On the diagram of Schröder permutations

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Abstract. Egge and Mansour have recently studied permutations which avoid 1243 and 2143 regarding the occurrence of certain additional patterns. Some of the open questions related to their work can easily be answered by using permutation diagrams. As for 132-avoiding permutations the diagram approach gives insights into the structure of $\{1243, 2143\}$-avoiding permutations that yield simple proofs for some enumerative results concerning forbidden patterns in such permutations.

1 Introduction

Let $\mathcal{S}_n$ be the set of all permutations of $\{1, \ldots, n\}$. Given a permutation $\pi = \pi_1 \cdots \pi_n \in \mathcal{S}_n$ and a permutation $\tau = \tau_1 \cdots \tau_k \in \mathcal{S}_k$, we say that π *contains the pattern* τ if there is a sequence $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ such that the elements $\pi_{i_1}\pi_{i_2} \cdots \pi_{i_k}$ are in the same relative order as $\tau_1\tau_2 \cdots \tau_k$. Otherwise, π *avoids the pattern* τ, or alternatively, π is τ-*avoiding*. The set of τ-avoiding permutations in $\mathcal{S}_n$ is denoted by $\mathcal{S}_n(\tau)$. For an arbitrary finite collection T of patterns we write $\mathcal{S}_n(T)$ to denote the permutations of $\{1, \ldots, n\}$ which avoid each pattern in T.

Egge and Mansour [2] studied permutations which avoid both 1243 and 2143. This work was motivated by the parallels to 132-avoiding permutations. In [6, Lem. 2 and Cor. 9] it was shown that the number of elements of $\mathcal{S}_n(1243, 2143)$ is counted by the $(n-1)$st Schröder number r_{n-1}. The (large) *Schröder numbers* may be defined by

$$r_0 := 1, \quad r_n := r_{n-1} + \sum_{i=0}^{n-1} r_i r_{n-1-i} \quad \text{for } n \geq 1.$$

For this reason, the authors of [2] called the permutations which avoid 1243 and 2143 *Schröder permutations*; we will do this as well. (The reference to Schröder numbers may be somewhat inexact because there are ten inequivalent pairs $(\tau_1, \tau_2) \in \mathcal{S}_4^2$ for which $|\mathcal{S}_n(\tau_1, \tau_2)| = r_{n-1}$, see [6, Theo. 3]. However, it is sufficient for our purposes.)

Schröder permutations are known to have a lot of properties which are analogous to properties of 132-avoiding permutations. A look at their diagrams shows why this is so. Given a permutation $\pi \in \mathcal{S}_n$, we obtain its *diagram* $D(\pi)$ as follows: first let π be represented by an $n \times n$-array with a dot in each of the squares (i, π_i) (numbering from the top left hand corner). Shadow all squares due south or due east of some dot and the dotted cell itself. The diagram $D(\pi)$ is defined as the region left unshaded after this procedure. A square that belongs to $D(\pi)$ we call a *diagram square*; a row (column) of the array that contains a diagram square is called a *diagram row* (*diagram column*). (The diagram is an important tool in the theory of the Schubert polynomial of a permutation. Schubert polynomials were extensively developed by Lascoux and Schützenberger. See [7] for a treatment of this work.)

By the construction, each of the connected components of $D(\pi)$ is a Young diagram. Their corners are defined to be the elements of the *essential set* $\mathcal{E}(\pi)$ of the permutation π. In [4], Fulton introduced this set which together with a rank function was used as a tool for an algebraic treatment of Schubert polynomials. For any element $(i, j) \in \mathcal{E}(\pi)$, its *rank* is defined to be the number of dots northwest of (i, j), and is denoted by $\rho(i, j)$. Furthermore, by $\mathcal{E}_r(\pi)$ we denote the set of all elements of $\mathcal{E}(\pi)$ whose rank equals r.

It is clear from the construction that the number of dots in the northwest is the same for all diagram squares which are connected. Consequently, we can extend the rank function to the set of all diagram squares. The concept should be clear from the figure.

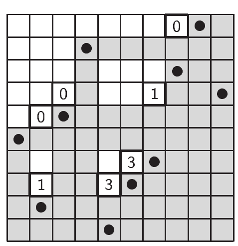


Figure 1 Diagram and ranked essential set of $\pi = 9\,4\,8\,10\,3\,1\,7\,6\,2\,5 \in \mathcal{S}_{10}$.

It is a fundamental property of the ranked essential set of a permutation π, that it uniquely determines π. This result was first proved by Fulton, see [4, Lem. 3.10b]; alternatively, an algorithm for retrieving the permutation from its ranked essential set was provided in [3]. Answering a question of Fulton, Eriksson and Linusson gave in [3] a characterization of all ranked sets of squares that can arise as the ranked essential set of a permutation.

To recover a permutation from its diagram is trivial: row by row, put a dot in the leftmost shaded square such that there is exactly one dot in each column.

In [8], we used permutation diagrams to give combinatorial proofs for some enumerative results concerning forbidden subsequences in 132-avoiding permutations. Now we develop analogues of these bijections. In particular, we will discuss some open problems which have been raised in [2].

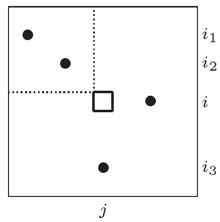
The following section begins with a characterization of Schröder permutation diagrams. Then we will give a surjection that takes any Schröder permutation to a 132-avoiding permutation having the same number of inversions. On the other hand, a simple way to generate all Schröder permutation diagrams from those corresponding to 132-avoiding permutations is described.
Section 3 deals with additional restrictions on Schröder permutations. As was done for 132-avoiding permutations we will characterize from the diagram the occurrence of increasing and decreasing subsequences of prescribed length, as well as of some modifications. This yields simple combinatorial proofs for some results appearing in [2].
In the same reference a bijection between Schröder permutations and lattice paths was given. Section 4 shows how the path can immediately be obtained from the diagram of the corresponding permutation.
The paper ends with some remarks about potential generalizations of its results.

