1,k_1}]' \cdots [i_{j-1,1}, i_{j-1,2}, \cdots, i_{j-1,k_{j-1}}]' \\
& \sum_{\beta \in K_2} sgn(\beta)\ \beta\ [i_{j,1}, i_{j,2}, \cdots, i_{j,m-1}]'[i_{j,m}, \cdots, i_{j,k_j}]'[i_{j+1,1}, \cdots, i_{j+1,m}]' \\
& [i_{j+1,m+1}, \cdots, i_{j+1,k_{j+1}}]'[i_{j+2,1}, i_{j+2,2}, \cdots, i_{j+2,k_{j+2}}]'\ \cdots [i_{h,1}, i_{h,2}, \cdots, i_{h,k_h}]' \\
& [i_{j+1,1}, i_{j+1,2}, \cdots, i_{j+1,m}, i_{j,m}, \cdots, i_{j,k_j}]'\ \beta^{-1}\ \beta\ (u_1|v_1)(u_2|v_2)\cdots(u_n|v_n) \\
= & m!\ (k_j - m + 1)!\ [i_{1,1}, i_{1,2}, \cdots, i_{1,k_1}]' \cdots [i_{j-1,1}, i_{j-1,2}, \cdots, i_{j-1,k_{j-1}}]' \\
& \sum_{\beta \in K_2} [\beta(i_{j,1}), \beta(i_{j,2}), \cdots, \beta(i_{j,m-1})]'\ [\beta(i_{j+1,m+1}), \cdots, \beta(i_{j+1,k_{j+1}})]' \\
& [\beta(i_{j+2,1}), \beta(i_{j+2,2}), \cdots, \beta(i_{j+2,k_{j+2}})]'\ \cdots [\beta(i_{h,1}), \beta(i_{h,2}), \cdots, \beta(i_{h,k_h})]' \\
& [\beta(i_{j+1,1}), \beta(i_{j+1,2}), \cdots, \beta(i_{j+1,m}), \beta(i_{j,m}), \cdots, \beta(i_{j,k_j})]' \\
& sgn(\beta)\ \beta\ (u_1|v_1)(u_2|v_2)\cdots(u_n|v_n) \\
= & m!\ (k_j - m + 1)! \sum_{\beta \in K_2} sgn(\beta)\ (Q_\beta, M_\beta)_{det}
\end{aligned}
\tag{2.9}
$$

where the shape of Q_β and M_β is higher than the shape of U and V. Setting equations (2.8) and (2.9) equal and solving for $(U, V)_{det}$ yields that $(U, V)_{det}$ can be written as a linear combination of bideterminants with smaller column sequences or higher shapes. By induction, (U, V) can be written as a linear combination of elements of $\mathcal{CSD}_n$ (see equation (2.7)). The proof when V is not column-strict is done in the same manner using equation (2.1). Thus we have the theorem for bideterminants.

The proof for bipermanents is similar. The only major difference between the two proofs is that we need to define the order $<_{sr}$ by setting

$$(U, V) <_{sr} (M, Q),$$

where U and V are tableaux of shape λ and M and Q are tableaux of shape μ whenever

1. $\lambda <_L \mu$; or

2. if $\lambda = \mu$ then $rs(U)\ rs(V) >_L rs(M)\ rs(Q)$. □

The following theorems are proven in [7] and [11]. Particularly, they show that both $\mathcal{CSD}$ and $\mathcal{CSP}$ are bases for the letter place algebra.

Theorem 2.2.

The collection

$$\mathcal{CSD} \;=\; \Big\{(M,Q)_{det} : M \in \mathcal{CS}, Q \in \mathcal{CS}, sh(M) = sh(Q)\Big\}$$

is linearly independent over $\mathbb{C}$*.*

Theorem 2.3.

The collection

$$\mathcal{CSP} \;=\; \Big\{(M,Q)_{per} : M \in \mathcal{CS}, Q \in \mathcal{CS}, sh(M) = sh(Q)\Big\}$$

is linearly independent over $\mathbb{C}$*.*

3. A Basis for $\mathbb{C}_S[X,Y]$.

Let $\mathcal{LP}_n^*$ be the subspace of a letter place algebra $\mathcal{LP}$ that consists of the linear span of the collection

$$\Big\{(1|d_1)(2|d_2)\cdots(n|d_n) : d_i \in \mathcal{A}, 1 \leq i \leq n\Big\}$$

where $d_i = (d_{i,1}, d_{i,2}) \in \mathcal{A}$. Define $\phi : \mathcal{LP}_n^* \to \mathbb{C}[X,Y]$ by linearly extending the map

$$\theta\big((1|d_1)(2|d_2)\cdots(n|d_n)\big) \;=\; x_1^{d_{1,1}} y_1^{d_{1,2}} x_2^{d_{2,1}} y_2^{d_{2,2}} \cdots x_n^{d_{n,1}} y_n^{d_{n,2}}.$$

With U an injective tableau and V a tableau (and $sh(U) = sh(V)$) with entries from $\mathcal{A}$, we will denote $\phi\big((U,V)_{det}\big)$ and $\phi\big((U,V)_{per}\big)$ by $[U,V]_{det}$ and $[U,V]_{per}$, respectively. The map ϕ is a (vector space) isomorphism. Thus we have

Theorem 3.1.

The collection

$$\Big\{[U,V]_{det} : U \in \mathcal{ST}_n, V \in \mathcal{CS}_n, sh(U) = sh(V)\Big\},$$

where $\mathcal{ST}_n$ *is the collection of all standard tableaux with* n *cells and* $\mathcal{CS}_n$ *is the collection of all column-strict tableaux with* n *cells with entries from* $\mathcal{A}$*, is linearly independent in* $\mathbb{C}[X,Y]$*.*

Let

$$\psi_S \;=\; \Big[[a_1, a_2, \cdots, a_j], [b_1, b_2, \cdots, b_{j'}]\Big]$$

be an ordered pair of two sequences where each of the a_i and b_i are positive integers and one of the following is true:

(i) $j \geq 2$, $j' \geq 2$, $a_1 = b_1 = 1$ and $j + j' = n + 1$;

(ii) $j = n$, $j' = 0$ and $\psi_S = \Big[[a_1, a_2, \cdots, a_n], \emptyset\Big]$, where $\emptyset$ denotes the empty sequence;

(iii) $j = 0$, $j' = n$ and $\psi_S \ = \ \Big[\emptyset, [b_1, b_2, \cdots, b_n]\Big]$.

For the remainder of this section, j and j' will denote the length of $[a_1, a_2, \ldots, a_j]$ and $[b_1, b_2, \cdots, b_{j'}]$, respectively. Note that the three possible cases lead to three different possible situations for ψ_S:

$$\psi_S \ = \ \Big[[1, a_2, \cdots, a_j], [1, b_2, \cdots, b_{n-j+1}]\Big], \tag{3.1}$$

$$\psi_S \ = \ \Big[[a_1, \cdots, a_n], \emptyset\Big] \tag{3.2}$$

or

$$\psi_S \ = \ \Big[\emptyset, [b_1, \cdots, b_n]\Big], \tag{3.3}$$

where each of the a_i and b_i are positive integers. Note that in equation (3.1), we set $a_1 = b_1 = 1$.

For $1 \leq k \leq j$, let

$$f_k \ = \ -1 \ + \ \sum_{i=1}^{k} a_i,$$

and for $1 \leq h \leq j'$, let

$$g_h \ = \ -1 \ + \ \sum_{i=1}^{h} b_i.$$

We will say that ψ_S is *dense* if and only if both of the following two conditions hold.

1. For all k such that $1 \leq k \leq j$ and any sequence $\alpha_k, \cdots, \alpha_j$ of nonnegative integers not all zero, either

$$f_k \ - \ \sum_{i=k}^{j} \alpha_i \ a_i < 0,$$

or

$$f_k \ - \ \sum_{i=k}^{j} \alpha_i \ a_i = f_p,$$

for some p.

2. For all h such that $1 \leq h \leq j'$ and any sequence $\beta_h, \cdots, \beta_{j'}$ of nonnegative integers not all zero, either

$$g_h \ - \ \sum_{i=h}^{j'} \beta_i \ b_i < 0,$$

or

$$g_h \;-\; \sum_{i=h}^{j'} \beta_i \; b_i = g_q,$$

for some q.

There are many classes of sequences that are dense. For example,

$$\psi_S \;=\; \Big[[1,k,k,k],[1,k,k,k,k]\Big]$$

for some positive integer k. Another such class is

$$\psi_S \;=\; \Big[[1,a_2,\cdots,a_j],[1,b_2,\cdots,b_{n-j+1}]\Big]$$

in which we require $a_2 \leq a_3$, $a_i | a_{i+1}$ (for $3 \leq i \leq j-1$), $b_2 \leq b_3$ and $b_i | b_{i+1}$ (for $3 \leq i \leq n-j$). Another example, would be

$$\psi_S \;=\; \Big[[1,1,1,1,3,8],[1,2,2,5,10]\Big].$$

With ψ_S given in (3.1), (3.2) or (3.3), we can construct a subset S_ψ of $\mathcal{A}$ in the following manner.

(i) For each $1 \leq i \leq j$, place $(0,f_i) = (0,-1+\sum_{k=1}^{i} a_k)$ in S_ψ;

(ii) For each $1 \leq i \leq j'$, place $(g_i,0) = (-1+\sum_{k=1}^{i} b_k,0)$ in S_ψ.

Equations (3.2) and (3.3) correspond to cases in which $\Delta_{S_\psi}(X,Y)$ uses either the variables $Y = \{y_1,\cdots,y_n\}$ or $X = \{x_1,\cdots,x_n\}$, respectively, but not both. The requirement that $a_1 = b_1 = 1$ in equation (3.1) comes from the fact that in this case $\Delta_{S_\psi}(X,Y)$ uses both sets of variables X and Y. Whenever we have $a_1 = b_1 = 1$, both (i) and (ii) place $(0,0)$ in S_ψ. Since S_ψ is a set (and not a multi-set), the net result is that there is exactly one $(0,0)$ in S_ψ.

We will say that S_ψ is dense if and only if ψ_S is dense. Note that given a finite subset $S \subset \mathcal{A}$, it is possible to reverse this process and construct an appropriate ψ_S.

Examples

Let $\psi_S = \Big[[1,1,1,1,1],\emptyset\Big]$. In this situation,

$$S_\psi \;=\; \Big\{(0,4),(0,3),(0,2),(0,1),(0,0)\Big\}.$$

Note that $\Delta_{S_\psi}(X,Y)$ is the Vandermonde determinant in the variables $\{y_1, y_2, y_3, y_4, y_5\}$. With $\psi_S = \Big[[1,2,4,4],[1,1,1,3,6]\Big]$,

$$S_\psi \;=\; \Big\{(0,10),(0,6),(0,2),(0,0),(1,0),(2,0),(5,0),(11,0)\Big\}.$$

If $\psi_S = \Big[\emptyset,[1,1,2,3,3,3]\Big]$, then

$$S_\psi \;=\; \Big\{(0,0),(1,0),(3,0),(6,0),(9,0),(12,0)\Big\}.$$

If $\psi_S = \Big[[2,2,4,4,4],\emptyset\Big]$, then

$$S_\psi \;=\; \Big\{(0,15),(0,11),(0,7),(0,3),(0,1)\Big\}.$$

It is worthwhile to contrast the above collections of S_ψ with the set S given in equation (1.12).

Similarly, if

$$S_\psi \;=\; \Big\{(0,12),(0,7),(0,2),(0,0),(1,0),(2,0),(5,0),(11,0),(17,0)\Big\}$$

then $\psi_S = \Big[[1,2,5,5],[1,1,1,3,6,6]\Big]$. $\square$

With ψ_S given in (3.1), (3.2) or (3.3), let Υ_{ψ_S} be the collection of all sequences of length n

$$\begin{aligned} \rho \;&=\; [h_j, h_{j-1}, \cdots, h_2, g, e_2, \cdots, e_{j'}] \\ &=\; [\rho_1, \cdots, \rho_{j-1}, \rho_j, \rho_{j+1}, \cdots, \rho_n] \end{aligned} \tag{3.4}$$

in which h_i, e_i and g are nonnegative integers such that $0 \le h_i \le a_i - 1$ (for $2 \le i \le j$), $0 \le e_h \le b_h - 1$ (for $2 \le h \le j'$) and $0 \le g < a_1$ (cases (3.1) or (3.2)) or $0 \le g < b_1$ (case (3.3)). The number of elements $d_{\psi_S} = |\Upsilon_{\psi_S}|$ of Υ_{ψ_S} is given by

$$d_{\psi_S} \;=\; |\Upsilon_{\psi_S}| \;=\; \left(\prod_{i=1}^{j} a_i\right)\left(\prod_{i=1}^{j'} b_i\right).$$

With ψ_S given in equation (3.1) or equation (3.2), set

$$\begin{aligned} \rho^- \;&=\; [a_j - 1 - h_j, \cdots,\; a_2 - 1 - h_2, a_1 - 1 - g,\; b_2 - 1 - e_2,\; \cdots, b_{j'} - 1 - e_{j'}] \\ &=\; [a_j - 1 - \rho_1, \cdots,\; a_2 - 1 - \rho_{j-1}, a_1 - 1 - \rho_j,\; b_2 - 1 - \rho_{j+1},\; \cdots, b_{j'} - 1 - \rho_n]. \end{aligned}$$

The analog for ρ^- when ψ_S is the case found in (3.3) is that we set

$$\begin{aligned}\rho^- &= [b_1 - 1 - g,\ b_2 - 1 - e_2,\ \cdots, b_n - 1 - e_n] \\ &= [b_1 - 1 - \rho_1,\ b_2 - 1 - \rho_2,\ \cdots, b_n - 1 - \rho_n].\end{aligned}$$

We will say that $i+1$ is *northwest* of i if $i+1$ is strictly north and weakly west of i. Given a standard tableau V of shape λ, $\rho = [\rho_1, \rho_2, \cdots, \rho_{j-1}, \rho_j, \rho_{j+1}, \cdots, \rho_n] \in \Upsilon_{\psi_S}$ and setting

$$\delta_V(i) = \begin{cases} 1 & \text{if } i+1 \text{ is northwest of } i \text{ in } V \\ 0 & \text{otherwise} \end{cases},$$

we define the cocharge tableau $C = C_\rho(V)$ of V to be the tableau of shape λ with entries from the alphabet $\mathcal{A}$, where

A. i. If ψ_S is given in (3.1) (and hence $a_1 = b_1 = 1$ and $\rho_j = 0$) then place $c_j = (c_{j,1}, c_{j,2}) = (0,0)$ in the cell containing j in V.

ii. If ψ_S is given in (3.2) then place $c_n = (c_{n,1}, c_{n,2}) = (0, \rho_n)$ in the cell containing n in V.

iii. If ψ_S is given in (3.3) then place $c_1 = (c_{1,1}, c_{1,2}) = (\rho_1, 0)$ in the cell containing 1 in V.

B. Assuming that we have placed $c_i = (c_{i,1}, c_{i,2})$ in the cell containing i in V (for some positive integer i such that $j \le i < n$), place $(c_{i,1} + \rho_{i+1} + \delta_V(i), 0)$ in the cell containing $i+1$ in V.

C. Assuming that we have placed $c_h = (c_{h,1}, c_{h,2})$ in the cell containing h in V (for some $2 \le h \le j$), place $(0, c_{h,2} + \rho_{h-1} + \delta_V(h-1))$ in the cell containing $h-1$ in V.

Given a column-strict tableau U (of shape λ) with entries from $\mathcal{A}$, we can label the entries from smallest to largest (with respect to $<_{\mathcal{A}}$) with the integers from 1 to n breaking ties by which entry is the farthest west (left). Let $u_i = (u_{i,1}, u_{i,2})$ be the entry of the cell in U labelled i. Note that if ψ_S is given by (3.1) and the entry labelled j, u_j, is not $(0,0)$ then $[I,U]_{per} \in \mathcal{I}_S(X,Y)$. Thus, without loss of generality, we may assume that $u_j = (0,0)$ when ψ_S is given by equation (3.1). Similarly, with ψ_S given in equation (3.2), we may assume that $u_n \le_{\mathcal{A}} (0,0)$. Additionally, with ψ_S given by equation (3.3), we may assume that $u_1 \ge_{\mathcal{A}} (0,0)$. Set $\mathcal{CS}_S \subset \mathcal{CS}_n$ to be the collection of all $U \in \mathcal{CS}_n$ such that

A. If ψ_S is given by equation (3.1) then $u_j = (0,0)$.
B. If ψ_S is given by equation (3.2) then $u_n \le_{\mathcal{A}} (0,0)$.
C. If ψ_S is given by equation (3.3) then $u_1 \ge_{\mathcal{A}} (0,0)$.

Let $U \in \mathcal{CS}_S$ and let $V = st(U)$ be the standard tableau of shape λ obtained by placing the label of the cells of U in its respective cell. With ψ_S given in (3.1), set

$$\rho_{i,U} = \begin{cases} (u_{i,2} - u_{i+1,2} - \delta_V(i)) \pmod{a_{j+1-i}} & \text{for } 1 \le i \le j-1; \\ 0 & i = j; \\ (u_{i,1} - u_{i-1,1} - \delta_V(i-1)) \pmod{b_{i+j'-n}} & \text{for } j+1 \le i \le n. \end{cases}$$

Note that we are considering (mod q) as a function that maps the integers into

$$\{0, 1, \cdots, q-1\}$$

and not as a relation.

