A generalization of Simion-Schmidt's bijection for restricted permutations

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Abstract. We consider the two permutation statistics which count the distinct pairs obtained from the final two terms of occurrences of patterns $\tau_1 \cdots \tau_{m-2}m(m-1)$ and $\tau_1 \cdots \tau_{m-2}(m-1)m$ in a permutation, respectively. By a simple involution in terms of permutation diagrams we will prove their equidistribution over the symmetric group. As a special case we derive a one-to-one correspondence between permutations which avoid each of the patterns $\tau_1 \cdots \tau_{m-2}m(m-1) \in \mathcal{S}_m$ and those which avoid each of the patterns $\tau_1 \cdots \tau_{m-2}(m-1)m \in \mathcal{S}_m$. For $m = 3$ this correspondence coincides with the bijection given by Simion and Schmidt in [11].

1 Introduction

Recently, a lot of work has been done investigating permutations with restrictions on the patterns they contain. Given a permutation $\pi \in \mathcal{S}_n$ and a permutation $\tau \in \mathcal{S}_m$, an *occurrence of* τ *in* π is an integer sequence $1 \leq i_1 < i_2 < \ldots < i_m \leq n$ such that the letters of the subword $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_m}$ are in the same relative order as the letters of τ. In this context, τ is called a *pattern*. If there is no occurrence at all we say that π *avoids* τ or, alternatively, π is τ*-avoiding*. We write $\mathcal{S}_n(\tau)$ to denote the set of τ-avoiding permutations in $\mathcal{S}_n$ and, more generally, $\mathcal{S}_n(T)$ for the set of all permutations of length n which avoid each pattern of the set T.
A central theme in the theory of pattern-avoiding permutations is to classify all patterns up to Wilf-equivalence. Two patterns τ_1 and τ_2 are called *Wilf-equivalent* if they are equally restrictive, that is, $|\mathcal{S}_n(\tau_1)| = |\mathcal{S}_n(\tau_2)|$ for all $n \in \mathbb{N}$. The first major result dealing with this problem states that 123 and 132 are Wilf-equivalent. (By obvious symmetry arguments, this implies that $\mathcal{S}_3$ is one Wilf-class.) The first explicit bijection between the sets $\mathcal{S}_n(123)$ and $\mathcal{S}_n(132)$ was presented by Simion and Schmidt [11]. We will generalize their correspondence.

In [9] and [10], the diagram of a permutation has been used to study certain forbidden patterns. Given a permutation $\pi \in \mathcal{S}_n$, we obtain its *diagram* as follows. Let π be represented by an $n \times n$ array with a dot in each of the squares (i, π_i). Shadow the squares in each row from the dot and eastwards and the squares in each column from the dot and southwards. The diagram is defined to be the region left unshaded after this procedure. By construction, the connected components of a diagram form Young diagrams. For a diagram square, its *rank* is defined to be the number of dots northwest of it. Clearly, connected diagram squares have the same rank.

In this paper, permutation diagrams play a major role again. Section 2 will show that diagram squares are closely related to occurrences of patterns $\tau \in \mathcal{S}_m$ with $\tau_{m-1}\tau_m = m(m-1)$ or $\tau_{m-1}\tau_m = (m-1)m$. The distinct pairs arising from the last two terms of all occurrences of $\tau_1 \cdots \tau_{m-2} m(m-1)$ in a permutation correspond to the diagram squares of rank at least $m-2$. On the other hand, it suffices to know the distinct pairs arising from the last two terms of all occurrences of $\tau_1 \cdots \tau_{m-2}(m-1)m$ in order to complete a permutation array which contains merely the diagram squares of rank at most $m-3$.
We will prove that the permutation statistics counting the number of these pairs have the same distribution over the symmetric group. In Section 3, a bijection on $\mathcal{S}_n$ will be established which respects these statistics. In particular, it will be shown that there are as many permutations in $\mathcal{S}_n$ which avoid all patterns $\tau \in \mathcal{S}_m$ with $\tau_{m-1} = m$ and $\tau_m = m-1$ as permutations which avoid all patterns $\tau \in \mathcal{S}_m$ with $\tau_{m-1} = m-1$ and $\tau_m = m$. For $m = 3$ the correspondence coincides with Simion-Schmidt's bijection.