2 A description of Schröder permutation diagrams

By [8, Theo. 2.2], 132-avoiding permutations are precisely those permutations for which the diagram corresponds to a partition, or equivalently, for which the rank of every element of the essential set equals 0. More exactly, the diagram of a permutation in $\mathcal{S}_n(132)$ is a Young diagram fitting in the shape $(n-1, n-2, \ldots, 1)$, that is, whose ith row is of length at most $n-i$. Analogously, we can characterize the elements of $\mathcal{S}_n(1243, 2143)$.

Theorem 2.1 *A permutation $\pi \in \mathcal{S}_n$ is a Schröder permutation if and only if every element of its essential set is of rank at most* 1.

Proof. If there exists an element $(i,j) \in \mathcal{E}(\pi)$ (or equivalently, any diagram square (i,j)) with $\rho(i,j) \geq 2$ then, by definition, at least two dots appear in the northwest of (i,j), say in the rows $i_1 < i_2$. Obviously, the subsequence $\pi_{i_1}\pi_{i_2}\pi_i\pi_{i_3}$ is of type 1243 (represented in the following figure) or 2143 where $\pi_{i_3} = j$:



On the other hand, it is clear from the construction that the occurrence of a pattern 1243 or 2143 in a permutation yields a diagram corner of rank at least 2. $\square$

We wish to describe more precisely the diagrams that arise as the diagram of a Schröder permutation. First we state two elementary properties of permutation diagrams.

Lemma 2.2 *Let $\pi \in \mathcal{S}_n$ be an arbitrary permutation.*

a) *We have $i + j \leq n + r$ for each $(i, j) \in \mathcal{E}_r(\pi)$.*

b) *Let (i, j) be a diagram square of rank 1 for which $(i - 1, j)$ and $(i, j - 1)$ do not belong to $D(\pi)$. Then $\pi_{i-1} = j - 1$. Furthermore, for any element $(i, j) \in \mathcal{E}_1(\pi)$ there exists no square $(i', j') \in \mathcal{E}(\pi)$ with $i' < i$ and $j' < j$.*

Proof. **a)** Let $(i, j) \in \mathcal{E}(\pi)$ be of rank r. Then exactly r indices $k < i$ satisfy $\pi_k < j$. By construction, we have $\pi_i > j$ and $i < \pi_j^{-1}$. Thus there exist $i - r$ integers $k \leq i$ with $\pi_k > j$. Clearly, the number of all elements $\pi_k > j$ in π equals $n - j$. This yields the restriction.

b) By definition, there is exactly one dot (representing a pair (i', j') where $\pi_{i'} = j'$) northwest of (i, j). From the condition that (i, j) forms the upper left-hand corner of a connected component of diagram squares it follows that $\pi_{i-1} < j$ and $\pi_{j-1}^{-1} < i$. Thus we have $i' = i - 1$ and $j' = j - 1$.

For the second assertion let $(i, j) \in \mathcal{E}_1(\pi)$. Suppose that there exists a diagram corner (i', j') with $i' < i$ and $j' < j$. Obviously, (i', j') must be of rank 0, and by the first part, it is different from $(i - 1, j - 1)$. Thus (i', j') is a corner of the Young diagram formed from the diagram squares that are connected with $(1, 1)$. Hence $\pi_{i'+1} \leq j'$ and $\pi_{j'+1}^{-1} \leq i'$. (Note that $i' + 1 < i$ and $j' + 1 < j$; otherwise (i, j) is not a diagram square.) Consequently, there are two dots northwest of (i, j), contradicting that $(i, j) \in \mathcal{E}_1(\pi)$. $\square$

Remark 2.3 Condition a) is a part of Eriksson's and Linusson's characterization of ranked essential sets, see [3, Theo. 4.1].

In the case of Schröder permutations, the second claim of b) means: there are no diagram corners (i, j) and (i', j') such that $i' < i$ and $j' < j$. By [4, Prop. 9.6], this is just the property that characterizes *vexillary* permutations. Fulton's description is an important example of characterization classes of permutations by the shape of their essential set. He gave a set of sufficient conditions that all except for one are also necessary. Later, Eriksson and Linusson strengthened that condition to obtain a set of necessary and sufficient conditions. Note that vexillary permutations can alternatively be characterized as 2143-avoiding ones, see [7, (1.27)]. Of course, every Schröder permutation is vexillary.

Consequently, we can answer the question: when is a subset of the n^2 squares of $\{1,\ldots,n\}^2$ the essential set of a Schröder permutation in $\mathcal{S}_n$? In particular, this yields a further combinatorial interpretation of Schröder numbers.

Proposition 2.4 *For $s \geq 0$ let $i_1 \geq i_2 \geq \ldots \geq i_s$ and $j_1 \leq j_2 \leq \ldots \leq j_s$ be positive integers, and let $r_1, r_2, \ldots, r_s$ be 0 or 1 such that*

$$i_1 - r_1 > i_2 - r_2 > \ldots > i_s - r_s > 0 \quad \textit{and} \quad 0 < j_1 - r_1 < j_2 - r_2 < \ldots < j_s - r_s. \tag{1}$$

For any $n \geq i_1 + j_s$ there is a unique permutation $\pi \in \mathcal{S}_n$ with $\mathcal{E}(\pi) = \{(i_1, j_1), \ldots, (i_s, j_s)\}$ and $\rho(i_k, j_k) = r_k$ for $k = 1, \ldots, s$. In particular, π avoids 1243 and 2143, and every Schröder permutation arises from a unique collection of such integers.

Proof. See [4, Prop. 9.6]. The condition $r_k \in \{0, 1\}$ follows from Theorem 2.1. □

In [3, Prop. 2.2], the condition $n \geq i_1 + j_s$ has been replaced by $i_k + j_k \leq n + r_k$, for $k = 1, \ldots, s$.