The analogues for $\rho_{i,U}$ when $\psi_S = \Big[[a_1, a_2, \cdots, a_n], \emptyset\Big]$ (equation (3.2)) or $\psi_S = \Big[\emptyset, [b_1, b_2, \cdots, b_n]\Big]$ (equation (3.3)) is that we set

$$\rho_{i,U} \;=\; \begin{cases} (u_{i,2} - u_{i+1,2} - \delta_V(i)) \pmod{a_{n+1-i}} & \text{for } 1 \le i \le j-1; \\ (u_{i,2}) \pmod{a_1} & i = n; \end{cases}$$

or

$$\rho_{i,U} \;=\; \begin{cases} (u_{i,1}) \pmod{b_1} & i = 1; \\ (u_{i,1} - u_{i-1,1} - \delta_V(i-1)) \pmod{b_i} & \text{for } 2 \le i \le n; \end{cases}$$

respectively.

Set ρ_U to be

$$\rho_U \;=\; [\rho_{1,U}, \rho_{2,U}, \cdots, \rho_{j-1,U}, \rho_{j,U}, \rho_{j+1,U}, \cdots, \rho_{n,U}]. \tag{3.5}$$

If $U \in \mathcal{CS}_S$ and $C = C_{\rho_U}(st(U))$ and $c_i = (c_{i,1}, c_{i,2})$ is the entry in C that replaced i in $st(U)$, set

$$\gamma_i = \begin{cases} (0, u_{i,2} - c_{i,2}) & \text{if } \psi_S = \Big[[1, a_2, \cdots, a_j], [1, b_2, \cdots, b_{n-j+1}]\Big] \text{ and } 1 \le i \le j-1; \\ (u_{i,1} - c_{i,1}, 0) & \text{if } \psi_S = \Big[[1, a_2, \cdots, a_j], [1, b_2, \cdots, b_{n-j+1}]\Big] \text{ and } j+1 \le i \le n; \\ (0, 0) & \text{if } \psi_S = \Big[[1, a_2, \cdots, a_j], [1, b_2, \cdots, b_{n-j+1}]\Big] \text{ and } i = j; \\ (0, u_{i,2} - c_{i,2}) & \text{if } \psi_S = \Big[[a_1, a_2, \cdots, a_n], \emptyset\Big] \text{ and } 1 \le i \le n; \\ (u_{i,1} - c_{i,1}, 0) & \text{if } \psi_S = \Big[\emptyset, [b_1, b_2, \cdots, b_n]\Big] \text{ and } 1 \le i \le n; \end{cases}$$

and

$$\gamma_{\rho_U,U} = [\gamma_1, \gamma_2, \cdots, \gamma_{j-1}, \gamma_j, \gamma_{j+1}, \cdots, \gamma_n]. \tag{3.6}$$

Note that ρ_U is a sequence of integers while $\gamma_{\rho_U,U}$ is a sequence of elements of $\mathcal{A}$.

Example

Suppose that

$$\psi_S \;=\; \Big[[1, 1, 3, 3, 3], [1, 2, 5, 5, 5, 5]\Big]$$

and

$$S_\psi \;=\; \Big\{(0, 10), (0, 7), (0, 4), (0, 1), (0, 0), (2, 0), (7, 0), (12, 0), (17, 0), (22, 0)\Big\}.$$

Let U be the column-strict tableau

$$U \;=\; \begin{array}{lllll} (17,0) & & & & \\ (0,5) & (0,0) & (6,0) & (13,0) & \\ (0,9) & (0,6) & (0,1) & (0,0) & (0,0) \end{array} .$$

Then the labelling of the entries of U would be

$$U \;=\; \begin{array}{lllll} (17,0)_{10} & & & & \\ (0,5)_3 & (0,0)_5 & (6,0)_8 & (13,0)_9 & \\ (0,9)_1 & (0,6)_2 & (0,1)_4 & (0,0)_6 & (0,0)_7 \end{array}$$

where the labels are the subscripts to the entries,

$$V = st(U) \;=\; \begin{array}{lllll} 10 & & & & \\ 3 & 5 & 8 & 9 & \\ 1 & 2 & 4 & 6 & 7 \end{array} ,$$

$$\begin{aligned} \rho_U \;=\; & [9-6 \pmod 3,\; 6-5-1 \pmod 3, 5-1 \pmod 3,\; 1-0-1 \pmod 1, \\ & 0,\; 0-0 \pmod 2,\; 0-0 \pmod 5,\; 6-0-1 \pmod 5, \\ & 13-6 \pmod 5, 17-13-1 \pmod 5], \\ =\; & [0,0,1,0,0,0,0,0,2,3], \end{aligned}$$

$$C_{\rho_U}(st(U)) \;=\; \begin{array}{lllll} (7,0) & & & & \\ (0,2) & (0,0) & (1,0) & (3,0) & \\ (0,3) & (0,3) & (0,1) & (0,0) & (0,0) \end{array}$$

and

$$\gamma_{\rho_U,U} \;=\; [(0,6),(0,3),(0,3),(0,0),(0,0),(0,0),(0,0),(5,0),(10,0),(10,0)].$$

Note that

$$V^t \;=\; st(U)^t \;=\; \begin{array}{lll} 7 & & \\ 6 & 9 & \\ 4 & 8 & \\ 2 & 5 & \\ 1 & 3 & 10 \end{array} ,$$

$$\rho_U^- \;=\; [2,2,1,0,0,1,4,4,2,1]$$

and

$$C_{\rho_U^-}(st(U)^t) \;=\; \begin{array}{lll} (7,0) & & \\ (2,0) & (14,0) & \\ (0,0) & (11,0) & \\ (0,4) & (0,0) & \\ (0,7) & (0,2) & (15,0) \end{array} . \;\square$$

Note that $(1,0)$ replaced 8 in $st(U)$ when we constructed $C_{\rho_U}(st(U))$. $(11,0)$ replaced 8 in $st(U)^t$ when we constructed $C_{\rho_U^-}(st(U)^t)$. Now, $(1,0)+(11,0)=(12,0)$ which is the eighth entry of S_ψ as we read from left to right. In fact, the sums for the replacement for i equals the i^{th} entry of S_ψ for $1 \le i \le 10$. This leads us to our next lemma.

Lemma 3.2.

Let $S_\psi = \{s_1, s_2, \cdots, s_n\}$, V a standard tableau and $\rho \in \Upsilon_{\psi_S}$. If c_i replaced i in $C = C_\rho(V)$ and if c_i' replaced i in $C' = C_{\rho^-}(V^t)$ then $c_i + c_i' = s_i$.

Proof

Assume ψ_S is of case (3.1) and hence $\rho_j = 0$. Then $s_j = (0,0)$, $c_j = (0,0)$, $c_j' = (0,0)$ and $s_j = c_j + c_j'$. Assume for some $i \ge j$ that $c_i + c_i' = s_i$. Now,

$$c_{i+1} \;=\; (c_{i,1} + \rho_i + \delta_V(i), 0)$$

and

$$c_{i+1}' \;=\; (c_{i,1}' + \rho_i^- + \delta_{V^t}(i), 0).$$

Note that $\delta_V(i) + \delta_{V^t}(i) = 1$, $\rho_i + \rho_i^- = b_{i+j'-n} - 1$ and $c_{i,1} + c_{i,1}' = s_{i,1}$. Therefore, we have

$$\begin{aligned} c_{i+1} \;+\; c_{i+1}' &= \big(c_{i,1} \;+\; \rho_i \;+\; \delta_V(i) \;+\; c_{i,1}' \;+\; \rho_i^- \;+\; \delta_{V^t}(i), 0\big) \\ &= (s_{i,1} \;+\; b_{i+j'-n}, 0) \\ &= (s_{i+1,1}, 0) \end{aligned} .$$

The other cases are similar. □

The following lemma is proven in [4] (see equations (6.5) and (6.6) in Theorem 6.2).

Lemma 3.3.

Let $S_\psi = \{s_1, s_2, \cdots, s_n\}$, let Q be a standard tableau of shape λ, let C be a column-strict tableau with entries from $\mathcal{A}$ also of shape λ and let c_i be the entry in C that corresponds to the cell containing i in Q. Then

$$[Q, C]_{per}(\partial_X, \partial_Y)\ \Delta_{S_\psi}(X, Y) \;=\; \sum_{\phi \in S_n} sgn(\phi)\ d_\phi\ [Q^t, E_\phi]_{det},$$

where d_ϕ is an integer and E_ϕ is the tableau of shape λ that has entry $s_{\phi^{-1}(i)} - c_i$ in the cell that contains i in Q^t. Furthermore, if $s_{\phi^{-1}(i)} - c_i \notin \mathcal{A}$ for any $1 \le i \le n$, then $d_\phi = 0$; otherwise $d_\phi > 0$.

Example

Let $\psi_S = \Big[[1,1,4],[1,2]\Big]$. Therefore,

$$S_\psi \;=\; \Big\{(0,5),(0,1),(0,0),(2,0)\Big\}$$

and

$$\Delta_{S_\psi}(X,Y) \;=\; \begin{vmatrix} y_1^5 & y_1 & 1 & x_1^2 \\ y_2^5 & y_2 & 1 & x_2^2 \\ y_3^5 & y_3 & 1 & x_3^2 \\ y_4^5 & y_4 & 1 & x_4^2 \end{vmatrix}.$$

With $\rho = [2,0,0,1]$ and

$$M \;=\; \begin{array}{lll} 2 & & \\ 1 & 3 & 4 \end{array}$$

then

$$C \;=\; C_\rho(M) \;=\; \begin{array}{lll} (0,0) & & \\ (0,3) & (0,0) & (1,0) \end{array}.$$

Therefore, ignoring those tableaux with negative entries and letting

$$Q \;=\; \begin{array}{lll} 4 & & \\ 1 & 2 & 3 \end{array},$$

we have

$$\begin{aligned} &[Q,C]_{per}(\partial_X,\partial_Y)\;\Delta_{S_\psi}(X,Y) \\ &= 120 \left[\begin{array}{ll} 3 & \\ 2 & \\ 1 & 4 \end{array}, \begin{array}{ll} (1,0) & \\ (0,0) & \\ (0,2) & (0,1) \end{array}\right]_{det} \;+\; 120 \left[\begin{array}{ll} 3 & \\ 2 & \\ 1 & 4 \end{array}, \begin{array}{ll} (1,0) & \\ (0,1) & \\ (0,2) & (0,0) \end{array}\right]_{det}. \end{aligned}$$

Essentially the coefficient "120" in the preceding equation comes from the fact that

$$\partial_{x_4}\;\partial_{y_1}^3\; y_1^5 y_2 x_4^2 \;=\; 120\; y_1^2 y_2 x_4.$$

Now

$$M^t \;=\; \begin{array}{ll} 4 & \\ 3 & \\ 1 & 2 \end{array},$$

$$\rho^- \;=\; [1,0,0,0]$$

and

$$C_{\rho^-}(M^t) \;=\; \begin{array}{ll} (1,0) & \\ (0,0) & \\ (0,2) & (0,1) \end{array}.$$

Note that $C_{\rho^-}(M^t)$ is one of the tableaux that appears in $[Q,C]_{per}(\partial_X,\partial_Y)\;\Delta_{S_\psi}(X,Y)$. The reason that this occurs in the subject of Theorem 3.4 $\Box$

The *type* $\tau(W)$ of a tableau U is a listing of the entries of U in decreasing order with respect to $<_{\mathcal{A}}$. For example, with

$$U = \begin{array}{llll} (2,0) & (4,0) & & \\ (0,0) & (3,0) & (3,0) & \\ (0,1) & (0,1) & (1,0) & (4,0) \end{array}$$

then

$$\tau(U) = (4,0),(4,0),(3,0),(3,0),(2,0),(1,0),(0,0),(0,1),(0,1).$$

Recall that the column sequence of a tableau U is defined to be a listing of the entries of each column of U from bottom to top starting with the column farthest west and continuing east. For example, the column sequence of U is

$$cs(U) = (0,1),(0,0),(2,0),(0,1),(3,0),(4,0),(1,0),(3,0),(4,0).$$

With $>_L$ denoting the lexicographic ordering, we will say that

$$U <_{stc} V \tag{3.7}$$

when

1. $sh(U^t) <_L sh(V^t)$ (specifically, V is higher than U); or
2. if $sh(U) = sh(V)$ then $\tau(U) <_L \tau(V)$; or
3. if $sh(U) = sh(V)$ and $\tau(U) = \tau(V)$ then $cs(U) >_L cs(V)$.

Furthermore, we will state that $(U,V) <_{stc} (P,Q)$ when

1. $U <_{stc} P$; or
2. if $U = P$ then $V <_{stc} Q$.

Define

$$\mathcal{CO}_{S_\psi} = \Big\{ C_\rho(V) : \; V \in \mathcal{ST}_n \text{ and } \rho \in \Upsilon_{\psi_S} \Big\} \tag{3.8}$$

and

$$\mathcal{B}_{\psi_S} = \Big\{ [Q,C]_{per} : Q \in \mathcal{ST}_n, \, C \in \mathcal{CO}_{S_\psi} \text{ and } sh(Q) = sh(C) \Big\}. \tag{3.9}$$

Theorem 3.4.

Let Q be a standard tableau of shape λ and let $C \in \mathcal{CO}_{S_\psi}$ with $sh(C) = sh(Q)$. Furthermore, suppose that $C = C_\rho(V)$ where V is a standard tableau and $U = C_{\rho^-}(V^t)$. Then

$$[Q,C]_{per}(\partial_X,\partial_Y)\; \Delta_{S_\psi}(X,Y) = e_{Q^t,U}\; [Q^t,U]_{det} + \sum_{\substack{(Q^t,U) <_{stc} (P,M) \\ P \in \mathcal{ST}_n, M \in \mathcal{CS}_n}} e_{P,M}\; [P,M]_{det}, \tag{3.10}$$

where $e_{Q^t,U} \neq 0$. Therefore, the collection $\mathcal{B}_{\psi_S}$ is linearly independent in $\mathbb{C}_S[X,Y]$.

Proof

Recall that in Theorem 3.1, we showed that the collection

$$\Big\{ [M,U]_{det} : M \in \mathcal{ST}_n, U \in \mathcal{CS}_n, sh(M) = sh(U) \Big\}$$

is linearly independent in $\mathbb{C}[X,Y]$. Thus equation (3.10) immediately implies that the collection

$$\Big\{[Q,C]_{per}(\partial_X,\partial_Y)\ \Delta_{S_\psi}(X,Y) : [Q,C]_{per}\in\mathcal{B}_{\psi_S}\Big\}$$

is also a linearly independent set in $\mathbb{C}[X,Y]$. This would imply that the collection $\mathcal{B}_{\psi_S}$ is linearly independent in $\mathbb{C}_S[X,Y]$ since

$$\begin{aligned} & \alpha_1 p_1\ +\ \alpha_2 p_2\ +\ \cdots\ +\ \alpha_k p_k \in \mathcal{I}_S(X,Y)\\ \Leftrightarrow\ & \Big(\alpha_1 p_1\ +\ \alpha_2 p_2\ +\ \cdots\ +\ \alpha_k p_k\Big)(\partial_X,\partial_Y)\ \Delta_{S_\psi}(X,Y)\ =\ 0\\ \Leftrightarrow\ & \alpha_1 p_1(\partial_X,\partial_Y)\ \Delta_{S_\psi}(X,Y)\ +\ \cdots\ +\ \alpha_k p_k(\partial_X,\partial_Y)\ \Delta_{S_\psi}(X,Y)\ =\ 0 \end{aligned}$$

where $\alpha_i\in\mathbb{C}$ and $p_i\in\mathcal{B}_{\psi_S}$.

