Generalizing the Ramsey Problem through Diameter

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Abstract

Given a graph G and positive integers d, k, let $f_d^k(G)$ be the maximum t such that every k-coloring of $E(G)$ yields a monochromatic subgraph with diameter at most d on at least t vertices. Determining $f_1^k(K_n)$ is equivalent to determining classical Ramsey numbers for multicolorings. Our results include

- determining $f_d^k(K_{a,b})$ within 1 for all d, k, a, b
- for $d \geq 4$, $f_d^3(K_n) = \lceil n/2 \rceil + 1$ or $\lceil n/2 \rceil$ depending on whether $n \equiv 2 \ (mod\ 4)$ or not
- $f_3^k(K_n) > \frac{n}{k-1+1/k}$

The third result is almost sharp, since a construction due to Calkin implies that $f_3^k(K_n) \leq \frac{n}{k-1} + k - 1$ when $k-1$ is a prime power. The asymptotics for $f_d^k(K_n)$ remain open when $d = k = 3$ and when $d \geq 3, k \geq 4$ are fixed.

1 Introduction

The Ramsey problem for multicolorings asks for the minimum n such that every k-coloring of the edges of K_n yields a monochromatic K_p. This problem has been generalized in many ways (see, e.g., [2, 6, 7, 9, 12, 13, 14]). We begin with the following generalization due to Paul Erdős [8] (see also [11]):

Problem 1 *What is the maximum t with the property that every k-coloring of $E(K_n)$ yields a monochromatic subgraph of diameter at most two on at least t vertices?*

A related problem is investigated in [14], where the existence of the Ramsey number is proven when the host graph is not necessarily a clique. Call a subgraph of diameter at most d a *d-subgraph.*

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Theorem 2 (Tonoyan [14]) *Let $D, k \geq 1$, $d \geq D$, $n \geq 2$. Then there is a smallest integer $t = R_{D,k}(n, d)$ such that every graph G with diameter D on at least t vertices has the following property: every k-coloring of $E(G)$ yields a monochromatic d-subgraph on at least n vertices.*

We study a problem closely related to Tonoyan's result that also generalizes Problem 1 to larger diameter.

Definition 3 *Let G be a graph and d, k be positive integers. Then $f_d^k(G)$ is the maximum t with the property that every k-coloring of $E(G)$ yields a monochromatic d-subgraph on at least t vertices.*

The asymptotics for $f_d^k(G)$ when $G = K_n$ and $d = 2$ (Erdős' problem) were determined in [10].

Theorem 4 (Fowler [10]) *$f_2^2(K_n) = \lceil 3n/4 \rceil$ and if $k \geq 3$, then $f_2^k(K_n) \sim n/k$ as $n \to \infty$.*

In this paper, we study $f_d^k(G)$ when G is a complete graph or a complete bipartite graph. In the latter case, we determine its value within 1.

Theorem 5 (Section 3) *Let $k, a, b \geq 2$. Then $f_2^k(K_{a,b}) = 1 + \lceil \max\{a, b\}/k \rceil$, and for $d \geq 3$,*

$$\left\lceil \frac{1}{ab} \left(\left\lceil \frac{ab^2}{k} \right\rceil + \left\lceil \frac{a^2 b}{k} \right\rceil \right) \right\rceil \leq f_d^k(K_{a,b}) \leq \left\lceil \frac{a}{k} \right\rceil + \left\lceil \frac{b}{k} \right\rceil .$$

Determining $f_d^k(K_n)$ (for $d \geq 3$) seems more difficult. We succeed in doing this only when $d > k = 3$.

Theorem 6 (Section 4) *Let $d \geq 4$. Then*

$$f_d^3(K_n) = \begin{cases} n/2 + 1 & n \equiv 2 \ (mod\ 4) \\ \lceil n/2 \rceil & otherwise \end{cases}$$

When $d = 3$ we are able to obtain bounds for $f_d^k(K_n)$.

Theorem 7 (Section 5) *Let $k \geq 2$. Then $f_3^k(K_n) > n/(k - 1 + 1/k)$.*

In section 5 we also describe an unpublished construction of Calkin which implies that $f_3^k(K_n) \leq n/(k-1) + k - 1$ when $k - 1$ is a prime power. This shows that the bound in Theorem 7 is not far off from being best possible. In section 6 we summarize the known results for $f_d^k(K_n)$. Our main tool for Theorems 5 and 7 is developed in Section 2.