Corollary 2.5

a) *The $(n-1)$st Schröder number r_{n-1} counts the number of triples of the integer sequences $i_1 \geq i_2 \geq \ldots \geq i_s > 0$ and $0 < j_1 \leq j_2 \leq \ldots \leq j_s$, and the binary sequence $r_1, \ldots, r_s$ satisfying (1) and $i_k + j_k \leq n + r_k$ for all k.*

b) *The nth Catalan number C_n counts the number of pairs of integer sequences $i_1 > i_2 > \ldots > i_s > 0$ and $0 < j_1 < j_2 < \ldots < j_s$ such that $i_k + j_k \leq n$ for all k. In particular, the number of such pairs of sequences of length s is counted by the Narayana number $N(n, s+1)$.*

Proof. The special case of 132-avoiding permutations ($\rho(i,j) = 0$ for each element (i,j) of the essential set) in 2.4 yields b). It is well-known that $|\mathcal{S}_n(132)| = C_n = \frac{1}{n+1}\binom{2n}{n}$ for all n. The second result of part b) where $N(n, s+1) = \frac{1}{n}\binom{n}{s}\binom{n}{s+1}$ appeared in [8, Rem. 2.6c]. □

Some of the results of this paper are given in terms of essential sets. Therefore we will describe first how one can retrieve a Schröder permutation from its ranked essential set.

For Schröder permutations the retrieval algorithm due to Eriksson and Linusson simplifies considerably and can therefore be carried out without the technical notation used in [3] for treating the general case.

Let $\pi \in \mathcal{S}_n$ be a Schröder permutation, and $E := \mathcal{E}(\pi)$ its essential set. Hence E is a subset of labeled squares in $\{1, 2, \ldots, n\}^2$ satisfying Proposition 2.4. Let the elements of E be represented as white labeled squares in an $n \times n$-array. (All squares that do not belong to E are shaded.)

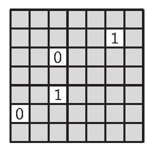


Figure 2a Ranked essential set of $\pi = 4\,7\,5\,2\,6\,3\,1 \in \mathcal{S}_7(1243, 2143)$.

(1) Colour white all squares (i', j') with $i' \leq i$ and $j' \leq j$ where $(i, j) \in E$ is a square labeled with 0. In this way we obtain *the* connected component of all diagram squares which are of rank 0. (Recall that the rank function can be extended to the set of all diagram squares.)

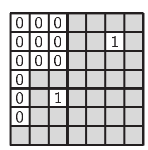


Figure 2b All diagram squares of rank 0 are known.

(2) Put a dot in each shaded square (i, j) for which every square (i', j') with $i' \leq i$ and $j' \leq j$, different from (i, j), is a diagram square of rank 0. Obviously, these dots just represent the left-to-right minima of the permutation. (A *left-to-right minimum* of a permutation π is an element π_i which is smaller than all elements to its left, i.e., $\pi_i < \pi_j$ for every $j < i$.)

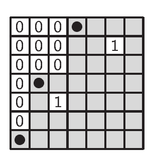


Figure 2c All dotted squares connected with a diagram square of rank 0 are known.

(3) For each dot contained in a square (i, j), colour white all squares (i'', j'') with $i < i'' \leq i'$ and $j < j'' \leq j'$ where $(i', j') \in E$ is a square labeled with 1. By this step, all

diagram squares of rank 1 are obtained. (Note that all squares which are situated in the area southeast of a given dot belong to the same connected component.)

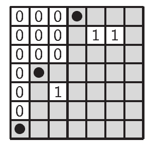


Figure 2d The diagram is completed.

(4) Row by row, if no dot exists in the row, put a dot in the leftmost shaded square such that there is exactly one dot in each column. Now the permutation can read off from the array.

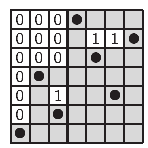


Figure 2e The permutation $\pi = 4\,7\,5\,2\,6\,3\,1$ is recovered.

The following transformation explains the close connection between 132-avoiding permutations and Schröder permutations.

Proposition 2.6 *Let $\pi \in \mathcal{S}_n$ be a Schröder permutation. Let $\mathcal{E}^*(\pi)$ be the set which we obtain from $\mathcal{E}(\pi)$ by replacing every element $(i,j) \in \mathcal{E}_1(\pi)$ by $(i-1, j-1)$ and defining it to be of rank* 0. *Then $\mathcal{E}^*(\pi)$ is an essential set. In particular, $\mathcal{E}^*(\pi)$ is the essential set of a* 132*-avoiding permutation.*

Proof. Let $\mathcal{E}(\pi) = \{(i_1, j_1), \ldots, (i_s, j_s)\}$. We may assume that $\rho(i_k, j_k) = 1$ for any k, otherwise the assertion is trivial. Set $i'_k := i_k - 1,\ j'_k := j_k - 1,\ r'_k := r_k - 1 = 0$, and check Proposition 2.4 for $E = \mathcal{E}(\pi) \cup \{(i'_k, j'_k)\} \setminus \{(i_k, j_k)\}$. Evidently, all the conditions are satisfied (we have $i_k - r_k = i'_k - r'_k,\ j_k - r_k = j'_k - r'_k$). □

Example 2.7 Let $\pi = 4\,7\,5\,2\,6\,3\,1 \in \mathcal{S}_7(1243, 2143)$. Then the transformation $\mathcal{E}(\pi) \mapsto \mathcal{E}^*(\pi)$ yields the essential set of $\sigma = 6\,4\,5\,3\,2\,7\,1 \in \mathcal{S}_7(132)$:

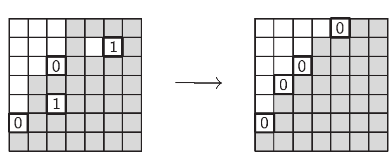
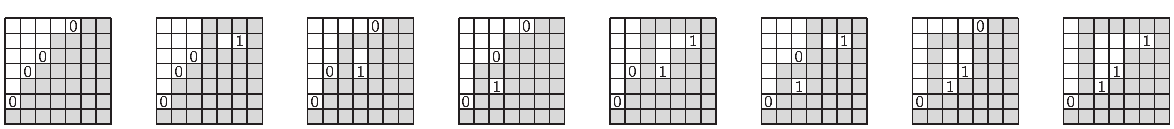


Figure 3 On the left the diagram of π; on the right the diagram of σ.