2 Diagrams and occurrences of patterns

For $m \geq 2$ define the pattern sets

$$A_m = \{\tau \in \mathcal{S}_m : \tau_{m-1} = m,\, \tau_m = m-1\} \quad \text{and} \quad B_m = \{\tau \in \mathcal{S}_m : \tau_{m-1} = m-1,\, \tau_m = m\}.$$

For a permutation $\pi \in \mathcal{S}_n$, denote by $O_m^A(\pi)$ and $O_m^B(\pi)$ the sets of pairs (i_{m-1}, i_m) obtained from an occurrence $(i_1, \ldots, i_{m-1}, i_m)$ of a pattern belonging to A_m and B_m, respectively. Furthermore, we define $\mathsf{a}_m(\pi) = |O_m^A(\pi)|$ and $\mathsf{b}_m(\pi) = |O_m^B(\pi)|$. In case of $\mathsf{a}_m(\pi) = 0$ (or $\mathsf{b}_m(\pi) = 0$), π avoids each pattern of A_m (or B_m). Note that $\mathsf{a}_2(\pi)$ counts the inversions in π while $\mathsf{b}_2(\pi)$ counts how often π contains the pattern 12. For $m > 2$, the numbers $\mathsf{a}_m(\pi)$ and $\mathsf{b}_m(\pi)$, respectively, are not equal in general to the total numbers of occurrences of A_m-patterns or B_m-patterns in π.
For example, the pattern 1243 occurs in $\pi = 8\,1\,4\,2\,6\,3\,5\,7 \in \mathcal{S}_8$ at the positions $(2,3,5,7)$, $(2,4,5,6)$, and $(2,4,5,7)$; $(3,4,5,7)$ is the only occurrence of 2143. Furthermore, π contains eight increasing subsequences of length 4 whose last two elements are at the positions $(5,8)$, $(6,7)$, $(6,8)$ or $(7,8)$. Finally, there are three occurrences of the pattern 2134, namely, $(3,4,5,8)$, $(3,4,7,8)$, and $(3,6,7,8)$. Hence $\mathsf{a}_4(\pi) = 2$ and $\mathsf{b}_4(\pi) = 4$.

The number $\mathsf{a}_m(\pi)$ can be read off immediately from the ranked diagram of π.

Proposition 1 *Let $\pi \in \mathcal{S}_n$ be a permutation. Then $\mathsf{a}_m(\pi)$ equals the number of diagram squares of rank at least $m-2$. In particular, π avoids all patterns of A_m if and only if every diagram square is of rank at most $m-3$.*

Proof. It follows from the diagram construction that any diagram square (i,j) of rank at least $m-2$ corresponds to an occurrence of a pattern of A_m whose final terms are just i,k where $\pi_k = j$. $\square$

By definition, the number $\mathsf{b}_m(\pi)$ counts the number of non-inversions on the positions of π whose letters are greater than at least $m-2$ letters to their left. (Here a pair (i,j) is called a *non-inversion* if $i<j$ and $\pi_i < \pi_j$.) All the information about a permutation is encoded in the diagram squares of rank at most $m-3$ and the elements of $O_m^B(\pi)$.

Proposition 2 *For each $m \geq 2$, a permutation $\pi \in \mathcal{S}_n$ can be recovered completely from the diagram squares having rank at most $m-3$ and the pairs $(i,j) \in O_m^B(\pi)$.*

Proof. For any $m \geq 2$, let D be the set of all diagram squares of rank at most $m-3$. The proof is based on the following procedure.
First represent the elements of D as white squares in an $n \times n$ array, shaded elsewhere. Starting from the top and proceeding row by row, put a dot in the leftmost shaded square such that there is exactly one dot in each column. By definition of permutation diagrams, this yields the array representation of a permutation that coincides with π at all positions i for which there are at most $m-3$ integers $j<i$ with $\pi_j < \pi_i$. As mentioned before the Proposition, the pairs $(i,j) \in O_m^B(\pi)$ are exactly the non-inversions of the subword consisting of all letters of π having at least $m-2$ smaller letters to their left. Thus we obtain the array representation of π by marking all dots having more than $m-3$ dots northwest and rearranging these dots in a way that the marked dot contained in the ith row lies strictly to the left of the marked dot contained in the jth row if and only if $(i,j) \in O_m^B(\pi)$. $\square$