2 The Main Lemma

In this section we prove a statement about 3-subgraphs in colorings of bipartite graphs. Although this is later used in the proofs of Theorems 5 and 7, we feel it is of independent interest.

Suppose that G is a graph and $c : E(G) \to [k]$ is a k-coloring of its edges. For each $i \in [k]$ and $x \in V(G)$, let $N_i(x) = \{y \in N(x) : c(xy) = i\}$ and $d_i(x) = |N_i(x)|$. For $uv \in E(G)$, let the *weight* of uv be

$$w(uv) = d_{c(uv)}(u) + d_{c(uv)}(v).$$

Lemma 8 *Let G be a subgraph of $K_{a,b}$ with e edges and $d \geq 3$. Then*

$$f_d^k(G) \geq \left\lceil \frac{1}{e}\left(\left\lceil \frac{e^2}{ak}\right\rceil + \left\lceil \frac{e^2}{bk}\right\rceil\right)\right\rceil \geq \left\lceil \frac{e}{ak}\right\rceil + \left\lceil \frac{e}{bk}\right\rceil - 1.$$

Proof: Suppose that $K_{a,b}$ has bipartition A, B with $X = V(G) \cap A$, and $Y = V(G) \cap B$. Let $c : E(G) \to [k]$ be a k-coloring. Observe that an edge with weight w gives rise to a 3-subgraph on w vertices. We prove the stronger statement that G has an edge with weight at least

$$\left\lceil \frac{1}{e}\left(\left\lceil \frac{e^2}{ak}\right\rceil + \left\lceil \frac{e^2}{bk}\right\rceil\right)\right\rceil.$$

We obtain a lower bound on the sum of all the edge-weights.

$$\begin{aligned}\sum_{uv\in E(G)} w(uv) &= \sum_{x\in X}\sum_{i\in[k]}\sum_{y\in N_i(x)} w(xy)\\ &= \sum_{x\in X}\sum_{i\in[k]}\sum_{y\in N_i(x)} d_i(x)+d_i(y)\\ &= \sum_{x\in X}\sum_{i\in[k]} [d_i(x)]^2 + \sum_{y\in Y}\sum_{i\in[k]} [d_i(y)]^2\\ &\geq \left\lceil \frac{e^2}{ak}\right\rceil + \left\lceil \frac{e^2}{bk}\right\rceil, \qquad (2)\end{aligned}$$

where (1) follows from the Cauchy-Schwarz inequality applied to each double sum. Since there is an edge with weight at least as large as the average, we have

$$f_d^k(G) \geq \left\lceil \frac{1}{e}\left(\left\lceil \frac{e^2}{ak}\right\rceil + \left\lceil \frac{e^2}{bk}\right\rceil\right)\right\rceil \geq \left\lceil \frac{e}{ak} + \frac{e}{bk}\right\rceil \geq \left\lceil \frac{e}{ak}\right\rceil + \left\lceil \frac{e}{bk}\right\rceil - 1. \qquad \square$$

A slight variation of the proof of Lemma 8 also yields the following more general result.

Lemma 9 *Suppose that G is a graph with n vertices and e edges. Let $c : E(G) \to [k]$ be a k-coloring of $E(G)$ such that every color class is triangle-free. Then G contains a monochromatic 3-subgraph on at least $4e/(nk)$ vertices.*

3 Bipartite Graphs

Proof of Theorem 5: Let $c : E(K_{a,b}) \to [k]$ be a k-coloring. The lower bound for the case $d = 2$ is obtained by considering a pair (v, i) for which $d_i(v)$ is maximized. The set $v \cup N_i(v)$ induces a monochromatic 2-subgraph. The lower bound when $d \geq 3$ follows from Lemma 8. For the upper bounds we provide the following constructions.

Let $K_{a,b}$ have bipartition $X = \{x_1, \ldots, x_a\}$ and $Y = \{y_1, \ldots, y_b\}$, and assume that $a \leq b$. Partition X into k sets $X_1, \ldots, X_k$, each of size $\lceil a/k \rceil$ or $\lfloor a/k \rfloor$, and partition Y into k sets $Y_1, \ldots, Y_k$, each of size $\lceil b/k \rceil$ or $\lfloor b/k \rfloor$. Furthermore, let both these partitions be "consecutive" in the sense that $X_1 = \{x_1, x_2, \ldots, x_r\}$, $X_2 = \{x_{r+1}, x_{r+2} \ldots, x_{r+s}\}$, etc. Finally, for each nonnegative integer t, let $Y_i + t = \{y_{l+t} : y_l \in Y_i\}$, where subscripts are taken modulo b.