Let $\phi : \mathcal{S}_n(1243, 2143) \to \mathcal{S}_n$ be the map which takes any Schröder permutation π to the permutation whose essential set equals $\mathcal{E}^*(\pi)$. Obviously, ϕ is a surjection to $\mathcal{S}_n(132)$.
It follows from Lemma 2.2b and the retrieval procedure that $D(\pi)$ and $D(\phi(\pi))$ have the same number of squares. By [7, (1.21)], for any permutation $\pi \in \mathcal{S}_n$ the number of diagram squares is equal to the number of inversions $\mathsf{inv}(\pi)$ of π. Thus we have $\mathsf{inv}(\pi) = \mathsf{inv}(\phi(\pi))$ for every $\pi \in \mathcal{S}_n(1243, 2143)$. Furthermore, Fulton observed in [4] that a permutation $\pi \in \mathcal{S}_n$ has a descent at position i if and only if there exists a diagram corner in the ith row of the $n \times n$-array representing π. (An integer $i \in \{1, \ldots, n-1\}$ for which $\pi_i > \pi_{i+1}$ is called a *descent* of $\pi \in \mathcal{S}_n$. The number of descents of π is denoted by $\mathsf{des}(\pi)$.) Lemma 2.2b implies that $\mathsf{des}(\pi) \leq \mathsf{des}(\phi(\pi))$ for each $\pi \in \mathcal{S}_n(1243, 2143)$.
The left-to-right minima of a permutation $\pi \in \mathcal{S}_n$ are represented by such dots (i, j) for which $(i-1, j)$ or $(i, j-1)$ are diagram squares of rank 0. To include the case $\pi_1 = 1$ we assume that $(0, 1)$ is a diagram square of rank 0. Consequently, every left-to-right minimum of $\pi \in \mathcal{S}_n(1243, 2143)$ is also such a one for $\phi(\pi)$.

In [8, Theo. 5.1] we have shown that the number of subsequences of type 132 in an arbitrary permutation is equal to the sum of ranks of all diagram squares. For Schröder permutations this value is just the number of all diagram squares of rank 1.
The conversion of the above transformation is a simple way to construct Schröder permutations which contain a prescribed number of occurrences of the pattern 132.
Given the essential set of a 132-avoiding permutation $\sigma \in \mathcal{S}_n(132)$ (recall that $\mathcal{E}(\sigma)$ is the corner set of a Young diagram fitting in $(n-1, n-2, \ldots, 1)$; all elements are of rank 0), we replace some elements $(i, j) \in \mathcal{E}(\sigma)$ by $(i+1, j+1)$ and increase their label by 1. It follows from Proposition 2.4 and Lemma 2.2a that the resulting set is an essential set of a Schröder permutation in $\mathcal{S}_n$ if and only if we have $i + j < n$ for all replaced elements (i, j).
For instance, from $\sigma = 6\,4\,5\,3\,2\,7\,1 \in \mathcal{S}_7(132)$ we obtain:



$\pi_1 = \sigma$ $\quad \pi_2 = 4753261$ $\quad \pi_3 = 6357241$ $\quad \pi_4 = 6452731$ $\quad \pi_5 = 3756241$ $\quad \pi_6 = 4752631$ $\quad \pi_7 = 6257431$ $\quad \pi_8 = 2756431$

Figure 4 (All the) Schröder permutations obtained from σ.

Obviously, these are all the Schröder permutations which can be constructed in this way, that is, whose image with respect to ϕ equals σ. Note that $\mathsf{inv}(\sigma) = 15 = \mathsf{inv}(\pi_i)$.

Proposition 2.8 *Let $\sigma \in \mathcal{S}_n(132)$, and let s be the number of elements $(i, j) \in \mathcal{E}(\sigma)$ satisfying $i+j < n$. Then there exist 2^s Schröder permutations $\pi \in \mathcal{S}_n$ for which $\phi(\pi) = \sigma$.*

Proof. This follows from the preceding discussion. □

In [8, Cor. 3.7], we have enumerated the Young diagrams fitting in $(n-1, n-2, \ldots, 1)$ according to the number of their corners in the diagonal $i + j = n$. The number of such diagrams with $k \geq 0$ corners $(i, n-i)$ equals the ballot number $b(n-1, n-1-k) = \frac{k+1}{2n-1-k}\binom{2n-1-k}{n}$. Now we are interested in the distribution of corners outside that diagonal.

Proposition 2.9 *Let $c(n-1, k)$ be the number of Young diagrams fitting in the shape $(n-1, n-2, \ldots, 1)$ with $k \geq 1$ corners satisfying $i + j < n$. Then we have*

$$c(n-1, k) = \sum_{i=1}^{n-1-k} \frac{i}{n-i}\binom{n-i}{k}\binom{n-1}{k+i}.$$

Furthermore, there are 2^{n-1} such diagrams with no corner outside the diagonal $i + j = n$.

Proof. Consider the Young diagram as being contained in an $n \times n$-rectangle, and consider the lattice path from the upper right-hand to the lower left-hand corners of the rectangle that travels along the boundary of the diagram. Defining the rectangle diagonal to be the x-axis with origin in the lower left-hand corner, we obtain a Dyck path of length $2n$, that is, a lattice path from $(0, 0)$ to $(2n, 0)$ which never falls below the x-axis. (In [8], we have noted that the lattice path resulting from the diagram of a 132-avoiding permutation $\pi \in \mathcal{S}_n$ in this way is just the Dyck path corresponding to π according to a bijection proposed by Krattenthaler in [5].) In terms of Dyck paths, a diagram corner satisfying $i + j < n$ means a valley at a level greater than 0 (where the x-axis marks the 0-level). The distribution of the number of these valleys was given in [1, Sect. 6.11]. □

The previous two propositions immediately yield an explicit description for the Schröder numbers.

Corollary 2.10 *For $n \geq 0$ we have $r_n = 2^n + \sum_{k=1}^{n-1} 2^k c(n, k)$.*

Remark 2.11 Another one is $r_n = \sum_{k=0}^{n} \binom{2n-k}{k} C_{n-k}$ where $C_n = \frac{1}{n+1}\binom{2n}{n}$ denotes the nth Catalan number. This formula follows directly from an interpretation in terms of lattice paths, see [10, Exc. 6.19 and 6.39].