Lemma 3.3 implies

$$[Q,C]_{per}(\partial_X,\partial_Y)\ \Delta_{S_\psi}(X,Y)\ =\ \sum_{\phi\in S_n} sgn(\phi)\ d_\phi\ [Q^t,E_\phi]_{det},$$

where E_ϕ is the tableau of shape λ that has entry $s_{\phi^{-1}(i)}-c_i$ in the cell that contains i in Q^t. Furthermore, if any of the entries $s_{\phi^{-1}(i)}-c_i\notin\mathcal{A}$ then $d_\phi=0$. Suppose that

$$[Q,C]_{per}=\sum_{\phi\in R(Q)}\phi_{X,Y}\ \left(x_1^{c_{1,1}}x_2^{c_{2,1}}\cdots x_n^{c_{n,1}}y_1^{c_{1,2}}y_2^{c_{2,2}}\cdots y_n^{c_{n,2}}\right)$$

where $c_i=(c_{1,1},c_{1,2})$ is the entry in C that corresponds to the cell containing i in Q and $R(Q)=D(Q^t)$ is defined in equations (2.2) and (2.3). (Recall that the action $\phi_{X,Y}$ is defined in equation (1.5).) Set

$$q_{Q,C}\ =\ x_1^{c_{1,1}}x_2^{c_{2,1}}\cdots x_n^{c_{n,1}}y_1^{c_{1,2}}y_2^{c_{2,2}}\cdots y_n^{c_{n,2}}.$$

If $\phi_{X,Y}(q_{Q,C})=q_{Q,C}$ and $\phi\in R(Q)=D(Q^t)$ then $c_i=c_{\phi(i)}$ for $1\le i\le n$ and

$$s_{\phi^{-1}(i)}-c_i\ =\ s_{\phi^{-1}(i)}-c_{\phi^{-1}(i)}.$$

Lemma 3.2 implies that

$$s_{\phi^{-1}(i)}-c_{\phi^{-1}(i)}\ =\ u_{\phi^{-1}(i)}$$

where u_i is the i^{th} entry of $U=C_{\rho^-}(V^t)$. Therefore, with $\phi\in R(Q)=D(Q^t)$, we have

$$sgn(\phi)\ [Q^t,E_\phi]_{det}\ =\ sgn(\phi)\ sgn(\phi)\ [Q^t,U]_{det}\ =\ [Q^t,U]_{det}.$$

For ϕ such that $\phi_{X,Y}(q_{Q,C})\ne q_{Q,C}$ (i.e., $c_{\phi(i)}\ne c_i$ for some i), or $\phi\notin R(Q)$, it is not difficult to show that $E_\phi >_{stc} U$ (see, for example, Theorem 6.2 of [4] where it is worked out for the sequence

$$\psi_S\ =\ \Big[[1,k,k,\cdots,k],[1,k,\cdots,k]\Big]$$

in a fashion that is general enough to fit our particular situation). □

With

$$\psi_S \;=\; \Big[[1, a_2, \cdots, a_j], [1, b_2, \cdots, b_{n-j+1}]\Big],$$

for $1 \le t \le j-1$, set

$$\gamma_t \;=\; \left(0, \sum_{i=2}^{j+1-t} \alpha_i \; a_i\right), \tag{3.11}$$

for $j+1 \le r \le n$, set

$$\gamma_r \;=\; \left(\sum_{i=2}^{r+j'-n} \beta_i \; b_i, 0\right) \tag{3.12}$$

and set

$$\Gamma_{\psi_S} \;=\; \Big\{(\gamma_1, \gamma_2, \cdots, \gamma_{j-1}, (0,0), \gamma_{j+1}, \cdots, \gamma_n) : \alpha_2 + \cdots + \alpha_j + \beta_2 + \cdots + \beta_{j'} \;>\; 0\Big\}$$

where α_i and β_i are nonnegative integers. The analogues for Γ_{ψ_S} when

$$\psi_S \;=\; \Big[[a_1, a_2, \cdots, a_n], \emptyset\Big]$$

or

$$\psi_S \;=\; \Big[\emptyset, [b_1, b_2, \cdots, b_n]\Big]$$

is that we set

$$\Gamma_{\psi_S} \;=\; \Big\{(\gamma_1, \gamma_2, \cdots, \gamma_n) : \alpha_1 + \cdots + \alpha_n > \; 0\Big\},$$

with

$$\gamma_t \;=\; \left(0, \sum_{i=1}^{n+1-t} \alpha_i \; a_i\right)$$

for $1 \le t \le n$, and set

$$\Gamma_{\psi_S} \;=\; \Big\{(\gamma_1, \gamma_2, \cdots, \gamma_n) : \beta_1 + \cdots + \beta_n \;>\; 0\Big\}$$

where

$$\gamma_r \;=\; \left(\sum_{i=1}^{r} \beta_i \; b_i, 0\right)$$

for $1 \le r \le n$, respectively, where the α_i and β_i are nonnegative integers.

With $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ (where $\gamma_i = (\gamma_{i,1}, \gamma_{i,2}) \in \mathcal{A}$), set

$$m_\gamma(X, Y) \;=\; \sum_{\sigma \in S_n} \sigma_{X,Y} \;\left(x_1^{\gamma_{1,1}} y_1^{\gamma_{1,2}} x_2^{\gamma_{2,1}} y_2^{\gamma_{2,2}} \cdots x_n^{\gamma_{n,1}} y_n^{\gamma_{n,2}}\right).$$

Recall that the action $\sigma_{X,Y}$ is defined in equation (1.5). Note that $m_\gamma(X,Y)$ is a symmetric polynomial.

Theorem 3.5.

If ψ_S is dense and $\gamma \in \Gamma_{\psi_S}$ then $m_\gamma(X,Y) \in \mathcal{I}_S(X,Y)$.

Proof

Assume

$$\psi_S \ = \ \Big[[1, a_2, \cdots, a_j], [1, b_2, \cdots, b_{n-j+1}]\Big]$$

(once again, we have $a_1 = b_1 = 1$). Recall that we construct the subset S_ψ of $\mathcal{A}$ in the following manner.

(i) For each $1 \le h \le j$, place $(0, f_h) = (0, -1 + \sum_{k=1}^{h} a_k)$ in S_ψ;

(ii) For each $1 \le i \le n-j+1$, place $(g_i, 0) = (-1 + \sum_{k=1}^{i} b_k, 0)$ in S_ψ.

Now,

$$\begin{aligned} &\Delta_{S_\psi}(X,Y) \\ &= \sum_{\beta \in S_n} sgn(\beta)\ \beta_{X,Y}\left(x_1^0 y_1^{f_j}\ x_2^0 y_2^{f_{j-1}}\ \cdots\ x_{j-1}^0 y_{j-1}^{f_2}\ x_j^0 y_j^0\ x_{j+1}^{g_2} y_{j+1}^0\ \cdots\ x_n^{g_{n-j+1}} y_n^0\right). \end{aligned}$$

If $\gamma = (\gamma_1, \gamma_2, \cdots, \gamma_n)$ subject to equations (3.11) and (3.12), it is not difficult to show that

$$\begin{aligned} &m_\gamma(\partial_X, \partial_Y)\ \Delta_{S_\psi}(X,Y) \\ &= \sum_{\sigma \in S_n} c_\sigma \sum_{\beta \in S_n} sgn(\beta)\ \beta_{X,Y}\ \Big(x_1^{0-\gamma_{\sigma(1),1}} y_1^{f_j-\gamma_{\sigma(1),2}} \cdots x_{j-1}^{0-\gamma_{\sigma(j-1),1}} y_{j-1}^{f_2-\gamma_{\sigma(j-1),2}} \\ &\quad x_j^{0-\gamma_{\sigma(j),1}} y_j^{0-\gamma_{\sigma(j),2}}\ x_{j+1}^{g_{2,1}-\gamma_{\sigma(j+1),1}} y_{j+1}^{0-\gamma_{\sigma(j+1),2}}\ \cdots\ x_n^{g_{n-j+1,1}-\gamma_{\sigma(n),1}} y_n^{0-\gamma_{\sigma(n),2}}\Big) \end{aligned} \tag{3.13}$$

where $c_\sigma = 0$ if any of the exponents are negative (see, for example, Theorem 5.2 of [4]).

Suppose $1 \le h \le j-1$, $\sigma(h) = k$ where $j \le k \le n$ and $\gamma_k = \gamma_{\sigma(h)} \ne (0,0)$ (and, specifically, $\gamma_{\sigma(h),1} > 0$) then the exponent of x_h in equation (3.13) is

$$0 - \gamma_{\sigma(h),1}\ <\ 0$$

and therefore $c_\sigma = 0$. Thus, without loss of generality, we may assume that if $\gamma_{\sigma(h)} \neq (0,0)$ and $1 \leq h \leq j-1$, then $1 \leq \sigma(h) \leq j-1$. Similarly, we may assume that if $\gamma_{\sigma(r)} \neq (0,0)$ and $j+1 \leq r \leq n$, then $j+1 \leq \sigma(r) \leq n$. Let h be the largest integer such that $\gamma_{\sigma(h)} \neq (0,0)$ and $1 \leq h \leq j-1$. Thus for all q such that $h < q \leq j-1$, $\gamma_{\sigma(q)} = (0,0)$. So there are at least $j-h-1$ $(0,0)$ terms in the sequence $(\gamma_1, \gamma_2, \cdots, \gamma_{j-1})$. Now, $\gamma_i <_{\mathcal{A}} \gamma_{i+1}$ yields that

$$\gamma_{h+1} \;=\; \gamma_{h+2} \;=\; \cdots \;=\; \gamma_{j-1} \;=\; (0,0).$$

Since $\gamma_k = \gamma_{\sigma(h)} \neq (0,0)$ we must have $1 \leq k \leq h$. Also, note that

$$\gamma_{h+1,2} \;=\; \gamma_{h+2,2} \;=\; \cdots \;=\; \gamma_{j-1,2} \;=\; 0.$$

Recall that for $1 \leq k \leq j-1$ we have

$$\gamma_{k,2} \;=\; \sum_{i=2}^{j+1-k} \alpha_i \; a_i$$

for some sequence $(\alpha_2, \alpha_3, \cdots, \alpha_j)$ of nonnegative integers (see equation (3.11)). Thus

$$0 \;=\; \gamma_{h+1,2} \;=\; \alpha_2 \; a_2 \;+\; \cdots \;+\; \alpha_{j-h} \; a_{j-h},$$

$$\alpha_2 \;=\; \alpha_3 \;=\; \cdots \;=\; \alpha_{j-h} \;=\; 0,$$

and for all $1 \leq k \leq j-1$, we have

$$\gamma_{k,2} \;=\; \sum_{i=j+1-h}^{j+1-k} \alpha_i \; a_i.$$

The exponent of y_h in equation (3.13) is, with $k = \sigma(h)$,

$$\begin{aligned} f_{j+1-h} \;-\; \gamma_{\sigma(h),2} \;&=\; f_{j+1-h} \;-\; \gamma_{k,2} \\ &=\; f_{j+1-h} \;-\; \sum_{i=j+1-h}^{j+1-k} \alpha_i \; a_i. \end{aligned}$$

With $\gamma_{k,2} \neq 0$ and $1 \leq k \leq h$, gives us that $j+1-k \geq j+1-h$. Since ψ_S is dense, we must have either

$$f_{j+1-h} \;-\; \sum_{i=j+1-h}^{j+1-k} \alpha_i \; a_i < 0$$

or

$$f_{j+1-h} \;-\; \sum_{i=j+1-h}^{j+1-k} \alpha_i \; a_i = f_t$$

some t, where $t < j+1-h$. If the former case, $c_\sigma = 0$. In the latter, note that $t = j+1-r$ some r where $r > h$. Therefore, $\gamma_{\sigma(r)} = (0,0)$ (recall h is the largest integer $1 \le h \le j$ such that $\gamma_{\sigma(r)} \neq (0,0)$). In equation (3.13), the exponent of y_r is exactly the same as the exponent of y_h (both being exactly f_t) and that the exponent of both x_r and x_h is 0. Thus from equation (3.13) we have

$$\begin{aligned}
&\sum_{\sigma\in S_n} c_\sigma \sum_{\beta\in S_n} sgn(\beta)\ \beta_{X,Y}\ \Big(x_1^{0-\gamma_{\sigma(1),1}} y_1^{f_j-\gamma_{\sigma(1),2}} \cdots\ x_{j-1}^{0-\gamma_{\sigma(2),1}} y_{j-1}^{f_2-\gamma_{\sigma(2),2}} \\
&x_j^{0-\gamma_{\sigma(j+1),1}} y_j^{0-\gamma_{\sigma(j+1),2}}\ x_{j+1}^{g_{2,1}-\gamma_{\sigma(j+2),1}} y_{j+1}^{0-\gamma_{\sigma(j+2),2}} \cdots\ x_n^{g_{n-j+1,1}-\gamma_{\sigma(n),1}} y_n^{0-\gamma_{\sigma(n),2}} \Big) \\
&= \ 0.
\end{aligned}$$

(Recall that the action $\beta_{X,Y}$ is defined in equation (1.5).)

The other cases are similar. $\square$

In the case that $\psi_S = \Big[\emptyset, [1,1,1,\cdots,1]\Big]$ it is well-known that all symmetric polynomials $m_\gamma(X,Y)$ are in the ideal $\mathcal{I}_S(X,Y)$. Recall that in this case, $\Delta_S(X,Y)$ is the Vandermonde determinant in the variables X, $\mathcal{I}_S(X,Y)$ is the ideal generated by the monomials $\{y_1, y_2, \cdots, y_n\}$ (since $\partial_{y_i}\ \Delta_S(X,Y) = 0$) and the elementary symmetric polynomials in the variables X (see equation (1.12)). This previous theorem informs us that for a dense set ψ_S a certain subset of these symmetric polynomials still reside in $\mathcal{I}_S(X,Y)$. Specifically,

Corollary 3.6.

Assume that ψ_S is dense. If $U \in \mathcal{CS}_S$ and $U \notin \mathcal{CO}_{S_\psi}$ then

$$m_{\gamma_{\rho_U,U}}(X,Y) \in \mathcal{I}_S(X,Y).$$

Proof

Let $V = st(U)$. Assume

$$\psi_S \ = \ \Big[[1,a_2,\cdots,a_j],[1,b_2,\cdots,b_{n-j+1}]\Big]$$

and

$$\rho_U \ = \ [\rho_1, \rho_2, \cdots, \rho_{j-1}, \rho_j, \rho_{j+1}, \cdots, \rho_n].$$

Recall that $\gamma_{\rho_U,U}$ is defined in equation (3.6). To show that $m_{\gamma_{\rho_U,U}} \in \mathcal{I}_S(X,Y)$, we need to show that for $1 \le t \le j-1$,

$$\gamma_{t,2} = \sum_{i=2}^{j+1-t} \alpha_i\ a_i$$

and for $j+1 \le r \le n$,

$$\gamma_{r,1} = \sum_{i=2}^{r+j'-n} \beta_i \ b_i$$

and then apply Theorem 3.5. Since $U \in \mathcal{CS}_S$, we may assume that $u_j = (0,0)$ and $\gamma_j = (0,0)$. Lets assume that for some k such that $j \le k \le n$,

$$\gamma_{k,1} = \sum_{i=1}^{k+j'-n} \beta_i \ b_i.$$

Recall that (see equations (3.5) and (3.6))

$$\gamma_{k,1} \ = \ u_{k,1} \ - \ c_{k,1},$$

$$\rho_{k+1,U} = (u_{k+1,1} \ - \ u_{k,1} - \delta_V(k)) \pmod{b_{k+j'-n+1}}$$

and

$$c_{k+1,1} \ = \ c_{k,1} \ + \ \rho_{k+1,U} \ + \ \delta_V(k).$$

There exists some integer $\beta_{k+j'-n+1} \ge 0$ such that

$$\begin{aligned}
&\gamma_{k+1,1} \\
&= u_{k+1,1} \ - \ c_{k+1,1} \\
&= \beta_{k+j'-n+1} \ b_{k+j'-n+1} \ + \ u_{k,1} \ + \ \rho_{k+1,U} \ + \ \delta_V(k) \ - \ (c_{k,1} \ + \ \rho_{k+1,U} \ + \ \delta_V(k)) \\
&= \beta_{k+j'-n+1} \ b_{k+j'-n+1} \ + \ u_{k,1} \ - \ c_{k,1} \\
&= \beta_{k+j'-n+1} \ b_{k+j'-n+1} \ + \ \gamma_{k,1} \\
&= \beta_{k+j'-n+1} \ b_{k+j'-n+1} \ + \ \sum_{i=1}^{k+j'-n} \beta_i \ b_i \\
&= \sum_{i=1}^{k+j'-n+1} \beta_i \ b_i.
\end{aligned}$$

The other cases are similar. □

We will say that

$$U <_{str} V \tag{3.14}$$

when

1. $sh(U) <_L sh(V)$; or
2. if $sh(U) = sh(V)$ then $\tau(U) >_L \tau(V)$; or
3. if $sh(U) = sh(V)$ and $\tau(U) = \tau(V)$ then $rs(U) >_L rs(V)$.

Furthermore with (U, V) a pair of tableaux of shape μ and (P, Q) a pair of tableaux of shape ν, we will say that $(U, V) <_{str} (P, Q)$ whenever

1. $U <_{str} P$; or
2. if $U = P$ then $V <_{str} Q$.

Let U be a standard tableau and let R be a column-strict tableau of the same shape as U. Let r_i be the entry in the cell of R that corresponds to the cell containing i in U. In [4], the proof of Theorem 5.4 established that

$$m_\gamma(X,Y)\ [U,R]_{per} \ = \ \sum_{\alpha\in S_n} [U,V_\alpha]_{per} \tag{3.15}$$

where $sh(V_\alpha) = sh(R)$ and V_α has entry $r_i + \gamma_{\alpha(i)}$ in the cell that corresponds to the cell containing r_i in R. Note that if $r_i + \gamma_{\alpha(i)} \notin \mathcal{A}$, some i, then

$$[U,V_\alpha]_{per}(\partial_X,\partial_Y)\ \Delta_{S_\psi}(X,Y) \ = \ 0$$

and $[U,V_\alpha]_{per}(X,Y) \in \mathcal{I}_S(X,Y)$.