When $d = 2$ and $j \in [k]$, let the j^{th} color class be all edges between x_i and $Y_j + (i - 1)$ for each $i \in [n]$. Because $K_{a,b}$ is bipartite, the distance between a pair of nonadjacent vertices $x \in X$ and $y \in Y$ in the subgraph formed by the edges in color j is at least three. Thus a 2-subgraph of $K_{a,b}$ is a complete bipartite graph.

For $1 \leq i \leq k$, let α_i be the smallest subscript of an element in Y_i. Thus $\alpha_{i+1} - \alpha_i = |Y_i|$ since $Y_i = \{y_{\alpha_i}, y_{\alpha_i+1}, \ldots, y_{\alpha_{i+1}-1}\}$. Fix $l \in [k]$ and let H be a largest monochromatic complete bipartite graph in color l. Let $A = V(H) \cap X$ and $B = V(H) \cap Y$. Let r be the smallest index such that $x_r \in A$ and let s be the largest index such that $x_s \in A$. As $N_H(x_r) = \{y_{\alpha_l+r-1}, y_{\alpha_l+r}, \ldots, y_{\alpha_{l+1}+r-2}\}$ and $N_H(x_s) = \{y_{\alpha_l+s-1}, y_{\alpha_l+s}, \ldots, y_{\alpha_{l+1}+s-2}\}$, we have $N_H(x_r) \cap N_H(x_s) = \{y_{\alpha_l+s-1}, \ldots, y_{\alpha_{l+1}+r-2}\}$, where subscripts are taken modulo b. Consequently,

$$\begin{aligned}
|V(H)| &\leq |\{x_r, \ldots, x_s\}| + |\{y_{\alpha_l+s-1}, \ldots, y_{\alpha_{l+1}+r-2}\}| \\
&= (s - r + 1) + (\alpha_{l+1} + r - 2 - (\alpha_l + s - 1) + 1) \\
&= 1 + \alpha_{l+1} - \alpha_l \\
&= 1 + |Y_l| \\
&\leq 1 + \lceil b/k \rceil.
\end{aligned}$$

When $d > 2$ and $j \in [k]$, let the j^{th} color class consist of the edges between X_i and Y_{i-1+j} (subscripts taken modulo k) for each $i \in [k]$. The maximum size of a connected monochromatic subgraph is $\max_{i,i'}\{|X_i| + |Y_{i'}|\} = \lceil a/k \rceil + \lceil b/k \rceil$. □

Recall that the bipartite Ramsey number for multicolorings $b_k(H)$ is the minimum n such that every k-coloring of $E(K_{n,n})$ yields a monochromatic copy of H. Analogous to the case with the classical Ramsey numbers, determining these numbers is hard. Chvátal [5], and Bieneke-Schwenk [3] proved that when $H = K_{p,q}$, this number is at most $(q - 1)k^p + O(k^{p-1})$, and some exact results for the case $H = K_{2,q}$ were also obtained in [3].

It is worth noting that the function $f_2^k(K_{a,b})$ seems fundamentally different (and much easier to determine) from the numbers $b_k(K_{p,q})$, since we do not require our complete bipartite subgraphs to have a specified number of vertices in each partite set.

4 Diameter at least four

In this section we consider $f_d^k(K_n)$. Since a 1-subgraph is a clique, the problem is hopeless if $d = 1$. The case $d = 2$ was settled in [10], where nontrivial constructions were obtained that matched the trivial lower bounds asymptotically. We investigate the problem for larger d. We include the following slight strengthening of a well-known (and easy) fact for completeness (see problem 2.1.34 of [15]).

Proposition 10 *Every 2-coloring of $E(K_n)$ yields a monochromatic spanning 2-subgraph or a monochromatic spanning 3-subgraph in each color. Thus in particular, $f_d^2(K_n) = n$ for $d \geq 3$.*

Proof: Suppose that the coloring uses red and blue. We may assume that both the red subgraph and the blue subgraph have diameter at least three. Thus there exist vertices r_1, r_2 (respectively, b_1, b_2) with the shortest red r_1, r_2-path (respectively, blue b_1, b_2-path) having length at least three. We will show that the blue subgraph has diameter at most three.