3 Forbidden subsequences in Schröder permutations

In this section we will demonstrate that diagrams can be used to obtain simple proofs for enumerative results concerning certain restrictions of Schröder permutations. Most of the numbers $|\mathcal{S}_n(1243, 2143, \tau)|$ appearing below are known from their analytical derivation in [2].

For the following investigation, only one case is really of interest: the essential set of $\pi \in \mathcal{S}_n(1243, 2143)$ contains both elements of rank 0 and 1. If $\mathcal{E}_1(\pi) = \emptyset$ then π avoids 132, and that case has been treated in [8]. If there is no diagram corner of rank 0 then we have $\pi_1 = 1$, and $\pi_2 \cdots \pi_n$ can be identified with a permutation in $\mathcal{S}_{n-1}(132)$. In particular, these permutations contain as many subsequences of type 21 (inversions) as of type 132. (Note that the number of the first equals the number of all diagram squares, and the number of the latter counts all diagram squares of rank 1).

We start by considering increasing subsequences. In [8, Theo. 4.1b] we proved that a permutation $\pi \in \mathcal{S}_n(132)$ avoids the pattern $12 \cdots k$ if and only if its diagram contains $(n+1-k, n-k, \ldots, 1)$. (Recall that in the case of 132-avoiding permutations the diagram corresponds to a Young diagram fitting in the shape $(n-1, n-2, \ldots, 1)$.) This condition will be useful for Schröder permutations as well.

Theorem 3.1 *Let $\pi \in \mathcal{S}_n(1243, 2143)$ be a Schröder permutation. Then π avoids $12 \cdots k$ for any $k \geq 1$ if and only if $\phi(\pi)$ avoids $12 \cdots k$.*

Proof. We may assume that the essential set $\mathcal{E}(\pi)$ contains at least one element, say (i, j), of rank 1; otherwise the assertion is trivial. The proof of 2.6 implies that the set $\mathcal{E}'(\pi) := \mathcal{E}(\pi) \cup \{(i-1, j-1)\} \setminus \{(i, j)\}$ is the essential set of a Schröder permutation again. The rank of $(i-1, j-1)$ is defined as 0. (Successive application of this transformation yields the set $\mathcal{E}^*(\pi)$ stated in Proposition 2.6.) Now we consider which consequences for the corresponding permutation result from this transformation.
Let $\sigma \in \mathcal{S}_n(1243, 2143)$ such that $\mathcal{E}(\sigma) = \mathcal{E}'(\pi)$. Then σ differs from π at exactly three positions. Let π_{i_1} be the element represented by the only dot to the northwest of (i, j). Furthermore let $\pi_{i_2} = j$. Then we have $\sigma_i = \pi_{i_1}$, $\sigma_{i_2} = \pi_i$, $\sigma_{i_1} = \pi_{i_2}$, and $\sigma_k = \pi_k$ for all k, different from i, i_1, i_2. The proof of this fact follows from the retrieval procedure given in Section 2. (For a better understanding it is helpful to consider simultaneously

the example following the proof.)

1) The element π_{i_1} is a left-to-right minimum of π. All the squares due north or due west of the dot (i_1, π_{i_1}) are diagram squares of rank 0. Let i' be the smallest integer greater than i_1 such that a corner of rank 0 appears in row i'. (If such a corner does not exist, set $i' = \infty$.) By Lemma 2.2b, we have $i \leq i'$. In the array representing σ, the square $(i-1, j-1)$ forms a corner of the 0-component, and the next one appears in row i'. Thus the dot representing σ_i is contained in column π_{i_1}.
2) By the transformation, all squares (i'', j'') for which $i_1 < i'' \leq i$ and $\pi_{i_1} < j'' \leq j$ are moved northwestwards. Let j' be the column index of the corner of rank 0 which appears in row $i_1 - 1$. (Note that for $i_1 > 1$ such a one has to exist since π_{i_1} is a left-to-right minimum. For $i_1 = 1$ set $j' = \infty$.) By Lemma 2.2b again, we have $j \leq j'$. Hence in the array of σ the square (i_1, j) is dotted.
3) Now all dots $(i', \sigma_{i'})$ with $i' \leq i$ are fixed. It follows from the construction that these dots are just $(i', \pi_{i'})$ if $i' \neq i_1, i$. Since all diagram squares south of row i appear at the same position in $D(\sigma)$, the only possible position for the missing dot in row i_2 is (i_2, π_i). For all other indices k we have $\sigma_k = \pi_k$. (Note that $\pi_i > \pi_{i_1}$ and $\pi_i > \pi_{i_2}$.)

Consequently, if π contains any increasing subsequence of length k, the permutation σ contains such a sequence as well, and vice versa: by definition and step 2), the elements π_{i_1} and σ_{i_1} are left-to-right minima of π and σ, respectively. If these elements occur in an increasing subsequence then they occur as the first term. Obviously, all the elements $\pi_{i_1+1}, \ldots, \pi_{i-1}$ are greater than π_{i_2} $(= \sigma_{i_1})$. Furthermore we have $\pi_k < \pi_{i_2}$ for $k = i+1, \ldots, i_2 - 1$. (Note that there exists no diagram square in the area southeast of (i, j).) Thus, and since $\pi_{i_1} < \pi_{i_2} < \pi_i$, each increasing subsequence in $\pi_{i_1}\pi_{i_1+1}\cdots\pi_{i_2}$ corresponds to an increasing subsequence of the same length in $\sigma_{i_1}\sigma_{i_1+1}\cdots\sigma_{i_2}$.
Using the arguments successively (until the permutation $\phi(\pi)$ is obtained) proves the assertion of the theorem. $\square$

Example 3.2 For $\pi = 5\ 9\ 8\ 10\ 4\ 2\ 6\ 7\ 3\ 1 \in \mathcal{S}_{10}(1243, 2143)$ we obtain the essential set $\mathcal{E}(\pi) = \{(9,1), (8,3), (5,3), (4,4), (4,7), (2,8)\}$ where $(4,7)$ is of rank 1. Replacing this element yields the essential set of $\sigma = 7\ 9\ 8\ 5\ 4\ 2\ 6\ 10\ 3\ 1 \in \mathcal{S}_{10}(1243, 2143)$ as shown in Figure 5.