Example

Let

$$\psi_S \ = \ \Big[[1,1,4],[1,2]\Big]$$

and

$$R = \begin{matrix} (0,0) & & \\ (0,5) & (0,0) & (2,0) \end{matrix}.$$

Note that $R \notin \mathcal{CO}_{S_\psi}$. Now,

$$\rho_R = [0,0,0,0],$$

$$st(R) \ = \ \begin{matrix} 2 & & \\ 1 & 3 & 4 \end{matrix},$$

$$C_\rho(st(R)) \ = \ \begin{matrix} (0,0) & & \\ (0,1) & (0,0) & (0,0) \end{matrix}$$

and

$$\gamma \ = \ \gamma_{\rho_R,R} \ = \ [(0,4),(0,0),(0,0),(2,0)].$$

Now, with

$$U = \begin{matrix} 4 & & \\ 1 & 2 & 3 \end{matrix}$$

and recalling that

$$\left[U, \begin{matrix} (0,0) & & \\ (0,5) & (0,0) & (2,0) \end{matrix}\right]_{per} \ = \ \left[U, \begin{matrix} (0,0) & & \\ (0,5) & (2,0) & (0,0) \end{matrix}\right]_{per}$$

(specifically, any row rearrangements of the entries of R does not change the resulting polynomial) yields

$$
\begin{aligned}
& m_\gamma(X,Y) \left[U, \begin{array}{lll} (0,0) & & \\ (0,1) & (0,0) & (0,0) \end{array}\right]_{per} \\
&= 4 \left[U, \begin{array}{lll} (0,0) & & \\ (0,5) & (0,0) & (2,0) \end{array}\right]_{per} + \ 2 \left[U, \begin{array}{lll} (2,0) & & \\ (0,5) & (0,0) & (0,0) \end{array}\right]_{per} \\
&+ 4 \left[U, \begin{array}{lll} (2,0) & & \\ (0,1) & (0,4) & (0,0) \end{array}\right]_{per} + \ 4 \left[U, \begin{array}{lll} (0,0) & & \\ (0,1) & (0,4) & (2,0) \end{array}\right]_{per} \\
&+ 4 \left[U, \begin{array}{lll} (0,0) & & \\ (2,1) & (0,4) & (0,0) \end{array}\right]_{per} + \ 4 \left[U, \begin{array}{lll} (0,4) & & \\ (0,1) & (0,0) & (2,0) \end{array}\right]_{per} \\
&+ 2 \left[U, \begin{array}{lll} (0,4) & & \\ (2,1) & (0,0) & (0,0) \end{array}\right]_{per}.
\end{aligned}
$$

There are some properties of this expansion that need to be considered. First, the polynomial

$$
[U,R]_{per} \ = \ \left[U, \begin{array}{lll} (0,0) & & \\ (0,5) & (0,0) & (2,0) \end{array}\right]_{per}
$$

is the smallest of all the bipermanents with entries strictly from $\mathcal{A}$ under $<_{str}$ with non-zero coefficients. Second, the polynomials

$$
\left[U, \begin{array}{lll} (0,4) & & \\ (2,1) & (0,0) & (0,0) \end{array}\right]_{per}, \left[U, \begin{array}{lll} (0,0) & & \\ (2,1) & (0,4) & (0,0) \end{array}\right]_{per}
$$

are in the ideal $\mathcal{I}_S(X,Y)$. Third, since ψ_S is dense, $m_\gamma \in \mathcal{I}_S(X,Y)$ and

$$
m_\gamma(X,Y) \ \left[U, \begin{array}{lll} (0,0) & & \\ (0,1) & (0,0) & (0,0) \end{array}\right]_{per} \in \mathcal{I}_S(X,Y).
$$

Thus, in $\mathbb{C}_S[X,Y]$ we can solve for $[U,R]_{per}$

$$
\begin{aligned}
& \left[U, \begin{array}{lll} (0,0) & & \\ (0,5) & (0,0) & (2,0) \end{array}\right]_{per} \\
\equiv \ -\frac{1}{2} & \left[U, \begin{array}{lll} (2,0) & & \\ (0,5) & (0,0) & (0,0) \end{array}\right]_{per} - \left[U, \begin{array}{lll} (2,0) & & \\ (0,1) & (0,4) & (0,0) \end{array}\right]_{per} \\
- & \left[U, \begin{array}{lll} (0,0) & & \\ (0,1) & (0,4) & (2,0) \end{array}\right]_{per} - \left[U, \begin{array}{lll} (0,4) & & \\ (0,1) & (0,0) & (2,0) \end{array}\right]_{per} \pmod{\mathcal{I}_S(X,Y)}. \ \square
\end{aligned}
$$

The previous example is a special case of Theorem 3.7. In [4], the following was essentially proven (see Theorem 5.4 and Corollary 5.5). It should be noted that [4] deals with (using our notation) the sequence $\Big[[1,k,\cdots,k],[1,k,\cdots,k]\Big]$.

Theorem 3.7.

Let $f : \mathcal{CS}_S \to \mathcal{CS}_S$ be an operator on column-strict tableaux such that if $O = f(V)$ then $sh(O) = sh(V)$ and $st(O) = st(V)$. Furthermore, suppose that f is such that if v_i and o_i are the entries in V and O, respectively, that are labelled i and that $\gamma_i = v_i - o_i \in \mathcal{A}$ for $1 \leq i \leq n$ then $\gamma_i \leq_{\mathcal{A}} \gamma_{i+1}$ for $1 \leq i \leq n-1$. Then, with $\gamma = (\gamma_1, \cdots, \gamma_n)$ and U a standard tableau, we have

$$\begin{aligned} m_\gamma(X,Y)\ [U,O]_{per} &= c_{U,V}\ [U,V]_{per} \ + \sum_{\substack{(U,V) <_{str} (P,Q) \\ P \in \mathcal{ST}_n, Q \in \mathcal{CS}_n}} c_{P,Q}\ [P,Q]_{per} \\ &\quad + \sum_{\substack{P' \in \mathcal{ST}_n \\ Q' \notin \mathcal{CS}_n}} c_{P',Q'}\ [P',Q']_{per} \end{aligned} \tag{3.16}$$

where $c_{U,V} \neq 0$ and each Q' is a tableau with at least one entry that is not in $\mathcal{A}$.

Note that there is a slight difference between the order $>_{str}$ defined in equation (3.14) and the order $>_{tr}$ defined in [4]. The proof in [4], however, admits the previous theorem.

Note that Theorem 2.1 implies that the collection $\mathcal{CS}_S$ spans $\mathbb{C}_S[X,Y]$. (Recall that if $V \in \mathcal{CS}_n$ and $V \notin \mathcal{CS}_S$ then $[U,V]_{per} \in \mathcal{I}_S(X,Y)$.) Assume that $V \in \mathcal{CS}_S$. Set $f = C_{\rho_V}$ and hence $O = C_{\rho_V}(V)$. Therefore, in Theorem 3.7, we have $\gamma = \gamma_{\rho_V,V}$. Recall that Corollary 3.6 implies that whenever S is dense, $m_\gamma(X,Y) \in \mathcal{I}_S(X,Y)$. Also, in equation (3.16), since each Q' has an entry that is not in the alphabet $\mathcal{A}$, we must have

$$[P',Q']_{per}(\partial_X, \partial_Y)\ \Delta_{S_\psi}(X,Y) \ = \ 0$$

and $[P',Q']_{per} \in \mathcal{I}_S(X,Y)$. Therefore, solving for $[U,V]_{per}$ in the ring $\mathbb{C}_S[X,Y]$, we have

$$[U,V]_{per} \ \equiv \ \left(-\frac{1}{c_{U,V}} \sum_{\substack{(U,V) <_{str} (P,Q) \\ P \in \mathcal{ST}_n, Q \in \mathcal{CS}_n}} c_{P,Q}\ [P,Q]_{per} \right) \quad (\mathrm{mod}\ \mathcal{I}_S(X,Y)). \tag{3.17}$$

It is not difficult to see that if the type of V is either too large or too small, then $[U,V]_{per} \in \mathcal{I}_S(X,Y)$. Therefore, Theorem 3.7 and equation (3.17) imply an algorithm for the expansion of any element of $P(X,Y) \in \mathbb{C}_S[X,Y]$ into a linear combination of the collection $\mathcal{B}_{\psi_S}$. Thus Theorem 3.4 and Theorem 3.7 imply that

$$\mathcal{B}_{\psi_S} \ = \ \Big\{ [Q,C]_{per} : Q \in \mathcal{ST}_n,\ C \in \mathcal{CO}_{S_\psi} \text{ and } sh(Q) = sh(C) \ \Big\}.$$

is a basis for $\mathbb{C}_S[X,Y]$.

If R_{u_1,u_2} is the homogeneous subspace of dimension u_1 in X and dimension u_2 in Y of a space R, then we define the Hilbert series $\mathcal{H}(R)$ to be

$$\mathcal{H}(R) \;=\; \sum_{u_1,u_2} dim(R_{u_1,u_2})\; t^{u_1}\; q^{u_2}.$$

We can summarize the above discussion in the following theorem.

Theorem 3.8.

If S is dense then $\mathcal{B}_{\psi_S}$ is a basis for $\mathbb{C}_S[X,Y]$ and the Hilbert series $\mathcal{H}(\mathbb{C}_S[X,Y])$ is given by

$$\begin{aligned}\mathcal{H}(\mathbb{C}_S[X,Y]) \;&= \sum_{\lambda\vdash n}\;\sum_{(M,N)\in\mathcal{ST}_\lambda\times\mathcal{ST}_\lambda}\;\sum_{\rho\in\Upsilon_{\psi_S}} t^{|C_{\rho,1}(M)|}q^{|C_{\rho,2}(M)|}\\ &= \sum_{\lambda\vdash n} h_\lambda \sum_{M\in\mathcal{ST}_\lambda}\;\sum_{\rho\in\Upsilon_{\psi_S}} t^{|C_{\rho,1}(M)|}q^{|C_{\rho,2}(M)|} \end{aligned} \tag{3.18}$$

where $\mathcal{ST}_\lambda$ denotes the collection of standard tableaux of shape λ, h_λ denotes the number of standard tableaux of shape λ and $|C_{\rho,1}(M)|$ and $|C_{\rho,2}(M)|$ denote the sum of the first and second coordinates, respectively, of the entries of $C_\rho(M)$.

Example

Let $n=3$ and suppose that

$$\psi_S \;=\; \Big[[1,1],[1,2]\Big]. \tag{3.19}$$

Therefore,

$$\Upsilon_{\psi_S} \;=\; \Big\{[0,0,0],[0,0,1]\Big\}.$$

Now, letting N denote a standard tableau with three cells and $C_{\rho_1}(N)$ and $C_{\rho_2}(N)$ the cocharge tableaux with $\rho_1=[0,0,0]$ and $\rho_2=[0,0,1]$, respectively, we have

N	$C_{\rho_1}(N)$	$C_{\rho_2}(N)$
$\begin{array}{ccc}1&2&3\end{array}$	$\begin{array}{ccc}(0,0)&(0,0)&(0,0)\end{array}$	$\begin{array}{ccc}(0,0)&(0,0)&(1,0)\end{array}$
$\begin{array}{c}3\\2\\1\end{array}$	$\begin{array}{c}(1,0)\\(0,0)\\(0,1)\end{array}$	$\begin{array}{c}(2,0)\\(0,0)\\(0,1)\end{array}$
$\begin{array}{cc}2&\\1&3\end{array}$	$\begin{array}{cc}(0,0)&\\(0,1)&(0,0)\end{array}$	$\begin{array}{cc}(0,0)&\\(0,1)&(1,0)\end{array}$
$\begin{array}{cc}3&\\1&2\end{array}$	$\begin{array}{cc}(1,0)&\\(0,0)&(0,0)\end{array}$	$\begin{array}{cc}(2,0)&\\(0,0)&(0,0).\end{array}$

Recalling that $h_{1^3} = 1$, $h_{2,1} = 2$ and $h_3 = 1$ yields

$$\mathcal{H}(\mathbb{C}_S[X,Y]) \;=\; 1 + 3t + 3tq + t^2q + 2q + 2t^2 \;=\; (1+t)\;(1 + 2t + 2q + qt)\,. \;\square$$

The number of pairs (M,N) of standard tableaux with n cells and of the same shape is $n!$. This immediately implies the following corollary.

Corollary 3.9.

If ψ_S is dense then the dimension (as a vector space) of $\mathbb{C}_S[X,Y]$ is $d_{\psi_S}\; n!$ where

$$d_{\psi_S} \;=\; \left(\prod_{k=1}^{j} a_k\right)\left(\prod_{k=1}^{j'} b_k\right). \tag{3.20}$$

It is interesting to note that Theorem 3.4 did not require S_ψ to be dense. Thus the collection $\mathcal{B}_{\psi_S}$ is linearly independent for any S_ψ and $d_{\psi_S}\; n!$ is a lower bound for the dimension of $\mathbb{C}_S[X,Y]$ for any S_ψ. Note that when $\psi_S = \Big[[3,1,1,1],\emptyset\Big]$ (note that this particular ψ_S is **not** dense), in [6] it is shown that the dimension of $\mathbb{C}_S[X,Y]$ is 360. Since $360 \;>\; 3(4!)$, there do exist rings $\mathbb{C}_S[X,Y]$ with dimension larger than $d_{\psi_S}\; n!$. For a bitableaux bases for $\psi_S = \Big[[k,1,1,\cdots,1],\emptyset\Big]$ see [5].

Note that with

$$\rho^1 = (\rho_1, \cdots, \rho_{j-2}, \rho_{j-1}, \rho_j, \rho_{j+1}, \cdots, \rho_n)$$

and

$$\rho^2 = (\rho_1, \cdots, \rho_{j-2}, 0, \rho_j, \rho_{j+2}, \cdots, \rho_n)$$

(note that we are changing ρ_{j-1}) that

$$q^{|C_{\rho^1_{2,2}}(N)|} \;=\; \left(q^{j-1}\right)^{\rho_{j-1}}\; q^{|C_{\rho^2_{2,2}}(N)|}.$$

Therefore, in equation (3.18), we have

$$\begin{aligned}
&\sum_{\rho\in\Upsilon_{\psi_S}} t^{|C_{\rho,1}(N)|} q^{|C_{\rho,2}(N)|}\\
&= \Big(1 \;+\; q^{j-1} \;+\; (q^{j-1})^2 \;+\; \cdots \;+\; (q^{j-1})^{a_2-1}\Big) \sum_{\substack{\rho\in\Upsilon_{\psi_S}\\ \rho_{j-1}=0}} t^{|C_{\rho,1}(N)|} q^{|C_{\rho,2}(N)|}\\
&= \left(\frac{1-q^{(j-1)a_2}}{1-q^{j-1}}\right) \sum_{\substack{\rho\in\Upsilon_{\psi_S}\\ \rho_{j-1}=0}} t^{|C_{\rho,1}(N)|} q^{|C_{\rho,2}(N)|}.
\end{aligned}$$

Continuing this argument, let us set $d_{\psi_S}(q,t)$ to be

$$d_{\psi_S}(q,t) \;=\; \prod_{i=1}^{j}\left(\frac{1-q^{(j+1-i)a_i}}{1-q^{j+1-i}}\right)\prod_{i=1}^{j'}\left(\frac{1-t^{(j'+1-i)b_i}}{1-t^{j'+1-i}}\right). \tag{3.21}$$

Therefore, we have

Corollary 3.10.

Suppose ψ_S is dense and let $\Psi = \Big[[1^j],[1^{j'}]\Big]$. The Hilbert series $\mathcal{H}(\mathbb{C}_S[X,Y])$ is given by

$$\mathcal{H}(\mathbb{C}_S[X,Y]) \;=\; d_{\psi_S}(q,t)\;\mathcal{H}(\mathbb{C}_{S_\Psi}[X,Y]).$$

$\phi \in S_n$ acting on a basis for $\mathbb{C}_S[X,Y]$ yields a representation Φ of S_n. With I an injective tableau of shape λ, let ϕI denote the tableau of shape λ obtained by replacing i in I by $\phi(i)$. Let

$$\mathcal{N}_\lambda \;=\; \{N_1, N_2, \cdots, N_{h_\lambda}\}$$

denote the collection of standard tableaux of shape λ. Now

$$\begin{aligned}\phi[N_i,U]_{per} \;&=\; [\phi N_i, U]_{per}\\ &=\; \sum_{k=1}^{h_\lambda} e_{i,k}(\phi)\;[N_k,U]_{per} \;+\; \sum_{\substack{(P,Q)\in\mathcal{B}_{\psi_S}\\ sh(Q)>_L sh(U)}} g_{P,Q}\;[P,Q]_{per}.\end{aligned}$$

It is not difficult to show that the matrix $(e_{i,k}(\phi))$ is the irreducible S_n representation corresponding to shape λ (see [1]).

Thus for a given $\rho \in \Upsilon_{\psi_S}$, the action of $\phi \in S_n$ on the collection

$$\Big\{\Big[N_i, C_\rho(N_k)\Big]_{per} : N_i, N_k \in \mathcal{N}_\lambda\Big\}$$

yields a representation where the S_n-irreducible representation of shape λ has multiplicity h_λ (the number of standard tableaux of shape λ). Thus for each $\rho \in \Upsilon_{\psi_S}$ there corresponds one copy of the S_n-regular representation. Therefore, we have

Corollary 3.11.

If S is dense then the multiplicity of the regular representation of S_n in the representation Φ is d_{ψ_S}.

Now, let R_{u_1,u_2} denote the homogeneous component of $\mathbb{C}_S[X,Y]$ of degree u_1 in X and u_2 in Y. With $b \in \mathcal{B}_{\psi_S}$ (recall, $\mathcal{B}_{\psi_S}$ is a basis for $\mathbb{C}_S[X,Y]$) and if $b \in R_{u_1,u_2}$ then

Theorem 2.1 and Theorem 3.7 imply that $\sigma b \in R_{u_1,u_2}$. Now, Theorem 3.7 implies that if $p \in R_{u_1,u_2}$,

$$p \;\equiv\; \sum_{i=1}^{f} d_i\; b_i \qquad (\mathrm{mod}\; \mathcal{I}_S(X,Y))$$

and $d_i \neq 0$ then $b_i \in R_{u_1,u_2}$. Specifically, if $p \in R_{u_1,u_2}$, then $\sigma p \in R_{u_1,u_2}$. Let *char* (R_{u_1,u_2}) denote the character of the action of $\sigma \in S_n$ on $R_{r,s}$. The *graded character* of $\mathbb{C}_S[X,Y]$ is defined as

$$char_{q,t}(\mathbb{C}_S[X,Y]) \;=\; \sum_{u_1,u_2} char\;(R_{r,s})\; t^{u_1}\; q^{u_2}. \tag{3.22}$$

Corollary 3.12.

With S dense, the graded character $char_{q,t}(\mathbb{C}_S[X,Y])$ of $\mathbb{C}_S[X,Y]$ is given by:

$$\begin{aligned} char_{q,t}(\mathbb{C}_S[X,Y]) \;&=\; \sum_{\lambda\vdash n}\chi^\lambda \sum_{\rho\in\Upsilon_{\psi_S}}\;\sum_{M\in\mathcal{ST}_\lambda} t^{|C_{\rho,1}(M)|}q^{|C_{\rho,2}(M)|} \qquad (3.23)\\ &=\; d_{\psi_S}(q,t)\;\sum_{\lambda\vdash n}\chi^\lambda \sum_{\substack{M\in\mathcal{ST}_\lambda\\ \rho=(0^{j-1},0,0^{n-j})}} t^{|C_{\rho,1}(M)|}q^{|C_{\rho,2}(M)|}\\ &=\; d_{\psi_S}(q,t)\; char_{q,t}(\mathbb{C}_{S_\Psi}[X,Y]) \end{aligned}$$

where $\mathcal{ST}_\lambda$ denotes the collection of standard tableaux of shape λ and $|C_{\rho,1}(M)|$ and $|C_{\rho,2}(M)|$ denote the sum of the first and second coordinates, respectively, of the entries of $C_\rho(M)$, χ^λ denotes the irreducible S_n character corresponding to shape λ and $\Psi = \Big[[1^j],[1^{j'}]\Big]$.