Remark 3 An efficient way to arrange the marked dots is the following one. Let $r_1 < r_2 < \ldots < r_s$ be the indices of rows containing a marked dot, and $c_1 > c_2 > \ldots > c_s$ the indices of columns with a marked dot. Furthermore, let e_i be the number of pairs in $O_m^B(\pi)$ whose first component equals r_i. For $i = 1, \ldots, s$, set $c_i' = c_{e_i+1}$, delete c_{e_i+1} from the sequence c, and renumber the sequence terms. Put the dots in the squares (r_i, c_i') where $1 \leq i \leq s$. Note that the second component of the elements of $O_m^B(\pi)$ has no relevance for this procedure.

Example 4 Let $\pi = 3\,8\,5\,10\,2\,4\,1\,9\,6\,7 \in \mathcal{S}_{10}$ and $m=5$. The leftmost array shows the ranked permutation diagram of π. All the occurrences of B_5-patterns end with $(9,10)$. Thus we obtain:

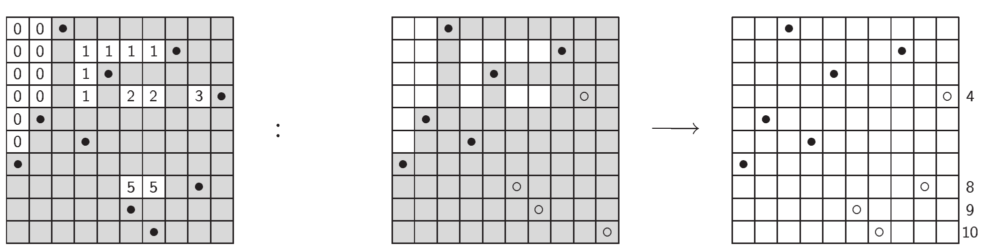

Figure 1 Recovering of a permutation

Black dots represent the elements of π which exceed at most two elements to their left. Note that $(9, 10)$ is the only non-inversion on the elements represented by white dots in the right-hand array. (The sorting routine yields $c' = (10, 9, 6, 7)$ since $e = (0, 0, 1, 0)$.)

3 The bijection

The properties of permutation diagrams given in the previous section are essential for the construction of a bijection Φ_m which proves

Theorem 5 $|\{\pi \in \mathcal{S}_n : \mathsf{a}_m(\pi) = k\}| = |\{\pi \in \mathcal{S}_n : \mathsf{b}_m(\pi) = k\}|$ *for all n and k.*

Let $\pi \in \mathcal{S}_n$ be a permutation. Denote by D_1 the set of its diagram squares of rank at most $m-3$, and by D_2 the set of the remaining diagram squares.
Now define $\sigma = \Phi_m(\pi)$ to be the permutation in $\mathcal{S}_n$ whose set of diagram squares of rank at most $m-3$ equals D_1, and which has as many occurrences $(i_1, \ldots, i_{m-1}, i_m)$ of B_m-patterns as there are squares $(i_{m-1}, *)$ in D_2.
Before analysing this map, let us give an example.

Example 6 Consider $\pi = 3\,8\,5\,10\,2\,4\,1\,9\,6\,7 \in \mathcal{S}_{10}$ again. For $m = 5$, the map Φ_m takes π to the permutation $\sigma = 3\,8\,5\,9\,2\,4\,1\,6\,10\,7$:

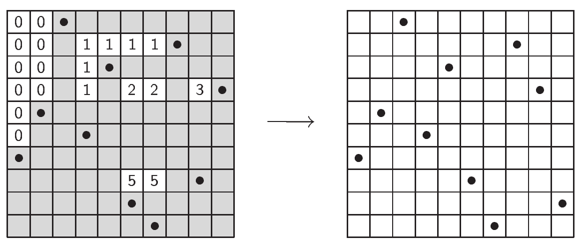

Figure 2 Bijection Φ_5, applied to $\pi = 3\,8\,5\,10\,2\,4\,1\,9\,6\,7$

The diagram squares having rank at most 2 coincide for π and σ. Furthermore, we obtain $O_5^B(\sigma) = \{(4, *), (8, *), (8, *)\}$. (Note that the second components will be not equal in general to the column indices of the diagram squares of π having rank at least 3.) The construction of σ's array is completed as described in the proof of Proposition 2. With the notations introduced in Remark 3, we have $r = (4, 8, 9, 10)$, $c = (10, 9, 7, 6)$, $e = (1, 2, 0, 0)$, and hence $c' = (9, 6, 10, 7)$.