The pattern considered now is closely related to the increasing subsequences. In the special case of 132-avoiding permutations the following characterization is identical with [8, Theo. 4.1c].

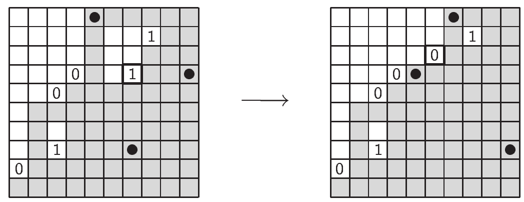
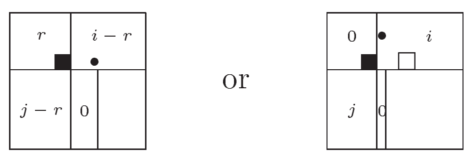


Figure 5 On the left the diagram of π; on the right the diagram of σ. Only the given dots change their position.

Theorem 3.3 *Let $\pi \in \mathcal{S}_n(1243, 2143)$ be a Schröder permutation. Then π avoids $213\cdots k$ if and only if every element $(i,j) \in \mathcal{E}(\pi)$ satisfies $i + j \geq n + 3 - k + \rho(i,j)$.*

Proof. Let $(i,j) \in \mathcal{E}(\pi)$. The n dots representing π are arranged as follows (the labels are the numbers of dots contained in the displayed regions where $r := \rho(i,j)$):



If there is no corner (i,j') such that $j < j'$ (see the left-hand picture) then for all elements π_k with $k > i$ and $\pi_k > j$ we have $\pi_k > \pi_i$. Clearly, $\pi_{i+1} < \pi_i$. On the other hand, if there exists such a corner (i,j') (see the right-hand picture; necessarily, $\rho(i,j) = 0$ and $\rho(i,j') = 1$) then the dot northwest of (i,j') is contained in column $j+1$, otherwise Lemma 2.2b fails to hold. (Note that this dot marks an inner corner of the 0-component.) In this case, all elements π_k with $k > i$ and $\pi_k > j$ satisfy $\pi_k > j + 1$. Clearly, $\pi_{i+1} \leq j$. In both cases the elements represented by dots in the lower right-hand region appear in increasing order since π is 2143-avoiding. If $i + j < n + 3 - k + \rho(i,j)$ then their number $n - (i + j) + \rho(i,j)$ is at least $k - 2$.

To prove the converse, suppose that every element of the essential set satisfies the above condition. Then we have $\pi_i + i > n + 3 - k$ for all $i \in \mathsf{D}(\pi)$. Hence for each descent i of π there exist at most $k - 3$ elements π_j with $j > i$ and $\pi_j > \pi_i$. Since π is 2143-avoiding these elements form an increasing sequence. Thus there is no pattern $2134\cdots k$ in π. $\square$

Now we are in the position to answer the first of a collection of open problems raised in [2]. Theorem 6.5 of this reference implies the Wilf-equivalence of $\{1243, 2143, 12\cdots k\}$ and $\{1243, 2143, 213\cdots k\}$, that is, $|\mathcal{S}_n(1243, 2143, 12\cdots k)| = |\mathcal{S}_n(1243, 2143, 213\cdots k)|$ for all n and k. The authors asked for a combinatorial proof of this fact. There is a simple bijection in terms of diagrams whose essence was already used to prove the analogue for 132-avoiding permutations.

Corollary 3.4 *There is a bijection* $\omega : \mathcal{S}_n(1243, 2143) \to \mathcal{S}_n(1243, 2143)$ *such that for all* $k \geq 1$ *and any permutation* $\pi \in \mathcal{S}_n(1243, 2143)$*, we have that* π *avoids* $12 \cdots k$ *if and only if* $\omega(\pi)$ *avoids* $213 \cdots k$*.*

Proof. Let $\pi \in \mathcal{S}_n(1243, 2143)$ be a Schröder permutation which avoids $12 \cdots k$. By Theorem 3.1 and [8, Theo. 4.1b], the diagram of $\phi(\pi) \in \mathcal{S}_n(132)$ contains $(n+1-k, n-k, \ldots, 1)$. Since all the corners of $(n+1-k, n-k, \ldots, 1)$ are on the diagonal $i+j = n+2-k$ we have $i + j \geq n + 2 - k + 2\rho(i, j)$ for all $(i, j) \in \mathcal{E}(\pi)$. Hence the diagram corners of rank 1 satisfy the condition of Theorem 3.3 anyway. Thus every diagram corresponding to a $12 \cdots k$-avoiding Schröder permutation is uniquely determined by its corners outside the shape $(n + 1 - k, n - k, \ldots, 1)$, that is, by all corners except for those satisfying $i + j = n + 2 - k$. Consequently, the diagram of $\omega(\pi)$ we define to be this one whose corners are the corners of $D(\pi)$ which are not contained in $(n+1-k, n-k, \ldots, 1)$. Ranks are kept up.

Conversely, given any Schröder permutation $\sigma \in \mathcal{S}_n(1243, 2143)$ all of whose diagram corners satisfy $i+j \geq n+3-k+\rho(i, j)$ we construct the permutation $\omega^{-1}(\sigma)$ as follows: let E be the corner set of the diagram obtained as the union of $D(\phi(\sigma))$ and $(n+1-k, n-k, \ldots, 1)$. (Note that this is a Young diagram since $D(\phi(\pi))$ is such a one.) Then we form the essential set of $\omega^{-1}(\sigma)$ from the pairs $(i, j) \in E$ for which $(i + 1, j + 1) \notin \mathcal{E}_1(\sigma)$, and all elements of $\mathcal{E}_1(\sigma)$. The first are defined to be of rank 0, while the latter have rank 1. Obviously, the resulting set is an essential set of a $12 \cdots k$-avoiding Schröder permutation. $\square$

Example 3.5 The maximum length of an increasing subsequence in the Schröder permutation $\pi = 4631572 \in \mathcal{S}_7(1243, 2143)$ equals 3. Taking $k = 4$, we obtain $\omega(\pi) = 1634572$:

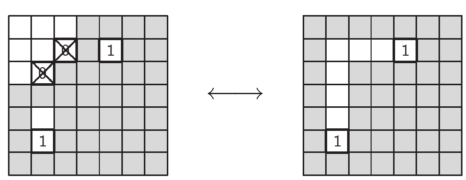


Figure 6 Construction of $\omega(\pi)$: the corners crossed out satisfy $i + j = n + 2 - k$.