Example

With ψ_S given in (3.19), we have

$$\begin{aligned} char_{q,t}(\mathbb{C}_S[X,Y]) \;&=\; (1\;+\;t)\chi^{(3)}\;+\;(tq\;+\;t^2q)\chi^{(1,1,1)}\;+\;(q\;+\;tq\;+\;t\;+\;t^2)\chi^{(2,1)}\\ &=\; (1+t)\Big(\chi^{(3)}\;+\;tq\;\chi^{(1,1,1)}\;+\;(t+q)\;\chi^{(2,1)}\Big). \;\square \end{aligned}$$

With

$$\psi_S \;=\; \Big[\emptyset,[b_1,b_2,\cdots,b_n]\Big],$$

for $1 \leq i \leq n$, set

$$\lambda_i \;=\; -i\;+\;\sum_{h=1}^{i} b_h,$$

and

$$\lambda \;=\; \lambda_{\psi_S} \;=\; (\lambda_1, \lambda_2, \cdots, \lambda_n).$$

Note that in [6], it is shown that the graded Frobenius characteristic of $\mathbb{C}_S[X,Y]$ for any ψ_S is given by

$$\mathcal{F}(\mathbb{C}_S[X,Y]) \;=\; \Xi_\lambda(t)\; \tilde{H}_{1^n}(x;q,t) \tag{3.24}$$

where $\Xi_\lambda(t)$ is the Hilbert series of the graded vector space of skew Schur functions $s_{\lambda/\mu}(x_1,\cdots,x_n)$ as μ varies in λ_{ψ_S} and $\tilde{H}_{1^n}(x;q,t)$ is a variant of the Macdonald polynomial (see Theorem 2.1 in [6]). Since the Frobenius characteristic of the factor space $\mathbb{C}_{S_\Psi}[X,Y]$ where $\Psi = [\emptyset,[1^n]]$ is $\tilde{H}_{1^n}(x;q,t)$, we get

Corollary 3.13.

If

$$\psi_S \;=\; \Big[\emptyset, [b_1, b_2, \cdots, b_n]\Big]$$

is dense, then

$$\Xi_{\lambda_{\psi_S}}(t) \;=\; d_{\psi_S}(0,t).$$

4. The Symmetric Module $\mathbb{C}^{+}_{S,T}[X,Y,Z,W]$.

Let $\mathcal{LP}^{+}_n$ be the module of polynomials over $\mathbb{C}$ of length n in the indeterminants $(\varpi_i|\varsigma_k)$ where $\varpi_i = (\varpi_{i,1}, \varpi_{i,2})$ and $\varsigma_k = (\varsigma_{k,1}, \varsigma_{k,2})$ are elements from the alphabet $\mathcal{A}$. Recall that the ring $\mathbb{C}^{S_n}[X,Y,Z,W]$ is defined by

$$\mathbb{C}^{S_n}[X,Y,Z,W] \;=\; \Big\{P(X,Y,Z,W) \in \mathbb{C}[X,Y,Z,W] : \sigma\; P \;=\; P \;\forall\; \sigma \in S_n\Big\}.$$

Additionally, the ideal $\mathcal{I}^{+}_{S,T}(X,Y,Z,W)$ and the factor space $\mathbb{C}^{+}_{S,T}[X,Y,Z,W]$ are defined by setting

$$\mathcal{I}^{+}_{S,T}(X,Y,Z,W) = \Big\{P \in \mathbb{C}^{S_n}[X,Y,Z,W] : P(\partial_X, \partial_Y, \partial_Z, \partial_W)\; \Delta_S(X,Y)\; \Delta_T(Z,W) = 0\Big\}$$

and

$$\mathbb{C}^{+}_{S,T}[X,Y,Z,W] \;=\; \mathbb{C}^{S_n}[X,Y,Z,W]/\mathcal{I}^{+}_{S,T}(X,Y,Z,W).$$

Note that the factor space $\mathbb{C}^{+}_{S,T}[X,Y,Z,W]$ is a generalization of the factor spaces found in [16] and [2].

Define the homomorphism $\phi^{+} : \mathcal{LP}^{+}_n \rightarrow \mathbb{C}^{S_n}[X,Y,Z,W]$ by linearly extending the map

$$\theta^{+}\Bigg((\varpi_1|\varsigma_1)\;(\varpi_2|\varsigma_2)\;(\varpi_3|\varsigma_3)\cdots(\varpi_n|\varsigma_n)\Bigg)$$
$$= \sum_{\sigma\in S_n} \sigma_{X,Y,Z,W}\;\; \left(x_1^{\varpi_{1,1}} x_2^{\varpi_{2,1}} \cdots x_n^{\varpi_{n,1}} y_1^{\varpi_{1,2}} y_2^{\varpi_{2,2}} \cdots y_n^{\varpi_{n,2}} z_1^{\varsigma_{1,1}} \cdots z_n^{\varsigma_{n,1}} w_1^{\varsigma_{1,2}} \cdots w_n^{\varsigma_{n,2}}\right).$$

Now, let U and V be two column-strict tableaux of shape $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_k)$ with entries from $\mathcal{A}$ and let I be an injective tableau of shape λ. Furthermore, suppose that u_i and v_i are the entries in the cells of U and V respectively that correspond to the cell containing i in I and let

$$f_I \;=\; \lambda_1!\ \lambda_2!\ \cdots\ \lambda_k!. \tag{4.1}$$

Noting that by acting by α^{-1} on both the subscripts and superscripts yields

$$\begin{aligned}
& x_1^{u_{\alpha(1),1}} x_2^{u_{\alpha(2),1}} \cdots x_n^{u_{\alpha(n),1}}\ \ y_1^{u_{\alpha(1),2}} \cdots y_n^{u_{\alpha(n),2}} \\
=\ & x_{\alpha^{-1}(1)}^{u_{1,1}} x_{\alpha^{-1}(2)}^{u_{2,1}} \cdots x_{\alpha^{-1}(n)}^{u_{n,1}}\ \ y_{\alpha^{-1}(1)}^{u_{1,2}} \cdots y_{\alpha^{-1}(n)}^{u_{n,2}}
\end{aligned}$$

and, with $D(I)$ defined in equation (2.3), we have

$$\begin{aligned}
&\phi^+\Big((U,V)_{det}\Big) \\
&= \phi^+\left(\sum_{\alpha\in D(I)} sgn(\alpha)\ (u_{\alpha(1)}|v_1)\ (u_{\alpha(2)}|v_2)\ \cdots\ (u_{\alpha(n)}|v_n)\right) \\
&= \sum_{\sigma\in S_n} \sigma_{X,Y,Z,W}\Bigg(\sum_{\alpha\in D(I)} sgn(\alpha)\ x_{\alpha(1)}^{u_{1,1}} x_{\alpha(2)}^{u_{2,1}} \cdots x_{\alpha(n)}^{u_{n,1}}\ y_{\alpha(1)}^{u_{1,2}} \cdots y_{\alpha(n)}^{u_{n,2}} \\
&\qquad z_1^{v_{1,1}} \cdots z_n^{v_{n,1}}\ w_1^{v_{1,2}} \cdots w_n^{v_{n,2}}\Bigg) \\
&= \frac{1}{f_{I^t}} \sum_{\sigma\in S_n} \sigma_{X,Y,Z,W}\Bigg(\sum_{\beta\in D(I)} \beta_{X,Y,Z,W}\ \Bigg(\sum_{\beta^{-1}\alpha\in D(I)} sgn(\beta^{-1})\ sgn(\alpha) \\
&\qquad x_{\beta^{-1}\alpha(1)}^{u_{1,1}} \cdots x_{\beta^{-1}\alpha(n)}^{u_{n,1}}\ y_{\beta^{-1}\alpha(1)}^{u_{1,2}} \cdots y_{\beta^{-1}\alpha(n)}^{u_{n,2}} z_1^{v_{1,1}} \cdots z_n^{v_{n,1}}\ w_1^{v_{1,2}} \cdots w_n^{v_{n,2}}\Bigg)\Bigg) \\
&= \frac{1}{f_{I^t}} \sum_{\sigma\in S_n} \sigma_{X,Y,Z,W}\Bigg(\sum_{\alpha\in D(I)} sgn(\alpha)\ x_{\alpha(1)}^{u_{1,1}} \cdots x_{\alpha(n)}^{u_{n,1}}\, y_{\alpha(1)}^{u_{1,2}} \cdots y_{\alpha(n)}^{u_{n,2}} \\
&\qquad \sum_{\beta\in D(I)} sgn(\beta)\ z_{\beta(1)}^{v_{1,1}} \cdots z_{\beta(n)}^{v_{n,1}}\, w_{\beta(1)}^{v_{1,2}} \cdots w_{\beta(n)}^{v_{n,2}}\Bigg) \\
&= \frac{1}{f_{I^t}} \sum_{\sigma\in S_n} \sigma_{X,Y,Z,W}\ \Big([I,U]_{det}(X,Y)\ [I,V]_{det}(Z,W)\Big).
\end{aligned}$$

Similarly,

$$\phi^+\Big((U,V)_{per}\Big) \;=\; \frac{1}{f_I} \sum_{\sigma\in S_n} \sigma_{X,Y,Z,W}\ \Big([I,U]_{per}(X,Y)\ [I,V]_{per}(Z,W)\Big).$$

We will denote $\phi^+\Big((U,V)_{det}\Big)$ and $\phi^+\Big((U,V)_{per}\Big)$ by $[U,V]^+_{det}$ and $[U,V]^+_{per}$, respectively. It is not difficult to see that the definitions of $[V,U]^+_{per}$ and $[V,U]^+_{det}$ are completely independent of the choice of the injective tableau I. Furthermore, note that ϕ^+ is a vector space isomorphism between $\mathcal{LP}^+_n$ and $\mathbb{C}^{S_n}[X,Y,Z,W]$. Theorem 2.2 and Theorem 2.1, therefore, immediately imply the following two lemmas. (Recall that the collection $\mathcal{CS}_n$ consists of column-strict tableaux with entries from the alphabet $\mathcal{A}$.)

Lemma 4.1.

The collection

$$\mathcal{GBD}_n \;=\; \Big\{[U,V]^+_{det} : U,V \in CS_n, sh(U) = sh(V)\Big\}, \tag{4.2}$$

is linearly independent in $\mathbb{C}^{S_n}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$.

Lemma 4.2.

The collection

$$\mathcal{GBP}_n \;=\; \Big\{[U,V]^+_{per} : U,V \in CS_n, sh(U) = sh(V)\Big\}, \tag{4.3}$$

spans $\mathbb{C}^{S_n}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$.

Recall that with pairs of tableaux (D,E) and (F,G), we have defined

$$(D,E) <_{stc} (F,G) \tag{4.4}$$

when

1. $D <_{stc} F$ (see equation (3.7)); or
2. if $D = F$ then $E <_{stc} G$.

We want to prove that the collection

$$\mathcal{BSQ}_{S,T} \;=\; \Big\{[U,V]^+_{per} : U \in \mathcal{CO}_S, V \in \mathcal{CO}_T \text{ and } sh(U) = sh(V)\Big\} \tag{4.5}$$

is a basis for $\mathbb{C}^+_{S,T}[X,Y,Z,W]$. Suppose $[U,V]^+_{per} \in \mathcal{BSQ}_{S,T}$. Thus $U = C_{\rho_1}(P)$ and $V = C_{\rho_2}(Q)$ for some standard tableaux P and Q and sequences $\rho_1 \in \Upsilon_{\psi_S}$ and $\rho_2 \in \Upsilon_{\psi_T}$. Let $D = C_{\rho_1^-}(P)$ and $E = C_{\rho_2^-}(Q)$. Now,

$$\sigma_{X,Y,Z,W}\; \Delta_S(X,Y)\; \Delta_T(Z,W) \;=\; \Delta_S(X,Y)\; \Delta_T(Z,W),$$

(recall equation (1.4)) and thus

$$\begin{aligned}
&[U,V]^+_{per}(\partial_X,\partial_Y,\partial_Z,\partial_W)\; \Delta_S(X,Y)\; \Delta_T(Z,W)\\
&= \frac{1}{f_I}\sum_{\sigma\in S_n} \sigma_{X,Y,Z,W}\; \Big([I,U]_{per}(\partial_X,\partial_Y)\; [I,V]_{per}(\partial_Z,\partial_W)\Big)\; \Delta_S(X,Y)\; \Delta_T(Z,W)\\
&= \frac{1}{f_I}\sum_{\sigma\in S_n} \sigma_{X,Y,Z,W}\; \Big([I,U]_{per}(\partial_X,\partial_Y)\; \Delta_S(X,Y)\; [I,V]_{per}(\partial_Z,\partial_W)\; \Delta_T(Z,W)\Big).
\end{aligned}$$

Lemma 3.3 and Theorem 3.4 imply that

$$\begin{aligned}
&[U,V]^+_{per}(\partial_X,\partial_Y,\partial_Z,\partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W)\\
&= \frac{1}{f_I}\sum_{\sigma\in S_n}\sigma_{X,Y,Z,W}\ \Big([I,U]_{per}(\partial_X,\partial_Y)\ \Delta_S(X,Y)\ [I,V]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big)\\
&= \frac{1}{f_I}\ c_{D,E}\ [D,E]^+_{det}\ +\sum_{(D,E)<_{stc}(F,G)} c_{F,G}\ [F,G]^+_{det}.
\end{aligned}$$

where $c_{D,E}>0$. Since the collection $\mathcal{GBD}_n$ is linearly independent (Lemma 4.1) and using the fact that

$$\begin{aligned}
& c_1p_1\ +\ \cdots\ +\ c_mp_m \in \mathcal{I}^+_{S,T}(X,Y,Z,W)\\
\Leftrightarrow\ & (c_1p_1\ +\ \cdots\ +\ c_mp_m)(\partial_X,\partial_Y,\partial_Z,\partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W)=0\\
\Leftrightarrow\ & c_1p_1(\partial_X,\partial_Y,\partial_Z,\partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W)\ +\ \cdots\\
&\quad +\ c_mp_m(\partial_X,\partial_Y,\partial_Z,\partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W)=0,
\end{aligned}$$

where $c_i\in\mathbb{C}$ and $p_i\in\mathcal{BSQ}_{S,T}$, we have

Theorem 4.3.

The collection $\mathcal{BSQ}_{S,T}$ is linearly independent in $\mathbb{C}^+_{S,T}[X,Y,Z,W]$.

Let (P,Q) and (U,V) be pairs of tableaux. Recall that we have defined

$$(P,Q) <_{str} (U,V) \tag{4.6}$$

if

1. $P <_{str} U$ (see equation (3.14)); or
2. if $P=U$ then $Q <_{str} V$.

Now suppose that $[U,V]^+_{per}\in\mathcal{GBP}$ but $[U,V]^+_{per}\notin\mathcal{BSQ}_{S,T}$. Let $\gamma=\gamma_{\rho_U,U}(U)$ and $\nu=\gamma_{\rho_V,V}(V)$. Since $[U,V]^+_{per}\notin\mathcal{BSQ}_{S,T}$, we know that at least one of γ and ν is a sequence not made up of entirely $(0,0)'s$. Furthermore, let $st(U)$ and $st(V)$ be the standard tableaux constructed from the labellings of U and V respectively. We will set $M=C_{\rho_U}(st(U))$ and $N=C_{\rho_V}(st(V))$. Now,

$$\begin{aligned}
& m_\gamma(X,Y)\ m_\nu(Z,W)\ [M,N]^+_{per}\\
&= m_\gamma(X,Y)\ m_\nu(Z,W)\sum_{\sigma\in S_n}\sigma_{X,Y,Z,W}\ \ ([I,M]_{per}(X,Y)\ [I,N]_{per}(Z,W))\\
&= \sum_{\sigma\in S_n}\sigma_{X,Y,Z,W}\ \Big(m_\gamma(X,Y)\ [I,M]_{per}(X,Y)\ m_\nu(Z,W)\ [I,N]_{per}(Z,W)\Big)\\
&= \sum_{\sigma\in S_n}\sigma_{X,Y,Z,W}\ \left(\sum_{\alpha\in S_n}[I,Q_\alpha]_{per}(X,Y)\ \sum_{\beta\in S_n}[I,P_\beta]_{per}(Z,W)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\alpha\in S_n}\sum_{\beta\in S_n}\sum_{\sigma\in S_n} \sigma_{X,Y,Z,W}\ \Big([I,Q_\alpha]_{per}(X,Y)\ [I,P_\beta]_{per}(Z,W) \Big) \\
&= \sum_{\alpha\in S_n}\sum_{\beta\in S_n} [Q_\alpha, P_\beta]^+_{per}
\end{aligned}
$$

where Q_α and P_β are as defined in equation (3.15). Thus, using Theorem 3.7 and the straightening algorithm in Theorem 2.1, we have

$$
\begin{aligned}
& m_\gamma(X,Y)\ m_\nu(Z,W)\ [M,N]^+_{per} \\
&\equiv \left(c_{U,V}\ [U,V]^+_{per}\ +\ \sum_{\substack{(P,Q)>_{str}(U,V)\\ P,Q\in\mathcal{CS}_n}} c_{P,Q}\ [P,Q]^+_{per} \right)\ \Big(mod\ \mathcal{I}^+_{S,T}(X,Y,Z,W) \Big) \qquad (4.7)
\end{aligned}
$$

where $c_{U,V} > 0$. Now

$$
m_\gamma(X,Y)\ m_\nu(Z,W) \in \mathcal{I}^+_{S,T}(X,Y,Z,W)
$$

if both S and T are dense. Thus, equation (4.7) implies an explicit algorithm for expanding polynomials in $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ in terms of $\mathcal{BSQ}_{S,T}$ (with coefficients from $\mathbb{C}$ or $\mathcal{I}^+_{S,T}(X,Y,Z,W)$). Using Theorem 4.3, we now have

Theorem 4.4.