Let u, v be arbitrary vertices in K_n. If $\{u, v\} \cap \{r_1, r_2\} \neq \emptyset$, then the fact that there is no red r_1, r_2-path of length at most two guarantees a blue u, v-path of length at most two. We may therefore assume that $\{u, v\} \cap \{r_1, r_2\} = \emptyset$.

At least one of ur_1, ur_2 is blue, and at least one of vr_1, vr_2 is blue. Together with the blue edge r_1r_2, these three blue edges contain a u, v-path of length at most three. Since u and v are arbitrary, the blue subgraph has diameter at most three. Similarly, the vertices b_1, b_2 can be used to show that the red subgraph also has diameter at most three. □

We now turn to the case when $d, k \geq 3$. The following k-coloring of K_n has the property that the largest connected monochromatic subgraph has order $2\lceil n/(k+1) \rceil$ when k is odd and $2\lceil n/k \rceil$ when k is even. As we will see below, this is sharp when $k = 3$, but not for any other value of k when $k - 1$ is a prime power [4].

This construction was suggested independently by Erdős. It uses the well-known fact that the edge-chromatic number of K_n is n if n is odd and $n - 1$ if n is even.

Construction 11 When k is odd, partition $V(K_n)$ into $k + 1$ sets $V_1, \ldots, V_{k+1}$, each of size $\lfloor n/(k+1) \rfloor$ or $\lceil n/(k+1) \rceil$. Contract each V_i to a single vertex v_i, and the edges between any pair V_i, V_j to a single edge v_iv_j to obtain K_{k+1}. Let $c : E(K_{k+1}) \to [k]$ be a proper edge-coloring. Expand K_{k+1} back to the original K_n, coloring every edge between V_i and V_j with $c(v_iv_j)$. Color all edges within each V_i with color 1.

Because c is a proper edge-coloring, a monochromatic connected graph G can have $V(G) \cap V_i \neq \emptyset$ for at most two distinct indices $i \in [k]$. Thus $|V(G)| \leq 2\lceil n/(k+1) \rceil$. In the case $n \equiv 1 \pmod{k}$, only one V_i has size $\lceil n/(k+1) \rceil$ and all the rest have size $\lfloor n/(k+1) \rfloor$, so $|V(G)| \leq \lceil n/(k+1) \rceil + \lfloor n/(k+1) \rfloor$.

When k is even, partition $V(K_n)$ into k sets, color as described above with $k-1$ colors and change the color on any single edge to the kth color. □

Proof of Theorem 6: For the upper bounds we use Construction 11. When $n \equiv 0, 3 \pmod 4$, $2\lceil n/4 \rceil = \lceil n/2 \rceil$. When $n \equiv 2 \pmod 4$, $2\lceil n/4 \rceil = n/2 + 1$. When $n \equiv 1 \pmod 4$, the construction gives the improvement $\lceil n/4 \rceil + \lfloor n/4 \rfloor$ which again equals the claimed bound $\lceil n/2 \rceil$.

For the lower bound, consider a 3-coloring $c : E(K_n) \to [3]$. Pick any vertex v, and assume without loss of generality that $\max\{d_i(v)\} = d_1(v)$. Let $N = v \cup N_1(v)$ and let $N' = (\cup_{w \in N} N_1(w)) - N$. The subgraph in color 1 induced by $N \cup N'$ is a 4-subgraph, thus we are done unless $|N| + |N'| \leq n/2$, which we may henceforth assume.

Let $M = V(K_n) - N - N'$. Observe that color 1 is forbidden on edges between N and M. Since $M \subseteq N_2(v) \cup N_3(v)$, we may assume without loss of generality that the set $S = N_2(v) \cap M$ satisfies $|S| \geq |M|/2 \geq n/4$.

If every $x \in N$ has the property that there is a $y \in S$ with $c(xy) = 2$, then the subgraph in color 2 induced by $N \cup S$ is a 4-subgraph with at least $(n + 2)/3 + n/4$ vertices, and we are done. We may therefore suppose that there is an $x \in N$ such that $c(xx') = 3$ for every $x' \in S$. For $i = 2, 3$, let

$$A_i = \{u \in N \cup N' \cup (M - S) : \text{ there is a } u' \in S \text{ with } c(uu') = i\}.$$

By the definitions of N, M, and A_i, we have $A_2 \cup A_3 \supseteq N$. We next strengthen this to $A_2 \cup A_3 \supseteq N \cup N'$. If there is a vertex $z \in N'$ with $c(zy) = 1$ for every $y \in S$, then the subgraph in color 1 induced by $S \cup N \cup \{z\}$ is a monochromatic 4-subgraph on at least $n/4 + (n + 2)/3 + 1 \geq n/2 + 1$ vertices. Therefore we assume the $A_2 \cup A_3 \supseteq N \cup N'$.