Remarks 3.6

a) Since $\mathcal{E}_1(\pi) = \mathcal{E}_1(\omega(\pi))$ for all $\pi \in \mathcal{S}(1243, 2143, 12 \cdots k)$ the map ω takes any 132-avoiding permutation to a permutation which avoids 132 as well. Indeed, the restriction of ω on $\mathcal{S}(132, 12 \cdots k)$ is precisely the bijection given in [8, Cor. 4.4] that proves the Wilf-equivalence of $\{132, 12 \cdots k\}$ and $\{132, 213 \cdots k\}$.

b) It is clear from the construction that a $12\cdots k$-avoiding Schröder permutation also avoids $213\cdots k$ if and only if it is a fixed point of ω. The essential set of such a permutation can be constructed as follows: consider the corner set of a Young diagram which contains $(n+1-k, n-k, \ldots, 1)$, and fits in $(n-1, n-2, \ldots, 1)$. Now replace at least all elements (i,j) by $(i+1, j+1)$ for which $i+j = n+2-k$. Some further corners can be replaced if these satisfy $i+j<n$. The rank of all new corners is set as 1; let the others be of rank 0.

We will discuss the enumerative consequence only for $k=3$.

Corollary 3.7 $|\mathcal{S}_n(1243, 2143, 123, 213)| = 2^{n-1}$ *for all* $n \geq 1$.

Proof. Taking up again the idea of the previous remark, the diagram of a Schröder permutation $\pi \in \mathcal{S}_n(1243, 2143)$ which avoids both 123 as 213 arises from a Young diagram that contains $(n-2, n-3, \ldots, 1)$, and fits in $(n-1, n-2, \ldots, 1)$. Clearly, each such Young diagram is uniquely determined by its corners in the diagonal $i+j=n$. In particular, there are 2^{n-1} diagrams of this kind. (This implies $|\mathcal{S}_n(132, 123)| = 2^{n-1}$; see [9, Prop. 7] for another proof.)

From the corner set of each Young diagram the essential set of only one permutation $\pi \in \mathcal{S}_n(1243, 2143, 123, 213)$ can be generated because all the corners $(i, n-1-i)$ must be replaced, but all the corners $(i, n-i)$ must not be replaced. $\square$

Remark 3.8 To obtain a $\{132, 123, 213\}$-avoiding permutation in the way described in the proof, the Young diagram must not have any corner on the diagonal $i+j=n-1$. We can identify such a diagram by a binary sequence of length $n-1$ whose ith element is defined as 1 (or 0) if $(i, n-i)$ is a corner (or not). The condition that there is no diagram corner outside the diagonal $i+j=n$ means that the corresponding sequence contains no consecutive zeros. The number of such sequences is known to be equal to the $(n+1)$st Fibonacci number F_{n+1}. (The *Fibonacci numbers* are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$.) The result $|\mathcal{S}_n(132, 123, 213)| = F_{n+1}$ already appears in [9, Prop. 15].

The next result deals with the occurrence of decreasing subsequences of length k in Schröder permutations. The analogue for 132-avoiding permutations is simple: a permutation $\pi \in \mathcal{S}_n(132)$ avoids $k(k-1)\cdots 1$ if and only if $|\mathcal{E}(\pi)| \leq k-2$, see [8, Theo. 4.1a]. Now the condition is somewhat more difficult.

To state it, we first set some notation. For a permutation $\pi \in \mathcal{S}_n$ we denote by $r(\pi)$ and

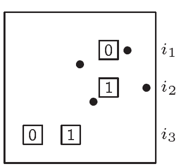
$c(\pi)$ the number of rows and columns, respectively, that contain a diagram corner. As mentioned above, we have $r(\pi) = \mathsf{des}(\pi)$, and $c(\pi) = \mathsf{des}(\pi^{-1})$. (Note that the transpose of $D(\pi)$ is just the diagram of π^{-1}.) It follows from Proposition 2.4 that any diagram row (column) contains at most two corners (necessarily of different rank) if $\pi \in \mathcal{S}_n(1243, 2143)$. Let $r_2(\pi)$ and $c_2(\pi)$ be the number of diagram rows and diagram columns, respectively, containing two corners.

Theorem 3.9 *Let $\pi \in \mathcal{S}_n(1243, 2143)$ be a Schröder permutation. Then π avoids $k(k-1)\cdots 1$ if and only if one of the following conditions holds:*

(i) $r(\pi) \leq k-2$ *or* $c(\pi) \leq k-2$*;*

(ii) $r(\pi) = k-1$*,* $r_2(\pi) = c_2(\pi) = 1$*, and there is no element* $(i,j) \in \mathcal{E}(\pi)$ *that such both row* i *and column* j *contain another corner.*

Proof. Suppose that π contains a decreasing subsequence of length k. Obviously, its inverse contains such a sequence as well. Consequently, both π and π^{-1} must have at least $k-1$ descents, that is, $r(\pi) \geq k-1$ and $c(\pi) \geq k-1$. Now let $r(\pi) = k-1$ and $r_2(\pi) = c_2(\pi) = 1$. (Then we also have $c(\pi) = k-1$.)



If the essential corners are arranged as in the picture opposite (where $i_2 \neq i_3$) we have $\pi_{i_1} > \pi_{i_1+1}$, $\pi_{i_2} > \pi_{i_2+1}$ (corners correspond to descents) but $\pi_{i_1} < \pi_{i_2}$, and $\pi_{i_1+1} < \pi_{i_2+1}$. The last relation follows from the fact that $(i_2, \pi_{i_2}+1)$ is a diagram square by the construction. If its rank was 0 then $r_2 > 1$. Since $\mathsf{des}(\pi) = k-1$ there is no decreasing subsequence of length k in π which contradicts the assumption. In the second case (corners (i_1, j_2), (i_2, j_1) of rank 0, corners (i_1, j_3), (i_3, j_1) of rank 1 where $i_1 < i_2 < i_3$ and $j_1 < j_2 < j_3$) the same argument may be used.