If S and T are dense then

$$
\mathcal{BSQ}_{S,T}\ =\ \Big\{ [U,V]^+_{per} : U\in\mathcal{CO}_S, V\in\mathcal{CO}_T \ and\ sh(U)=sh(V) \Big\}
$$

is a basis for $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$.

If R_{u_1,u_2,u_3,u_4} is a homogeneous subspace of dimension u_1 in X, u_2 in Y, u_3 in Z and u_4 in W, then we define the Hilbert series $\mathcal{H}(R)$ to be

$$
\mathcal{H}(R)\ =\ \sum_{u_1,u_2,u_3,u_4} dim(R_{u_1,u_2,u_3,u_4})\ t^{u_1}\ q^{u_2}\ r^{u_3}\ s^{u_4}.
$$

Recall that $d_{\psi_S}(q,t)$ is a polynomial defined in equation (3.21) and that Υ_{ψ_S} and Υ_{ψ_T} are collections of sequences defined in equation (3.4).

Corollary 4.5.

If S and T are dense then the Hilbert series $\mathcal{H}(\mathbb{C}^+_{S,T}[X,Y,Z,W])$ for $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ is given by

$$
\begin{aligned}
&\mathcal{H}(\mathbb{C}^{+}_{S,T}[X,Y,Z,W]) \\
&= \sum_{\lambda\vdash n}\sum_{\rho\in\Upsilon_{\psi_S}}\sum_{\rho'\in\Upsilon_{\psi_T}}\sum_{(M,N)\in\mathcal{ST}_\lambda\times\mathcal{ST}_\lambda} t^{|C_{\rho,1}(M)|}q^{|C_{\rho,2}(M)|}r^{|C_{\rho',1}(N)|}s^{|C_{\rho',2}(N)|} \\
&= d_{\psi_S}(q,t)\ d_{\psi_T}(s,r)\sum_{\lambda\vdash n}\sum_{\substack{(M,N)\in\mathcal{ST}_\lambda\times\mathcal{ST}_\lambda\\ \rho=(0^n),\rho'=(0^n)}} t^{|C_{\rho,1}(M)|}q^{|C_{\rho,2}(M)|}r^{|C_{\rho',1}(N)|}s^{|C_{\rho',2}(N)|}, \qquad (4.8)
\end{aligned}
$$

where $\mathcal{ST}_\lambda$ denotes the collection of standard tableaux of shape λ, $C_\rho(M)$ denotes a cocharge tableau corresponding to a tableau M and a sequence ρ, $|C_{\rho,1}(M)|$ and $|C_{\rho,2}(M)|$ denote the sums of the first and second coordinates of the cocharge tableau $C_\rho(M)$, respectively.

Example

With $S = T$ given in (3.19) we have

$$
\begin{aligned}
&\mathcal{H}(\mathbb{C}^{+}_{S,T}[X,Y,Z,W]) \\
&=\ (1+t)\,(1+r)\ +\ \left(q+tq+t+t^2\right)\left(s+rs+r+r^2\right)\ +\ \left(tq+t^2q\right)\left(rs+r^2s\right) \\
&=\ (1+t)\ \ (1+q)\ \ \Big(1+(t+q)\,(r+s)+tqrs\Big).
\end{aligned}
$$

5. The Skew-Symmetric Module $\mathbb{C}^{-}_{S,T}[X,Y,Z,W]$.

We will let $\mathcal{LP}^{-}_{n}$ denote the letter-place module over the alphabet $\mathcal{A}$ of length n that is *diagonally skew-symmetric*. Specifically, with $\varpi_i, \varsigma_i \in \mathcal{A}$, we have that

$$
(\varpi_1|\varsigma_1)\ (\varpi_2|\varsigma_2)\ (\varpi_3|\varsigma_3)\cdots(\varpi_n|\varsigma_n)\ =\ 0 \qquad (5.1)
$$

if $(\varpi_i|\varsigma_i) = (\varpi_k|\varsigma_k)$ for $i \neq k$ and that

$$
(\varpi_{\sigma_1}|\varsigma_{\sigma_1})\ (\varpi_{\sigma_2}|\varsigma_{\sigma_2})\ \cdots(\varpi_{\sigma_n}|\varsigma_{\sigma_n})\ =\ sgn(\sigma)\ (\varpi_1|\varsigma_1)\ (\varpi_2|\varsigma_2)\cdots(\varpi_n|\varsigma_n). \qquad (5.2)
$$

Note that Theorem 5 on page 28 of [11] implies that the collection

$$
\Big\{(U,V)_{det} : U\in\mathcal{CS}_n, V\in\mathcal{RS}_n, sh(U)=sh(V)\Big\}
$$

is a basis for $\mathcal{LP}^{-}_{n}$ (where $\mathcal{CS}_n$ and $\mathcal{RS}_n$ are the collections of column-strict and row-strict tableaux, respectively, with n cells with entries from the alphabet $\mathcal{A}$.) Recall that

$$
\mathbb{C}^{-}[X,Y,Z,W]\ =\ \Big\{P\in\mathbb{C}[X,Y,Z,W] : \sigma_{X,Y,Z,W}\ P\ =\ sgn(\sigma)\ P\ \forall\ \sigma\in S_n\Big\},
$$

$$\mathcal{I}^-_{S,T}(X,Y,Z,W) = \Big\{ P \in \mathbb{C}^-[X,Y,Z,W] : P(\partial_X, \partial_Y, \partial_Z, \partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W) = 0 \Big\}$$

and

$$\mathbb{C}^-_{S,T}[X,Y,Z,W] \ = \ \mathbb{C}^-[X,Y,Z,W]/\mathcal{I}^-_{S,T}(X,Y,Z,W).$$

Define $\phi^- : \mathcal{LP}^-_n \to \mathbb{C}^-[X,Y,Z,W]$ by linearly extending the map

$$\begin{aligned} &\theta^-\big((\varpi_1|\varsigma_1)\ (\varpi_2|\varsigma_2)\ (\varpi_3|\varsigma_3)\cdots(\varpi_n|\varsigma_n)\big) \\ &= \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \left(x_1^{\varpi_{1,1}}\cdots x_n^{\varpi_{n,1}} y_1^{\varpi_{1,2}}\cdots y_n^{\varpi_{n,2}} z_1^{\varsigma_{1,1}}\cdots z_n^{\varsigma_{n,1}} w_1^{\varsigma_{1,2}}\cdots w_n^{\varsigma_{n,2}}\right) \end{aligned}$$

where $\varpi_i = (\varpi_{i,1}, \varpi_{i,2}) \in \mathcal{A}$ and $\varsigma_k = (\varsigma_{k,1}, \varsigma_{k,2}) \in \mathcal{A}$.

Let G be the (group) direct product $G = S_n \times S_n$ and let $\mathbb{C}[G]$ be the group algebra on G with coefficients from $\mathbb{C}$. The action of $(\alpha,\beta) \in G$ on a polynomial $P(X,Y,Z,W)$ is defined by (recall equation (1.5))

$$(\alpha,\beta)\ P(X,Y,Z,W) \ = \ \alpha_{X,Y}\ \beta_{Z,W}\ P(X,Y,Z,W).$$

We will identify $\alpha_{X,Y} = (\alpha,\epsilon)$, $\beta_{Z,W} = (\epsilon,\beta)$ and $\sigma_{X,Y,Z,W} = (\sigma,\sigma)$, where ϵ is the identity element in the symmetric group S_n. With I a standard tableau, recall that $D(I) = R(I^t)$ (see equations (2.2) and (2.3)) and that f_{I^t} is the order of the group $D(I)$. In the group algebra $\mathbb{C}[G]$,

$$\begin{aligned} &\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \left[\sum_{\alpha\in D(I)} sgn(\alpha)\ \alpha_{X,Y}\right] \\ &= \frac{1}{f_{I^t}} \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Bigg[\sum_{\beta\in R(I^t)} sgn(\beta)\ \beta_{X,Y,Z,W} \\ &\qquad \sum_{\beta^{-1}\alpha\in D(I)} sgn(\beta^{-1})\ sgn(\alpha)\ \beta^{-1}_{X,Y}\ \alpha_{X,Y}\Bigg] \\ &= \frac{1}{f_{I^t}} \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\left[\left(\sum_{\alpha\in D(I)} sgn(\alpha)\ \alpha_{X,Y}\right)\left(\sum_{\beta\in R(I^t)} \beta_{Z,W}\right)\right]. \end{aligned} \tag{5.3}$$

Similarly,

$$\begin{aligned} &\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \left[\sum_{\beta\in R(I^t)} \beta_{Z,W}\right] \\ &= \frac{1}{f_{I^t}} \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \left[\sum_{\alpha\in D(I)} sgn(\alpha)\ \alpha_{X,Y,Z,W} \sum_{\alpha^{-1}\beta\in R(I^t)} \alpha^{-1}_{Z,W}\ \beta_{Z,W}\right] \\ &= \frac{1}{f_{I^t}} \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\left[\left(\sum_{\alpha\in D(I)} sgn(\alpha)\ \alpha_{X,Y}\right)\left(\sum_{\beta\in R(I^t)} \beta_{Z,W}\right)\right]. \end{aligned} \tag{5.4}$$

The equality of equation (5.3) and equation (5.4) yields the following important identity:

$$\begin{aligned}
&\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \left[\sum_{\alpha\in D(I)} sgn(\alpha)\ \alpha_{X,Y}\right]\\
&= \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \left[\sum_{\beta\in R(I^t)} \beta_{Z,W}\right]\\
&= \frac{1}{f_{I^t}} \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \left[\left(\sum_{\alpha\in D(I)} sgn(\alpha)\ \alpha_{X,Y}\right)\ \left(\sum_{\beta\in R(I^t)} \beta_{Z,W}\right)\right].
\end{aligned} \tag{5.5}$$

Let U and M be, respectively, a column-strict tableau and a row-strict tableau both of shape λ. Equation (5.5) implies

$$\phi^-\Big((U,M)_{det}\Big) \tag{5.6}$$

$$= \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W} \Bigg[\Bigg(\sum_{\alpha\in D(I)} sgn(\alpha)\ \alpha_{X,Y}\ x_1^{u_{1,1}}\cdots x_n^{u_{n,1}} y_1^{u_{1,2}}\cdots y_n^{u_{n,2}}\Bigg)$$

$$z_1^{m_{1,1}}\cdots z_n^{m_{n,1}} w_1^{m_{1,2}}\cdots w_n^{m_{n,2}}\Bigg] \tag{5.7}$$

$$= \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Bigg[x_1^{u_{1,1}}\cdots x_n^{u_{n,1}} y_1^{u_{1,2}}\cdots y_n^{u_{n,2}}$$

$$\Bigg(\sum_{\beta\in R(I^t)} \beta_{Z,W}\ z_1^{m_{1,1}}\cdots z_n^{m_{n,1}} w_1^{m_{1,2}}\cdots w_n^{m_{n,2}}\Bigg)\Bigg] \tag{5.8}$$

$$= \frac{1}{f_{I^t}} \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big([I,U]_{det}(X,Y)\ [I^t,M^t]_{per}(Z,W)\Big). \tag{5.9}$$

Note that since M is row-strict, M^t is column-strict. Let us denote $\phi^-\Big((U,M)_{det}\Big)$ by $[U,M]^-_{det}$. Set

$$\mathcal{CSRS}_n\ =\ \Big\{[U,M]^-_{det} : U\in\mathcal{CS}_n, M\in\mathcal{RS}_n, sh(U)=sh(M)\Big\}. \tag{5.10}$$

Since ϕ^-_n is a (vector space) isomorphism (up to sign), we have

Theorem 5.1.

The collection $\mathcal{CSRS}_n$ is linearly independent in $\mathbb{C}^-[X,Y,Z,W]$.

It is important to note that equations (5.7) and (5.8) imply that the straightening algorithm found in Theorem 2.1 can be used in $\mathcal{LP}_n^-$ to write every $[P,Q]_{det}^-$ as a linear combination of elements from $\mathcal{CSRS}$. Particularly, we have the following lemma.

Lemma 5.2.

Suppose that U and M are tableaux such that $sh(U)=\mu$ and $sh(M)=\nu$ where μ and ν are not necessarily equal. Let I_1 and I_2 be injective tableaux of shape μ and ν respectively. Let λ be the larger (lexicographically) of μ^t and ν^t. Then

$$\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big([I_1,U]_{det}(X,Y)\ [I_2^t,M^t]_{per}(Z,W)\Big)\ =\ \sum_{sh(P^t)\geq_L \lambda^t} c_{P,Q}\ [P,Q]_{det}^-$$

where $[P,Q]_{det}^-\in\mathcal{CSRS}_n$.

Proof

Suppose that $\lambda=\mu^t$. Then, assuming that

$$[I_2^t,M^t]_{per}(Z,W)\ =\ \sum_i c_i\ p_i(Z,W),$$

where each $p_i(Z,W)$ is a monomial in $\mathbb{C}[Z,W]$, and using equations (5.7) and (5.9), we have

$$\begin{aligned}
&\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \left[\Big([I_1,U]_{det}(X,Y)\ [I_2^t,M^t]_{per}(Z,W)\Big)\right]\\
&=\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \left[\Big(\sum_{\alpha\in D(I_1)} sgn(\alpha)\ \alpha_{X,Y}\ x_1^{u_{1,1}}\cdots x_n^{u_{n,1}}y_1^{u_{1,2}}\cdots y_n^{u_{n,2}}\Big)\right.\\
&\qquad\left.\sum_i c_i\ p_i(Z,W)\right]\\
&=\sum_i c_i\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \left[\Big(\sum_{\alpha\in D(I_1)} sgn(\alpha)\ \alpha_{X,Y}\ x_1^{u_{1,1}}\cdots x_n^{u_{n,1}}y_1^{u_{1,2}}\cdots y_n^{u_{n,2}}\Big)\right.\\
&\qquad\left.p_i(Z,W)\right]\\
&=\frac{1}{f_{I_1^t}}\sum_i c_i\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[[I_1,U]_{det}(X,Y)\ [I_1^t,Q_i]_{per}(Z,W)\Big].
\end{aligned}$$

The proof concludes by applying Theorem 2.1.

The proof when $\lambda=\nu^t$ is similar. □

With I an injective tableau of shape $sh(U) = sh(V)$, set

$$[U,V]^{-}_{per} \;=\; \frac{1}{f_I} \sum_{\sigma \in S_n} sgn(\sigma)\; \sigma_{X,Y,Z,W} \Big[[I,U]_{per}(X,Y)\; [I,V]_{per}(\partial_Z,\partial_W)\; \Delta_T(Z,W) \Big] \tag{5.11}$$

where f_I is the order of the group $R(I)$. Our goal is to show that the collection

$$\mathcal{BAQ}_{S,T} \;=\; \Big\{ [U,V]^{-}_{per} : U \in \mathcal{CO}_S, V \in \mathcal{CO}_T, sh(U) = sh(V) \Big\} \tag{5.12}$$

is a basis for $\mathbb{C}^{-}_{S,T}[X,Y,Z,W]$. To this end, we have the following lemma.

Lemma 5.3.

If the collection $\{p_1, p_2, \cdots, p_m\}$ is a basis for $\mathbb{C}_T[Z,W]$ then the collection

$$\Big\{ p_1(\partial_Z,\partial_W)\Delta_T(Z,W), \cdots, p_m(\partial_Z,\partial_W)\Delta_T(Z,W) \Big\} \tag{5.13}$$

is a basis for $\mathbb{C}_T[Z,W]$.

Proof

Define an inner product $<,>$ on $\mathbb{C}[Z,W]$ by setting

$$< P, Q > = \; P(\partial_Z,\partial_W)\; Q|_{Z=W=0} \tag{5.14}$$

where $|_{Z=W=0}$ indicates that we evaluate the resulting polynomial $P(\partial_Z,\partial_W)\; Q$ at

$$z_1 = \cdots = z_n = w_1 = \cdots = w_n = 0.$$

It is clear that for any polynomial $P \in \mathbb{C}[Z,W]$ that

$$< P, P > \geq \; 0$$

and

$$< P, P > = \; 0$$

if and only if $P = 0$.