Because of v and x, each of the sets $A_i \cup S$ induces a monochromatic 4-subgraph. Consequently, there is a monochromatic 4-subgraph of order at least $|S| + \max_i\{|A_i|\}$. By the previous observations, this is at least

$$|S| + \frac{|A_2| + |A_3|}{2} \geq \left\lceil \frac{|M|}{2} \right\rceil + \left\lceil \frac{|A_2 \cup A_3|}{2} \right\rceil \geq$$

$$\geq \left\lceil \frac{|M|}{2} \right\rceil + \left\lceil \frac{|N \cup N'|}{2} \right\rceil = \left\lceil \frac{|M|}{2} \right\rceil + \left\lceil \frac{n - |M|}{2} \right\rceil \geq \left\lceil \frac{n}{2} \right\rceil.$$

We now improve this bound by one when $n = 4l+2$. We obtain the improvement unless equality holds above, which forces $|M|$ to be even, $|S| = |M|/2$, and $A_2 \cup A_3 = N \cup N'$. Recall that $|N| + |N'| \leq n/2$, which implies that $|M| \geq n/2 = 2l + 1$. Because $|M|$ is even, we obtain $|M| \geq 2l + 2 = n/2 + 1$.

Since $A_2 \cup A_3 = N \cup N'$, every vertex in $M - S$ has no edge to S in color 2 or 3. Thus all edges between S and $M - S$ are of color 1, and the complete bipartite graph B with parts S and $M - S$ is monochromatic. Because $|S| = |M|/2$, both S and $M - S$ are nonempty. This implies that B is a monochromatic 2-subgraph with $|M| \geq n/2 + 1$ vertices. □

5 Diameter three and infinity

In this section we prove Theorem 7 and also present an unpublished construction of Calkin which improves the bounds given by Construction 11 when $k-1>3$ is a prime power.

Proof of Theorem 7: Given a k-coloring $c : E(K_n) \to [k]$, choose $v \in V(K_n)$, and assume that $d_i(v)$ is maximized when $i=1$. Consider the bipartite graph G with bipartition $A = v \cup N_1(v)$ and $B = V(K_n) - A$; set $a = |A|$. For $x \in A$ and $y \in B$, let $xy \in E(G)$ if $c(xy) \neq 1$. Let $\Delta = \max_{w \in A} |N_1(w) \cap B|$. Then $E(G) \geq a(n-a-\Delta)$.

For any $w \in A$ with $|N_1(w) \cap B| = \Delta$, the subgraph in color 1 induced by $A \cup N_1(w)$ is a 3-subgraph with at least $a + \Delta$ vertices. By definition, color 1 is absent in G and thus $E(G)$ is $(k-1)$-colored. Lemma 8 applied to G yields a 3-subgraph on at least $(n-a-\Delta)/(k-1) + a(n-a-\Delta)/((k-1)(n-a))$ vertices. Thus K_n contains a 3-subgraph of order at least

$$\min_{\substack{a,\,\Delta \\ a \,\geq\, 1+(n-1)/k \\ \Delta \,\leq\, n-a}} \max\left\{a+\Delta,\ \frac{a(n-a-\Delta)}{k-1}\left(\frac{1}{a}+\frac{1}{n-a}\right)\right\}.$$

We let Δ and a take on real values to obtain a lower bound on this minimum. Since one of these functions is increasing in Δ and the other is decreasing in Δ, the choice of Δ that minimizes the maximum (for fixed a) is that for which the two quantities are equal. This choice is

$$\Delta = \frac{(n-a)(n-a(k-1))}{kn-a(k-1)}\ ,$$

and both functions become $n^2/(kn-a(k-1))$. Since this is an increasing function for $1+(n-1)/k \leq a < kn/(k-1)$, and since we are assuming $a \leq n$, the minimum is obtained at $a = 1+(n-1)/k$. This yields a lower bound of $kn^2/((k^2-k+1)n-(k-1)^2)$. $\square$

Definition 12 *For a positive integer k, let $f_\infty^k(G)$ be the maximum t with the property that every k-coloring of $E(G)$ yields a monochromatic connected subgraph on at least t vertices.*