As discussed in the proof of Proposition 2, the equality of the diagram squares of rank at most $m-3$ for π and $\sigma = \Phi_m(\pi)$ means that $\sigma_i = \pi_i$ for all i for which there exist at most $m-3$ integers $j < i$ with $\pi_j < \pi_i$. In particular, the first $m-2$ letters coincide for π and σ. By diagram construction, each white square of rank greater than $m-3$ is a pair (i, π_j) for which there are at least $m-2$ integers $k < i$ with $\pi_k < \pi_j$. Obviously, we have $i < j$ and $\pi_j < \pi_i$. Hence both π_i and π_j are elements exceeding at least $m-2$ elements to their left. Consequently, the map Φ_m is well-defined, and bijective by Proposition 2.

It is easy to see that Φ_m yields the equidistribution of a_m and b_m over the symmetric group.

Proposition 7 *Let $\pi \in \mathcal{S}_n$ and $\sigma = \Phi_m(\pi)$, for any $m \geq 2$. Then $\mathsf{a}_m(\pi) = \mathsf{b}_m(\sigma)$.*

Proof. By Proposition 1, every pair $(i, j) \in O_m^A(\pi)$ corresponds to a diagram square of π having rank at least $m-2$, namely (i, π_j). It follows immediately from the definition of Φ_m that there is an occurrence of a B_m-pattern in σ which ends with (i, k) where k depends on j. □

Remarks 8

a) By the proof, every occurrence of an A_m-pattern in π corresponds in a one-to-one fashion to an occurrence of a B_m-pattern in $\Phi_m(\pi)$ where both sequences coincide at the $(m-1)$st position. Consequently, Φ_m is even an involution, and we have $\mathsf{b}_m(\pi) = \mathsf{a}_m(\Phi_m(\pi))$ for all $\pi \in \mathcal{S}_n$.

b) The bijection Φ_m has the advantage of fixing precisely the intersection of the sets $\mathcal{S}_n(A_m)$ and $\mathcal{S}_n(B_m)$.

c) The map Φ_2 simply takes a permutation $\pi \in \mathcal{S}_n$ to $\sigma \in \mathcal{S}_n$ with $\sigma_i = n + 1 - \pi_i$. Note that we have to arrange all n dots by the procedure given in Remark 3. Here e_i equals the number of diagram squares in the ith row or, equivalently, the number of integers j satisfying $i < j$ and $\pi_i > \pi_j$. The sorting routine yields the permutation $\sigma = c'$ whose occurrences of pattern 12 are precisely the inversions of π.

The case $k = 0$ in Theorem 5 gives the Wilf-equivalence of the pattern sets A_m and B_m, that is, there are as many permutations in $\mathcal{S}_n$ which avoid every pattern of A_m as those which avoid every pattern of B_m. An analytical proof of this result was given in [7].

Corollary 9 *For each $m \geq 2$, the sets A_m and B_m are Wilf-equivalent.*

For a permutation $\pi \in \mathcal{S}_n(A_m)$ the construction of $\sigma = \Phi_m(\pi)$ is particularly easy. By Proposition 1, every diagram square of π is of rank at most $m-3$. Therefore the bijection works as follows. Set $\sigma_i = \pi_i$ if there are at most $m-3$ integers $j < i$ satisfying $\pi_j < \pi_i$. Then arrange the remaining elements in decreasing order.
For example, the permutation $\pi = 2\,6\,\underline{7}\,1\,\underline{3}\,\underline{4}\,\underline{5} \in \mathcal{S}_7$ avoids both 1243 and 2143. We obtain $\Phi_4(\pi) = 2\,6\,\underline{7}\,1\,\underline{5}\,\underline{4}\,\underline{3} \in \mathcal{S}_7(B_4)$. (All the elements which exceed at least two elements to their left are underlined.)
In particular, for $m = 3$ all the left-to-right mimima of π are kept fixed, and the other elements are put at the empty positions in decreasing order. (A *left-to-right minimum* of a permutation π is an element π_i which is smaller than all elements to its left, i.e., $\pi_i < \pi_j$ for every $j < i$.) This is precisely the description of the bijection between $\mathcal{S}_n(132)$ and $\mathcal{S}_n(123)$ proposed by Simion and Schmidt in [11, Proposition 19].