Note that $p_i(\partial_Z,\partial_W)\Delta_T(Z,W) \neq 0$ since each p_i is an element of a basis for $\mathbb{C}_T[Z,W]$. Suppose that

$$Q \;=\; \sum_{k=1}^{n} c_k\; p_k(\partial_Z,\partial_W)\; \Delta_T(Z,W) \in \mathcal{I}_T(Z,W)$$

where not all of the c_j are equal to zero. Since the collection $\{p_1, p_2, \cdots, p_m\}$ is a basis for $\mathbb{C}_T[Z, W]$,

$$\sum_{k=1}^{n} c_k \ p_k \notin \mathcal{I}_T(Z, W)$$

and $Q \neq 0$. Let

$$P \ = \ \sum_{k=1}^{n} c_k \ p_k$$

and thus we have

$$P(\partial_Z, \partial_W) \ \Delta_T(Z, W) = Q.$$

Set

$$N \ = \ Q(\partial_Z, \partial_W) \ \Delta_T(Z, W) \ = \ 0$$

and hence $< P, N >= 0$. However,

$$\begin{aligned} < P, N > \ &= \ P(\partial_Z, \partial_W) \ Q(\partial_Z, \partial_W) \ \Delta_T(Z, W)|_{Z=W=0} \\ &= \ Q(\partial_Z, \partial_W) \ P(\partial_Z, \partial_W) \ \Delta_T(Z, W)|_{Z=W=0} \\ &= < Q, Q > \\ &> \ 0, \end{aligned}$$

a contradiction. Thus $Q(\partial_Z, \partial_W)\Delta_T(Z, W) \neq 0$, $Q \notin \mathcal{I}_T(Z, W)$ and the collection

$$\Big\{ p_1(\partial_Z, \partial_W)\Delta_T(Z, W), \cdots, p_m(\partial_Z, \partial_W)\Delta_T(Z, W) \Big\}$$

is linearly independent and hence a basis for $\mathbb{C}_T[Z, W]$. $\square$

It should be noted that Steinberg uses similar ideas in a different setting (see [18]).

Now,

$$\begin{aligned} & c_1 p_1 \ + \ \cdots \ + \ c_m p_m \in \mathcal{I}^-_{S,T}(X, Y, Z, W) \\ \Leftrightarrow \ & (c_1 p_1 \ + \ \cdots \ + \ c_m p_m)\,(\partial_X, \partial_Y, \partial_Z, \partial_W) \ \Delta_S(X, Y) \ \Delta_T(Z, W) = 0 \\ \Leftrightarrow \ & c_1 p_1(\partial_X, \partial_Y, \partial_Z, \partial_W) \ \Delta_S(X, Y) \ \Delta_T(Z, W) \ + \ \cdots \\ & \quad + \ c_m p_m(\partial_X, \partial_Y, \partial_Z, \partial_W) \ \Delta_{S_\psi}(X, Y) \ \Delta_T(Z, W) = 0. \end{aligned}$$

Thus we have shown

Lemma 5.4.

The collection $\{p_1, p_2, \cdots, p_m\}$ is linearly independent in $\mathbb{C}^-_{S,T}[X, Y, Z, W]$ if and only if the collection

$$\Big\{ p_1(\partial_X, \partial_Y, \partial_Z, \partial_W) \ \Delta_S(X, Y) \ \Delta_T(Z, W), \cdots, \\ p_m(\partial_X, \partial_Y, \partial_Z, \partial_W) \ \Delta_S(X, Y) \ \Delta_T(Z, W) \Big\}$$

is linearly independent in $\mathbb{C}[X, Y, Z, W]$.

This leads us to our next theorem.

Theorem 5.5.

The collection

$$\mathcal{BAQ}_{S,T} \;=\; \Big\{[U,V]^-_{per} : U \in \mathcal{CO}_S, V \in \mathcal{CO}_T, sh(U) = sh(V)\Big\}$$

is linearly independent in $\mathbb{C}^-_{S,T}[X,Y,Z,W]$.

Proof

To prove this theorem, we will show that the collection

$$\begin{aligned}&\mathcal{BAQD}_{S,T}\\ &= \Big\{[U,V]^-_{per}(\partial_X,\partial_Y,\partial_Z,\partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W) : [U,V]^-_{per} \in \mathcal{BAQ}_{S,T}\Big\} \quad (5.15)\end{aligned}$$

is linearly independent in $\mathbb{C}[X,Y,Z,W]$ and then apply Lemma 5.4. To prove the independence of the collection $\mathcal{BAQD}_{S,T}$, we will define a linear map

$$\eta : \mathbb{C}^-_{S,T}[X,Y,Z,W] \to \mathbb{C}[X,Y,Z,W]$$

and show that the matrix $(g_{i,h})$ defined by

$$\eta\Big([U_i,V_i]^-_{per}(\partial_X,\partial_Y,\partial_Z,\partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W)\Big) \;=\; \sum_h g_{i,h}\ [U_h,V_h^t]^-_{det}, \quad (5.16)$$

where

$$\Big\{[U_1,V_1]^-_{per},[U_2,V_2]^-_{per},\cdots,[U_m,V_m]^-_{per}\Big\}$$

is a listing of the elements of $\mathcal{BAQ}_{S,T}$ in increasing order with respect to $<_{str}$ (see equation (4.6)) and

$$\Big\{[U_1,V_1^t]^-_{det},[U_2,V_2^t]^-_{det},\cdots,[U_m,V_m^t]^-_{det}\Big\}$$

is a subset of $\mathcal{CSRS}_n$ (see equation (5.10) and Theorem 5.1) listed in increasing order with respect to $<_{stc}$ (see equation (4.4)), has rank $m = |\mathcal{BAQ}_{S,T}|$. (Note that the polynomials in the former collection are bipermanents and the polynomials in the latter collection are bideterminants.) Specifically, we will show that $(g_{i,h})$ is a nonsingular $m \times m$ matrix.

Suppose that $[U,V]^-_{per} \in \mathcal{BAQ}_{S,T}$. Now,

$$[U,V]^-_{per}(\partial_X,\partial_Y,\partial_Z,\partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W)$$

$$= \frac{1}{f_I}\Bigg[\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}$$

$$\Big[[I,U]_{per}(X,Y)\ [I,V]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big]\Bigg](\partial_X,\partial_Y,\partial_Z,\partial_W)\ \Delta_S(X,Y)\Delta_T(Z,W)$$

$$= \frac{1}{f_I}\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\Bigg[\Big[[I,U]_{per}(\partial_X,\partial_Y)\Delta_S(X,Y)\Big]$$

$$\Big[\big[[I,V]_{per}(\partial_Z,\partial_W)\Delta_T(Z,W)\big](\partial_Z,\partial_W)\Delta_T(Z,W)\Big]\Bigg] \tag{5.17}$$

where I is some injective tableau. Now, let

$$\mathcal{B}_{\psi_T} \ = \ \Big\{[O_1,V_1]_{per},[O_2,V_2]_{per},\cdots,[O_k,V_k]_{per}\Big\}$$

be a basis for $\mathbb{C}_T[Z,W]$ ordered (from smallest to largest) with respect to $<_{str}$ (see equation (4.6)). Therefore, the $k\times k$ matrix $(c_{i,h})$ defined by

$$\begin{aligned}&\Big[[O_i,V_i]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big](\partial_Z,\partial_W)\ \Delta_T(Z,W)\\ &= \sum_h c_{i,h}\ [O_h,V_h]_{per}(Z,W)\ +\ \sum_{\substack{D\in\mathcal{ST}_n\\ E\in\mathcal{CS}_n, E\notin\mathcal{CO}_T}} \xi_{D,E}\ [D,E]_{per},\end{aligned} \tag{5.18}$$

is unique and nonsingular since both $\mathcal{B}_{\psi_T}$ and

$$\Bigg\{\Big[[O_i,V_i]_{per}(\partial_Z,\partial_W)\Delta_T(Z,W)\Big](\partial_Z,\partial_W)\Delta_T(Z,W) : 1\le i\le m\Bigg\}$$

are bases for $\mathbb{C}_T[Z,W]$ (apply Lemma 5.3 twice). Note

$$\sum_{\substack{D\in\mathcal{ST}_n\\ E\in\mathcal{CS}_n, E\notin\mathcal{CO}_T}} \xi_{D,E}\ [D,E]_{per}\in\mathcal{I}_T(Z,W)$$

and $\xi_{D,E}\in\mathbb{C}\cup\mathcal{I}_T(Z,W)$.

Some additional properties of equation (5.18) need to be established. Particularly, let $q_{I,T}$ and r_{O_i,V_i} be monomials such that

$$\Delta_T(Z,W) = \sum_{\alpha\in S_n} sgn(\alpha)\ \alpha\ q_{I,T}$$

and

$$[O_i,V_i]_{per} \ = \ \sum_{\sigma\in R(O_i)} \sigma\ r_{O_i,V_i}.$$

We have (by equation (5.18))

$$
\begin{aligned}
&\Big[[O_i,V_i]_{per}(\partial_Z,\partial_W)\Delta_T(Z,W)\Big](\partial_Z,\partial_W)\ \Delta_T(Z,W)\\
&= \sum_{V_h\in\mathcal{CO}_T} c_{i,h}\ [O_h,V_h]_{per}(Z,W)\ +\sum_{\substack{D\in\mathcal{ST}_n\\ E\in\mathcal{CS}_n, E\notin\mathcal{CO}_T}} \xi_{D,E}\ [D,E]_{per}\\
&= \Bigg[\left(\sum_{\sigma\in R(O_i)} \sigma\ r_{O_i,V_i}\right)(\partial_Z,\partial_W)\\
&\quad \left(sgn(\sigma)\ \sigma \sum_{\alpha\in S_n} sgn(\alpha)\ \alpha\ (q_{I,T})\right)\Bigg](\partial_Z,\partial_W)\left(sgn(\sigma)\ \sigma \sum_{\beta\in S_n} sgn(\beta)\ \beta\ (q_{I,T})\right)\\
&= \sum_{\alpha\in S_n} sgn(\alpha)\sum_{\beta\in S_n} sgn(\beta) \sum_{\sigma\in R(O_i)} \sigma\left(\Big(\big(r_{O_i,V_i}(\partial_Z,\partial_W)\ \alpha\ (q_{I,T})\big)\ (\partial_Z,\partial_W)\Big)\ \beta\ (q_{I,T})\right).
\end{aligned}
$$

Note that the terms

$$
\sum_{\sigma\in R(O_i)} \sigma\left(\Big(\big(\ r_{O_i,V_i}(\partial_Z,\partial_W)\ \alpha\ (q_{I,T})\big)\ (\partial_Z,\partial_W)\Big)\ \beta\ (q_{I,T})\right)
$$

correspond to bipermanents of shape $sh(O_i)=sh(V_i)$ and thus the algorithm implied by Theorem 3.7 gives us that if $sh(V_h) <_L sh(V_i)$ then $c_{i,h}=0$. Thus $(c_{i,h})$ is a upper block triangular matrix and each block is nonsingular (since $(c_{i,h})$ is nonsingular). Note also that if $sh(E) < sh(V_i)$ then $\xi_{D,E}=0$ (see equation (3.10)).

Additionally, the straightening algorithm associated to Theorem 2.1 yields that

$$
\begin{aligned}
&\Big[[O_i,V_i]_{per}(\partial_Z,\partial_W)\Delta_T(Z,W)\Big](\partial_Z,\partial_W)\ \Delta_T(Z,W)\\
&= \sum_{V_h\in\mathcal{CO}_T} c_{i,h}\ [O_i,V_h]_{per}(Z,W)\ + \sum_{\substack{sh(O_m)>sh(O_i)\\ V_m\in\mathcal{CO}_T, O_m\in\mathcal{ST}_n}} c_{i,m}\ [O_m,V_m]_{per}(Z,W)\\
&+ \sum_{\substack{E\notin\mathcal{CO}_T\\ E\in\mathcal{CS}_n}} \xi_{O_i,E}\ [O_i,E]_{per}\ + \sum_{\substack{D\in\mathcal{ST}_n, sh(D)>_L sh(O_i)\\ F\notin\mathcal{CO}_T, F\in\mathcal{CS}_n}} \xi_{D,F}\ [D,F]_{per}. \qquad (5.19)
\end{aligned}
$$

Now, the coefficients (e_i) defined by

$$
\begin{aligned}
&[O_i,U_i]_{per}(\partial_X,\partial_Y)\ \Delta_S(X,Y)\\
&= e_i\ [O_i^t, C_{\rho^-_{U_i^t}}(U_i^t)]_{det}\ + \sum_{\substack{H_g>_{st} C_{\rho^-_{U_i^t}}(U_i^t)\\ H_g\in\mathcal{CS}_n}} d_{i,g}\ [O_i^t,H_g]_{det}\\
&+ \sum_{\substack{sh(P^t)>sh(U_i^t)\\ P\in\mathcal{ST}_n, Q\in\mathcal{CS}_n}} d_{P,Q}\ [P,Q]_{det} \qquad (5.20)
\end{aligned}
$$

are nonzero (see Lemma 3.3, Theorem 3.4, equation (3.10) and the algorithm associated to Theorem 2.1). Furthermore, the coefficients $(d_{i,g})$ and $(d_{P,Q})$ are unique.

Substituting (5.19) and (5.20) into (5.17), setting

$$\mathcal{V}_i \;=\; \{V_h \in \mathcal{CO}_T : sh(V_h) = sh(V_i)\},$$

$O_i = I$ and recalling equation (5.9) and Lemma 5.2 yields

$$\begin{aligned}
&[U_i, V_i]^-_{per}(\partial_X, \partial_Y, \partial_Z, \partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W) \\
&= \frac{1}{f_{O_i}} \sum_{\sigma \in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W} \Bigg[\Big[[O_i, U]_{per}(\partial_X, \partial_Y) \Delta_S(X,Y) \Big] \\
&\Big[[[O_i, V]_{per}(\partial_Z, \partial_W) \Delta_T(Z,W)](\partial_Z, \partial_W) \Delta_T(Z,W) \Big] \Bigg] \\
&= \frac{1}{f_{O_i}} \sum_{\sigma \in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W} \Bigg[\Big[e_i\ [O_i^t, C_{\rho^-_{U_i^t}}(U_i^t)]_{det}(X,Y) \\
&+ \sum_{\substack{H_g >_{stc} C_{\rho^-_{U_i^t}}(U_i^t) \\ H_g \in \mathcal{CS}_n}} d_{i,g}\ [O_i^t, H_g]_{det}(X,Y)\ +\ \sum_{\substack{sh(P) > sh(U_i^t) \\ P \in \mathcal{ST}_n, Q \in \mathcal{CS}_n}} d_{P,Q}\ [P,Q]_{det}(X,Y) \Big] \\
&\Big[\sum_{\substack{sh(O_i) = sh(V_h) \\ V_h \in \mathcal{CO}_T, O_m \in \mathcal{ST}_n}} c_{i,h}\ [O_i, V_h]_{per}(Z,W)\ +\ \sum_{\substack{sh(O_m) > sh(O_i) \\ V_m \in \mathcal{CO}_T}} c_{i,m}\ [O_m, V_m]_{per}(Z,W) \\
&+ \sum_{\substack{E \notin \mathcal{CO}_T \\ E \in \mathcal{CS}_n}} \xi_{O_i,E} [O_i, E]_{per}\ +\ \sum_{\substack{D \in \mathcal{ST}_n, sh(D) >_L sh(O_i) \\ F \notin \mathcal{CO}_T, F \in \mathcal{CS}_n}} \xi_{D,F} [D,F]_{per} \Big] \Bigg] \\
&= \frac{f_{O_i^t}}{f_{O_i}} \sum_{V_h \in \mathcal{V}_i} e_i\ c_{i,h}\ [C_{\rho^-_{U_i^t}}(U_i^t), V_h^t]^-_{det} \\
&+ \sum_{\substack{(M,N)) >_{stc} (C_{\rho^-_{U_i^t}}(U_i^t), V_h^t) \\ M \in \mathcal{CO}_S, N \in \mathcal{CO}_T}} d_{M,N}\ [M,N]^-_{det}\ +\ \sum_{\substack{Q \notin \mathcal{CO}_S\ or \\ E \notin \mathcal{CO}_T}} p_{Q,E}\ [Q, E^t]^-_{det},
\end{aligned} \tag{5.21}$$

where Q and E are column-strict tableaux. The coefficients e_i, $c_{i,h}$ and $d_{M,N}$ are uniquely defined in equation (5.21).