Clearly $f_d^k(G) \leq f_\infty^k(G)$ for each d, since a d-subgraph is connected. Construction 11 and Theorem 6 therefore immediately yield $f_\infty^3(K_n) = n/2+1$ or $\lceil n/2 \rceil$ depending on whether $n \equiv 2 \pmod 4$ or not (see also exercise 14 of Chapter 6 of [1]). For larger k, however, the following unpublished construction due to Calkin improves Construction 11

Construction 13 (Calkin) Let q be a prime power and $\mathbf{F}$ be a finite field on q elements. We exhibit a $q+1$-coloring of $E(K_{q^2})$. Let $V(K_{q^2}) = \mathbf{F} \times \mathbf{F}$. Color the edge $(i,j)(i',j')$ by the field element $(j'-j)/(i'-i)$ if $i \neq i'$, and color all edges $(i,j)(i,j')$ with a single new color. This coloring is well-defined since $(j'-j)/(i'-i) = (j-j')/(i-i')$. $\square$

Lemma 14 *Construction 13 produces a $q+1$-coloring of $E(K_{q^2})$ such that the subgraph of any given color consists of q vertex disjoint copies of K_q.*

Proof: This is certainly true of the color on edges of the form $(i,j)(i,j')$. Now fix a color $l \in \mathbf{F}$. Let $(x,y) \sim (x',y')$ if the edge $(x,y)(x',y')$ has color l. We will show that this relation is transitive.

Suppose that $(i,j) \sim (i',j')$ and $(i',j') \sim (i'',j'')$. Then

$$(j'-j)/(i'-i) = l = (j''-j')/(i''-i').$$

Consequently,

$$(j''-j) = (j''-j') + (j'-j) = l(i''-i') + l(i'-i) = l(i''-i)$$

and therefore $(i,j) \sim (i'',j'')$.

Since this relation on $V(K_{q^2}) \times V(K_{q^2})$ is an equivalence relation, the edges in color l form vertex disjoint complete graphs. For fixed i,j,l, there are exactly $q-1$ distinct $(x,y) \neq (i,j)$ for which $(x,y) \sim (i,j)$, because $x \neq i$ uniquely determines y. This completes the proof. $\square$

Lemma 14 together with Theorem 5 allows us to easily obtain good bounds for $f^k_\infty(K_n)$. The author believes that the following theorem was also proved independently by Calkin. Our proof of the lower bound given below uses Theorem 5.

Theorem 15 *Let $k-1$ be a prime power. Then $n/(k-1) \leq f^k_\infty(K_n) \leq n/(k-1)+k-1$.*

Proof: For the upper bound we use the idea of Construction 13. Let $\mathbf{F}$ be a finite field of $q = k-1$ elements. Partition $V(K_n)$ into $(k-1)^2$ sets $V_{i,j}$ of size $\lfloor n/(k-1)^2 \rfloor$ or $\lceil n/(k-1)^2 \rceil$, where $i,j \in \mathbf{F}$. Color all edges between $V_{i,j}$ and $V_{i',j'}$ by the field element $(j'-j)/(i'-i)$ if $i \neq i'$, and by a new color if $i = i'$. Color all edges within each $V_{i,j}$ by a single color in $\mathbf{F}$.

Lemma 14 implies that the order of the largest monochromatic connected subgraph is at most $\lceil n/(k-1)^2 \rceil (k-1) \leq n/(k-1) + k - 1$.

For the lower bound, consider a k-coloring of $E(K_n)$. We may assume that the subgraph H in some color l is not a connected spanning subgraph. This yields a partition $X \cup Y$ of $V(K_n)$ such that no edge between X and Y has color l (let X be a component of H). The bipartite graph B formed by the X,Y edges is colored with $k-1$ colors. Applying Theorem 5 to B yields a 3-subgraph of order at least $|X|/(k-1)+|Y|/(k-1) = n/(k-1)$. $\square$

6 Table of Results

Table of Results for $f_d^k(K_n)$

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<tr><td>2</td><td>$\left\lceil \frac{3n}{4} \right\rceil$, [10]</td><td colspan="4">$\sim \frac{n}{k}$, [10]</td></tr>
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Construction 11
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$> \frac{4n}{13}$, Theorem 7</td><td colspan="2">$\leq \frac{n}{k-1} + k - 1$, $k-1$ prime power, Construction 13
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