In [1], Babson and West proved the Wilf-equivalence of the singleton pattern sets $\{\tau(m-1)m\}$ and $\{\tau m(m-1)\}$ for every $\tau \in \mathcal{S}_{m-2}$ by means of a stronger Wilf-equivalence relation. The bijection presented here does not show this result. For example, the permutation $\pi = 2\,1\,5\,4\,3 \in \mathcal{S}_5(1243)$ is taken to $\sigma = 2\,1\,3\,4\,5 \notin \mathcal{S}_5(1234)$ by Φ_4. (Note that π does not avoid the pattern 2143 simultaneously; we have $\mathsf{a}_4(\pi) = 3$.)

4 Final remarks

The number of elements of $\mathcal{S}_n(B_m)$ (or, equivalently, $\mathcal{S}_n(A_m)$) was determined in [2]. Recently, research into the enumeration of permutations having a prescribed number of occurrences of certain patterns has been intensified. Similarly, we may ask for the number $|\{\pi \in \mathcal{S}_n : \mathsf{a}_m(\pi) = k\}|$ for any positive integer k. (By Theorem 5, the problem of determining $|\{\pi \in \mathcal{S}_n : \mathsf{b}_m(\pi) = k\}|$ is equivalent.) As seen from Proposition 1, this number counts the permutations in $\mathcal{S}_n$ whose diagram has exactly k squares of rank at least $m-2$. Here we only consider the special case $m = 3$ and $k = 1$.

Our proof uses tunnels in Dyck paths which were introduced very recently by Elizalde [5]. Recall that a *Dyck path* of length $2n$ is a lattice path in $\mathbb{Z}^2$ between $(0,0)$ and $(2n,0)$ consisting of up-steps $[1,1]$ and down-steps $[1,-1]$ that never falls below the x-axis. For any Dyck path d, a *tunnel* is defined to be a horizontal segment between two lattice points of d that intersects d only in these two points, and stays always below d. The *length* and *height* of a tunnel are measured in the lattice. Figure 3 shows a tunnel (drawn with a bold horizontal line) of length 4 and height 2.

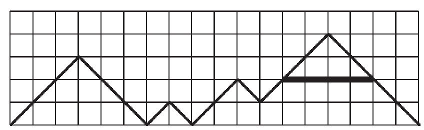

Figure 3 A tunnel in a Dyck path

The Dyck paths of length $2n$ have $\binom{2n-1}{n-3}$ tunnels of positive height and length at least 4. To verify this, note that there are nC_n tunnels in all since every tunnel is associated with an up-step, and the number of Dyck paths of length $2n$ equals the nth *Catalan number* $C_n = \frac{1}{n+1}\binom{2n}{n}$. The tunnels of height zero correspond precisely to *returns*, that is, down-steps landing on the x-axis. By [4], the total number of returns in Dyck paths of length $2n$ is equal to $\frac{3}{2n+1}\binom{2n+1}{n-1}$. Each tunnel of length 2 and positive height is just the connection line of a *high peak*. (A *high peak* of a Dyck path is an up-step followed by a down-step whose common lattice point is at a level greater than one.) Their number was also given in [4]; it equals $\binom{2n-1}{n-2}$.

Proposition 10 *We have* $|\{\pi \in \mathcal{S}_n : \mathsf{a}_3(\pi) = 1\}| = \binom{2n-1}{n-3}$ *for all* n.

Proof. We have to count the permutation diagrams having exactly one square, say (i,j), of rank $r \geq 1$. By definition of the rank function, there are exactly r dots northwest of (i,j). Hence the row i and the column j contain r shaded squares to the west and north of (i,j), respectively. (More exactly, these squares separate (i,j) from the connected component consisting of all diagram squares of rank zero.) It was shown in [10, Lemma 2.2] that the rank of a square (i,j) which belongs to the diagram of a permutation in $\mathcal{S}_n$ is at least $i+j-n$.