Recall that O_i is a standard tableau of shape $sh(U_i)$. Define

$$\eta : \mathbb{C}^-_{S,T}[X,Y,Z,W] \to \mathbb{C}[X,Y,Z,W]$$

by setting

$$\begin{aligned}
&\eta\Big([U_i,V_i]^-_{per}(\partial_X,\partial_Y,\partial_Z,\partial_W)\Big)\\
&= \frac{f_{O_i^t}}{f_{O_i}} \sum_{V_h\in\mathcal{V}_i} e_i\ c_{i,h}\ [C_{\rho^-_{U_i^t}}(U_i^t),V_h^t]^-_{det}\ +\sum_{\substack{(M,N^t))>_{stc}(C_{\rho^-_{U_i^t}}(U_i^t),V_h^t)\\ M\in\mathcal{CO}_S,N\in\mathcal{CO}_T}} d_{M,N}\ [M,N^t]^-_{det}\\
&= \sum_{h=1}^m g_{i,h}\ [U_h,V_h^t]^-_{det},
\end{aligned}$$

where the e_i, $c_{i,h}$ and $d_{M,N}$ are given in equation (5.21). The matrix $(g_{i,h})$ is a upper block triangular matrix where the blocks along the diagonal have the identity

$$g_{i,h} = e_i\ c_{i,h}\ \frac{f_{O_i^t}}{f_{O_i}}.$$

Recall that the matrix $(c_{i,h})$ is nonsingular. Thus the rank of $(g_{i,h})$ is $m = |\mathcal{BAQ}_{S,T}|$ and $\mathcal{BAQ}_{S,T}$ is linearly independent. $\square$

We turn our attention to showing that the collection $\mathcal{BAQ}_{S,T}$ spans $\mathbb{C}^-_{S,T}[X,Y,Z,W]$. We will first show that a collection $\mathcal{CSCO}_{S,T}$ spans $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ and then we will show that $\mathcal{BAQ}_{S,T}$ spans $\mathcal{CSCO}_{S,T}$. Notice that we have the following equalities concerning $[U,V]^-_{per}$.

$$\begin{aligned}
&[U,V]^-_{per}\\
&= \frac{1}{f_I}\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[[I,U]_{per}(X,Y)\ [I,V]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big]\\
&= \frac{1}{f_I}\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Bigg[\sum_{\alpha\in R(I)} \alpha_{X,Y}\ x_1^{u_{1,1}}\cdots x_n^{u_{n,1}} y_1^{u_{1,2}}\cdots y_n^{u_{n,2}}\\
&\quad \sum_{\beta\in R(I)}\Big[\beta_{Z,W}\ \partial_{z_1}^{v_{1,1}}\cdots\partial_{z_n}^{v_{n,1}}\partial_{w_1}^{v_{1,2}}\cdots\partial_{w_n}^{v_{n,2}}\Big]\ \Delta_T(Z,W)\Bigg]\\
&= \frac{1}{f_I}\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Bigg[\sum_{\alpha\in R(I)} \alpha_{X,Y}\ x_1^{u_{1,1}}\cdots x_n^{u_{n,1}} y_1^{u_{1,2}}\cdots y_n^{u_{n,2}}\\
&\quad \sum_{\alpha\beta\in R(I)}\Big[\alpha_{Z,W}\ \beta_{Z,W}\ \partial_{z_1}^{v_{1,1}}\cdots\partial_{z_n}^{v_{n,1}}\partial_{w_1}^{v_{1,2}}\cdots\partial_{w_n}^{v_{n,2}})\Big]\quad [sgn(\alpha)\ \alpha_{Z,W}\ \Delta_T(Z,W)]\Bigg]\\
&= \frac{1}{f_I}\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Bigg[\sum_{\alpha\in R(I)} sgn(\alpha)\ \alpha_{X,Y,Z,W}\Big[\ x_1^{u_{1,1}}\cdots x_n^{u_{n,1}} y_1^{u_{1,2}}\cdots y_n^{u_{n,2}}
\end{aligned}$$

$$\sum_{\beta\in R(I)} \beta_{Z,W}\ \partial_{z_1}^{v_{1,1}}\cdots\partial_{z_n}^{v_{n,1}}\partial_{w_1}^{v_{1,2}}\cdots\partial_{w_n}^{v_{n,2}}\ \Delta_T(Z,W)\Big]\Big]$$

$$= \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[\ x_1^{u_{1,1}}\cdots x_n^{u_{n,1}}y_1^{u_{1,2}}\cdots y_n^{u_{n,2}}$$

$$\sum_{\beta\in R(I)} \beta_{Z,W}\ \partial_{z_1}^{v_{1,1}}\cdots\partial_{z_n}^{v_{n,1}}\partial_{w_1}^{v_{1,2}}\cdots\partial_{w_n}^{v_{n,2}}\ \Delta_T(Z,W)\Big]. \tag{5.22}$$

Similarly, we have that

$$[U,V]^-_{per}\ =\ \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[\ \sum_{\alpha\in R(I)} \alpha_{X,Y}\ x_1^{u_{1,1}}\cdots x_n^{u_{n,1}}y_1^{u_{1,2}}\cdots y_n^{u_{n,2}}$$

$$\partial_{z_1}^{v_{1,1}}\cdots\partial_{z_n}^{v_{n,1}}\partial_{w_1}^{v_{1,2}}\cdots\partial_{w_n}^{v_{n,2}}\ \Delta_T(Z,W)\Big]. \tag{5.23}$$

With these equations, we can begin the process of showing that the collection $\mathcal{BAQ}_{S,T}$ spans $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$.

Lemma 5.6.

If S and T are dense then the collection

$$\mathcal{CSCO}_{S,T}\ =\ \{[U,V]^-_{per} : U\in\mathcal{CS}_S,\ V\in\mathcal{CO}_T,\ sh(U)=sh(V)\}$$

spans $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$.

Proof

Given any monomial $q(Z,W)\in\mathbb{C}[Z,W]$, Lemma 5.3 and Theorem 3.8 imply that

$$q(Z,W)\ =\ \sum_k c_k\ [I_k,V_k]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)$$

(identically), where $V_k\in\mathcal{CO}_T$, $I_k\in\mathcal{ST}_n$, $sh(V_k)=sh(I_k)$ and $c_k\in\mathcal{I}_T(Z,W)\cup\mathbb{C}$. Thus for any monomial

$$p(X,Y)\ =\ x_1^{q_1}x_2^{q_2}\cdots x_n^{q_n}y_1^{r_1}y_2^{r_2}\cdots y_n^{r_n},$$

we have

$$\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\big[p(X,Y)\ q(Z,W)\big]$$

$$= \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\big[p(X,Y)\ \sum_k c_k\ [I_k,V_k]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\big]$$

$$= \sum_k c_k \sum_{\sigma \in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\Big[x_1^{q_1} x_2^{q_2} \cdots x_n^{q_n} y_1^{r_1} \cdots y_n^{r_n}\ [I_k, V_k]_{per}(\partial_Z, \partial_W)\ \Delta_T(Z,W)\Big]$$

$$= \sum_k c_k\ [U_k, V_k]^-_{per} \tag{5.24}$$

by equation (5.22). Note that in equation (5.24) that it is possible for some of the U_k to not be column-strict. Using the algorithm associated to Theorem 2.1 and equations (5.22) and (5.23), we know that

$$\sum_k c_k\ [U_k, V_k]^-_{per}\ =\ \sum_i d_i\ [P_i, Q_i]^-_{per}$$

where all of the P_i and Q_i are column-strict. Note that if $r(Z,W) \in \mathcal{I}_T(Z,W)$ then we must have $r(Z,W)\ [P,Q]^-_{per} \in \mathcal{I}^-_{S,T}(X,Y,Z,W)$. Additionally, note that if either $P \notin \mathcal{CS}_S$ or $Q \notin \mathcal{CS}_T$ then $[P,Q]^-_{per} \in \mathcal{I}^-_{S,T}(X,Y,Z,W)$. Without loss of generality, we may assume that $P_i \in \mathcal{CS}_S$ and $Q_i \in \mathcal{CS}_T$. Thus the collection

$$\mathcal{CSCS}_{S,T}\ =\ \Big\{[P,Q]^-_{per} : P \in \mathcal{CS}_S,\ Q \in \mathcal{CS}_{T_\psi},\ sh(P) = sh(Q)\ \Big\}$$

spans $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$.

Let

$$S\ =\ \Big\{s_1, s_2, \cdots, s_n\Big\}$$

and

$$T\ =\ \Big\{t_1, t_2, \cdots, t_n\Big\}$$

(recall that S and T are the sets from which $\Delta_S(X,Y)$ and $\Delta_T(Z,W)$ are constructed listed in increasing order with respect to $<_{\mathcal{A}}$). Let $s' = min\{s_1, (0,1)\}$ and $t' = min\{t_1, (0,1)\}$. Let U and V be the tableaux of shape n with

$$U\ =\ s' \quad s' \quad \cdots \quad s'.$$

and

$$V\ =\ t' \quad t' \quad \cdots \quad t'.$$

For any pair of tableaux (M,N) such that $(M,N) >_{str} (U,V)$ (see equation (3.14) and equation (4.6)), we have that $[M,N]^-_{per} \in \mathcal{I}^-_{S,T}(X,Y,Z,W)$ since

$$\begin{aligned}
&[M,N]^-_{per}(\partial_X, \partial_Y, \partial_Z, \partial_W)\ \Delta_S(X,Y)\ \Delta_T(Z,W) \\
&= \frac{1}{f_I} \sum_{\sigma \in S_n} sgn(\sigma) \sigma_{X,Y,Z,W}\bigg[\Big[[I,M]_{per}(\partial_X, \partial_Y)\Delta_S(X,Y)\Big] \\
&\quad \Big[\big[[I,N]_{per}(\partial_Z, \partial_W)\Delta_T(Z,W)\big](\partial_Z, \partial_W)\Delta_T(Z,W)\Big]\bigg] \\
&= 0.
\end{aligned}$$

Now suppose that (P,Q) is the largest pair of column-strict tableaux in $\mathcal{CSCS}_{S,T}$ with respect to $<_{str}$ (see equation (4.6)) such that $[P,Q]^-_{per}$ is not in the linear span of $\mathcal{CSCO}_{S,T}$ (with coefficients from $\mathbb{C}$) and $[P,Q]^-_{per} \notin \mathcal{I}^-_{S,T}(X,Y,Z,W)$. Particularly, we must have that $Q \notin \mathcal{CO}_T$. Using Theorem 3.7 we have

$$\begin{aligned} &[I,Q]_{per}(\partial_Z,\partial_W) \\ &= m_{\gamma_2}(\partial_Z,\partial_W)\ [I,C_2]_{per}(\partial_Z,\partial_W)\ -\sum_{\substack{M>_{str}Q\\ sh(M)=sh(Q)}} c_M\ [I,M]_{per}(\partial_Z,\partial_W) \\ &\quad -\sum_{sh(M')>_L sh(Q)} c_{I',M'}\ [I',M']_{per}(\partial_Z,\partial_W) \end{aligned} \tag{5.25}$$

identically, where $C_2 = C_{\rho_Q}(st(Q))$ and $\gamma_2 = \gamma_{\rho_Q,Q}$. Now,

$$\begin{aligned} &[P,Q]^-_{per} \\ &= \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[[I,P]_{per}(X,Y)\ [I,Q]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big] \\ &= \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[[I,P]_{per}(X,Y)\ m_{\gamma_2}(\partial_Z,\partial_W)\ [I,C_2]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big] \\ &- \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[[I,P]_{per}(X,Y) \sum_{\substack{M>_{str}Q\\ sh(M)=sh(Q)}} c_M\ [I,M]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big] \\ &- \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[[I,P]_{per}(X,Y) \sum_{sh(M')>_L sh(Q)} c_{I',M'}\ [I',M']_{per}(\partial_Z,\partial_W)\Delta_T(Z,W)\Big]. \end{aligned}$$

Notice that

$$\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[[I,P]_{per}(X,Y)\ m_{\gamma_2}(\partial_Z,\partial_W)\ [I,C_2]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big]\ =\ 0$$

since $m_{\gamma_2}(\partial_Z,\partial_W)\ \Delta_T(Z,W) = 0$ and thus

$$\begin{aligned} &\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[[I,P]_{per}(X,Y)\ m_{\gamma_2}(\partial_Z,\partial_W)\ [I,C_2]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big] \\ &\in \mathcal{I}^-_{S,T}(X,Y,Z,W). \end{aligned}$$

Both

$$\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[[I,P]_{per}(X,Y) \sum_{\substack{M>_{str}Q\\ sh(M)=sh(Q)}} c_Q\ [I,M]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big]$$

and

$$\sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\Big[[I,P]_{per}(X,Y) \sum_{sh(M')>_L sh(Q)} c_{I',M'}\ [I',M']_{per}(\partial_Z,\partial_W)\Delta_T(Z,W)\Big]$$

lead to sums of bipermanents that are larger than (P,Q) with respect to $>_{str}$. The former is true since it is summed over M such that $M >_{str} Q$. The latter uses equation (5.22) and the fact that

$$\begin{aligned} & [I,P]_{per}(X,Y)\ [I',M']_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W) \\ = & \sum_i c_i\ p(x,y)\ \ [I',M']_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W). \end{aligned}$$

This implies that $[P,Q]^-_{per}$ is in the span of $\mathcal{CSCO}_{S,T}$ as well as the fact that $\mathcal{CSCO}_{S,T}$ spans $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$. $\square$

Let $[U,V]^-_{per} \notin \mathcal{BAQ}_{S,T}$ be the largest bipermanent with respect to $>_{str}$ (see equation (4.6)) with $U \in \mathcal{CS}_S$, and $V \in \mathcal{CO}_T$ such that $[U,V]^-_{per}$ is not in the linear span of $\mathcal{BAQ}_{S,T}$ with coefficients from $\mathbb{C}$. Now if $U \notin \mathcal{CO}_S$, by Theorem 3.7 we can find a

$$\gamma_1 \ = \ \gamma_{\rho_U,U}(X,Y) \in \Gamma_{S_\psi}$$

such that

$$\begin{aligned} & [I,U]_{per} \\ = & m_{\gamma_1}(X,Y)\ [I,C]_{per} - \sum_{\substack{M>_{str}U \\ sh(M)=sh(U)}} c_M\ [I,M]_{per} - \sum_{sh(M')>_L sh(U)} c_{M'}\ [I',M']_{per} \end{aligned} \tag{5.26}$$

where $C = C_{\rho_U}(st(U))$ and $c_M, c_{M'} \in \mathbb{C}$. Therefore,

$$\begin{aligned} & [U,V]^-_{per} \\ & = \frac{1}{f_I} \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[[I,U]_{per}(X,Y)\ [I,V]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big] \\ & = \frac{1}{f_I} \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W} \\ & \Big[m_{\gamma_1}(X,Y)\ [I,C]_{per}(X,Y)\ [I,V]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big] \\ & - \frac{1}{f_I} \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W} \\ & \Big[\sum_{\substack{M>_{str}U \\ sh(M)=sh(U)}} c_M\ [I,M]_{per}(X,Y)\ [I,V]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big] \\ & - \frac{1}{f_I} \sum_{\sigma\in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W} \\ & \Big[\sum_{sh(M')>_L sh(U)} c_{I',M'}\ [I',M']_{per}(X,Y)\ [I,V]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big]. \end{aligned}$$

Once again,

$$\sum_{\sigma \in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\Big[m_{\gamma_1}(X,Y)\ [I,C]_{per}(X,Y)\ [I,V]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big] \in \mathcal{I}^-_{S,T}(X,Y,Z,W).$$

The terms

$$\sum_{\sigma \in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W}\ \Big[\sum_{\substack{M >_{str} U \\ sh(M)=sh(U)}} c_M\ [I,M]_{per}(X,Y)\ [I,V]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big]$$

and

$$\sum_{\sigma \in S_n} sgn(\sigma)\ \sigma_{X,Y,Z,W} \Big[\sum_{sh(M') >_L sh(U)} c_{I',M'}\ [I',M']_{per}(X,Y)\ [I,V]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)\Big]$$

correspond to sums of bipermanents that are larger than $[U,V]^-_{per}$ with respect to $>_{str}$ by equation (5.23). Therefore, $[U,V]^-_{per}$ can be written as a linear combinations of elements of

$$\mathcal{BAQ}_{S,T}\ =\ \Big\{[U,V]^-_{per}:\ U \in \mathcal{CO}_S,\ V \in \mathcal{CO}_T \text{ and } sh(U)=sh(V)\ \Big\}$$

with coefficients in $\mathbb{C}$. Thus we have the following theorem.

Theorem 5.7.

If S and T are dense then the collection $\mathcal{BAQ}_{S,T}$ spans $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$. Hence, $\mathcal{BAQ}_{S,T}$ is a basis for $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ with coefficients from $\mathbb{C}$.

Now, the bidegree of $[I,C_\rho(U)]_{per}(\partial_Z,\partial_W)\ \Delta_T(Z,W)$ (with I standard) (as a polynomial in $\mathbb{C}[Z,W]$) equals the bidegree of $[I^t, C_{\rho^-}(U^t)]_{per}$ (recall Theorem 3.4). Recall that $d_{\psi_S}(q,t)$ is a polynomial defined in equation (3.21) and that Υ_{ψ_S} and Υ_{ψ_T} are collections of sequences defined in equation (3.4). Thus, Theorem 5.7 immediately yields the following.

Corollary 5.8.

If S and T are dense then the Hilbert series $\mathcal{H}(\mathbb{C}^-_{S,T}[X,Y,Z,W])$ of $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ is given by

$$\begin{aligned}
&\mathcal{H}(\mathbb{C}^-_{S,T}[X,Y,Z,W]) \\
&= \sum_{\lambda\vdash n}\sum_{\rho\in\Upsilon_{\psi_S}}\sum_{\rho'\in\Upsilon_{\psi_T}}\sum_{(M,N)\in\mathcal{ST}_\lambda\times\mathcal{ST}_\lambda} t^{|C_{\rho,1}(M)|}q^{|C_{\rho,2}(M)|}r^{|C_{\rho',1}(N^t)|}s^{|C_{\rho',2}(N^t)|} \\
&= d_{\psi_S}(q,t)\ d_{\psi_T}(s,r)\sum_{\lambda\vdash n}\sum_{\substack{(M,N)\in\mathcal{ST}_\lambda\times\mathcal{ST}_\lambda \\ \rho=(0^n),\rho'=(0^n)}} t^{|C_{\rho,1}(M)|}q^{|C_{\rho,2}(M)|}r^{|C_{\rho',1}(N^t)|}s^{|C_{\rho',2}(N^t)|}, \quad (5.27)
\end{aligned}$$

where $\mathcal{ST}_\lambda$ denotes the collection of standard tableaux of shape λ, $C_\rho(M)$ denotes a cocharge tableau corresponding to a tableau M and a sequence ρ, $|C_{\rho,1}(M)|$ and $|C_{\rho,2}(M)|$ denote the sums of the first and second coordinates of the cocharge tableau $C_\rho(M)$, respectively, and N^t denotes the transpose of the tableau N.

Example

With $S = T$ given in (3.19) we have

$$\begin{aligned}
&\mathcal{H}(\mathbb{C}^-_{S,T}[X,Y,Z,W])\\
&= (1+t)\left(rs+r^2s\right) + \left(t+q+tq+t^2\right)\left(r+s+rs+r^2\right) + \left(tq+t^2q\right)(1+r)\\
&= (1+t)\ (1+r)\ [rs + (t+q)\ (r+s) + tq]\,.
\end{aligned}$$

Note that the dimensions (as vector spaces) of $\mathbb{C}^+_{S,T}[X,Y,Z,W]$ and $\mathbb{C}^-_{S,T}[X,Y,Z,W]$ are equal. Both of these dimensions are equal to $d_{\psi_S}\ d_{\psi_T}\ n!$ (see equation (3.20)).

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