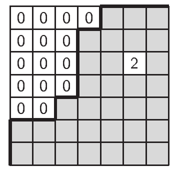

Figure 4 Diagram of a permutation $\pi \in \mathcal{S}_7$ satisfying $\mathsf{a}_3(\pi) = 1$

Consider now the Dyck path that goes from the lower-left corner to the upper-right corner of the array, and travels along the boundary of the connected component of the diagram squares of rank zero. (It follows from the lower bound for the rank of diagram squares that the path never goes below the diagonal.)
The square (i, j) corresponds in a one-to-one fashion to a tunnel of the Dyck path: the line that connects the path step contained in the ith row (up-step) with the path step contained in the jth column (down-step) is a tunnel of length $2r + 2 \geq 4$.
To see this, let (i, j') be the rightmost diagram square of rank zero in the ith row, and (i', j) the lowest square of rank zero in the jth column. (If there is no such square, define i' and j', respectively, to be zero.)

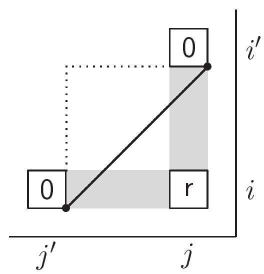

Figure 5 Correspondence between diagram square of rank $r \geq 1$ and Dyck path tunnel

As mentioned above, we have $i = i' + r + 1$ and $j = j' + r + 1$. Thus the segment between the considered lattice points is parallel to the southwest-northeast diagonal of the array. Furthermore, there is no diagram square (i'', j'') with $i' < i'' < i$ and $j' < j'' < j$ satisfying $i'' + j'' \geq i + j' = i' + j$. (The existence of such a square would mean that the path intersects the segment in further points.) Due to the diagram construction, each of the r dots northwest of (i, j) is contained in the $r \times r$ subarray (l, k) with $i' < l < i$ and $j' < k < j$. Therefore, this array represents a (132-avoiding) permutation in $\mathcal{S}_r$. In particular, for every of its diagram squares (i'', j'') – which are all of rank zero – we have $i'' + j'' \leq r$. Since $i + j' = i' + j' + r + 1 > r$, the segment is actually a tunnel. Its height is greater than zero; otherwise, we have $i + j' = n$ and hence $i + j > n + r$.
Consequently, there is a one-to-one correspondence between tunnels of length at least four and height at least one in Dyck paths of length $2n$ and permutations $\pi \in \mathcal{S}_n$ satisfying $\mathsf{a}_3(\pi) = 1$. $\square$

Remark 11 Thomas [12] gives the following alternative combinatorial proof of Proposition 10 dealing with the permutation statistic $\mathbf{b}_3$:
Let $\pi \in \mathcal{S}_n$ satisfy $\mathsf{b}_3(\pi) = 1$. Furthermore, let $(i, j) \in O_3^B(\pi)$, that is, there are integers $k < i < j$ with $\pi_k < \pi_i < \pi_j$. Consider now the permutation $\sigma \in \mathcal{S}_n$ which arises from

π by exchanging π_i with π_j. It is easy to see that σ avoids 123. What can we say about the elements σ_i and σ_j? They are successive right-to-left maxima of σ, and there is at least one element to the left of σ_i which is smaller than σ_j. (An element is called a *right-to-left maximum* of a permutation if it exceeds all the elements to its right.) In fact, for any $\sigma \in \mathcal{S}_n(123)$ these two properties characterize the pairs (σ_i, σ_j) whose transposition yields a permutation π for which $\mathbf{b}_3(\pi) = 1$. Consequently, we want to count right-to-left maxima of 123-avoiding permutations for which the set of elements to their right is not a complete interval $[1, k]$ for some k or the empty set.
In [6], Krattenthaler describes a bijection between 123-avoiding permutations in $\mathcal{S}_n$ and Dyck paths of length $2n$ having the property that any right-to-left maximum of the kind we consider corresponds to a valley at a level greater than zero. (A *valley* of a Dyck path is a down-step followed by an up-step.) By [4], these valleys are just counted by $\binom{2n-1}{n-3}$.

For comparison, Noonan [8] proved that the number of permutations in $\mathcal{S}_n$ containing 123 exactly once is given by $\frac{3}{n}\binom{2n}{n-3}$ while Bóna [3] showed that there are $\binom{2n-3}{n-3}$ permutations in $\mathcal{S}_n$ having exactly one 132-subsequence. By [9, Theorem 5.1], the latter permutations are characterized to be the ones having exactly one diagram square of rank 1 and only rank 0 squares otherwise.

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