On the other hand, if condition (i) holds then $\mathsf{des}(\pi) \leq k-2$ or $\mathsf{des}(\pi^{-1}) \leq k-2$ and hence $\pi \in \mathcal{S}_n(k \cdots 1)$. If (ii) is satisfied then (as shown in the first part of the proof) π cannot contain any decreasing subsequence of length k. $\square$

Here we will enumerate the permutations described in Theorem 3.9 only for $k = 3$. To satisfy condition (ii) is impossible in this case. Thus a Schröder permutation is 321-avoiding if and only if all its diagram corners are either in the same row or in the same column. This characterization was already given in [3, Prop. 5.4] for 321-avoiding vexillary permutations. (Note that the essential set of a vexillary permutation can contain elements of rank greater than 1; for example, 1243 is such a permutation.)

Egge and Mansour have shown (derived from the generating function in [2, Prop. 7.4]) that

$$|\mathcal{S}_n(1243, 2143, 321)| = \binom{n-1}{0} + \binom{n-1}{1} + 2\binom{n-1}{2} + 2\binom{n-1}{3} \quad \text{for all } n \geq 1.$$

Their fourth problem asked for a combinatorial proof. Here it is.

Corollary 3.10 $|\mathcal{S}_n(1243, 2143, 321)| = n + 2\binom{n}{3}$ *for all* $n \geq 1$.

Proof. Let π be a Schröder permutation avoiding 321. We distinguish the three cases mentioned at the beginning of the section.

If $\pi \in \mathcal{S}_n(132)$ is not the identity permutation then its diagram is a rectangle whose lower right-hand corner (i, j) satisfies $i + j \leq n$. There are

$$\sum_{i=1}^{n-1}(n-i) = \binom{n}{2}$$

such diagrams. (The enumeration of $\{132, 321\}$-avoiding permutations was first done in [9, Prop. 11].) If there exists no element of rank 0 in $\mathcal{E}(\pi)$, the permutation $\pi_2 - 1 \cdots \pi_n - 1$ belongs to $\mathcal{S}_{n-1}(132, 321)$.

It remains to consider the case that $D(\pi)$ has corners of rank 0 and 1. Since these squares are in the same row or column there is exactly one corner of each rank. Without loss of generality, we may assume that both the elements of $\mathcal{E}(\pi)$ are in the same row. (For the result in terms of columns consider the transpose that corresponds to the inverse of π.) Let $(i, j) \in \mathcal{E}_0(\pi)$ and $(i, j') \in \mathcal{E}_1(\pi)$. From Proposition 2.4 the conditions $1 < i$, $j + 1 < j'$, and $i + j' \leq n + 1$ results. Given the pair (i, j), the integer j' can be chosen in $n - i - j$ ways where $i \in \{2, \ldots, n-2\}$, and $j \in \{1, \ldots, n-1-i\}$. Clearly,

$$\sum_{i=2}^{n-2}\sum_{j=1}^{n-1-i}(n-i-j) = \frac{1}{2}\sum_{i=1}^{n-3} i(i+1) = \binom{n-1}{3}.$$

Summarizing, we obtain $|\mathcal{S}_n(1243, 2143, 321)| = 1 + \binom{n}{2} + \binom{n-1}{2} + 2\binom{n-1}{3}$. (Note that the term 1 stands for the identity; the factor 2 accounts for rows and columns in the third case.) $\square$

The last pattern we will discuss is a special case of an important class as well. In [8, Theo. 4.5], we characterized 132-avoiding permutations which avoid the additional pattern $s(s+1)\cdots k12\cdots(s-1)$ where $s \in \{2, \ldots, k\}$, and $k \geq 3$. The condition given there was of a technical nature but for $s = 2$ and $k = 3$ it is equivalent to the following simple one:

a permutation $\pi \in \mathcal{S}_n(132)$ avoids 231 if and only if all its diagram rows are of distinct length, that means, all diagram rows contain a corner. Analogously to that, 231-avoiding Schröder permutations can be described.

Proposition 3.11 *A Schröder permutation* $\pi \in \mathcal{S}_n(1243, 2143)$ *avoids* 231 *if and only if*

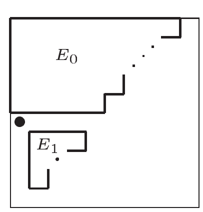
(i) *every diagram row contains exactly one element of the essential set,*

(ii) *and every diagram column contains at most an element of the essential set.*

Proof. Suppose that $\pi \in \mathcal{S}_n(1243, 2143)$ contains a 231-pattern: let $i_1 < i_2 < i_3$ such that $\pi_{i_3} < \pi_{i_1} < \pi_{i_2}$. Since diagram corners and permutation descents correspond to each other, there is an integer i with $i_1 \leq i < i_2$ such that row i contains no corner. We assume that this row does not belong to the diagram, otherwise condition (i) fails to hold. Since $(i_2, \pi_{i_3}) \in D(\pi)$ we have $\pi_i = 1$, and all diagram squares appearing below the ith row are of rank 1. By the construction, $(i+1, 2)$ is the upper left-hand corner of the component which contains (i_2, π_{i_3}) (and all the other diagram squares of rank 1). By Lemma 2.2b there is no corner in the strict northwest of another one. Therefore, and since $\pi_{i_1} < \pi_{i_2}$, the diagram corners contained in row $i-1$ and $i+1$, respectively, have to be in the same column.
For the other direction, we suppose that there is a diagram square (i_1, j) for which $(i_1, j+1) \notin D(\pi)$, and a square $(i_2, j) \in \mathcal{E}(\pi)$ with $i_1 < i_2$. (Then one of the conditions (i) and (ii) is not satisfied.) It is easy to see that $\pi_{i_1}\pi_{i_2}\pi_{i_3}$ where $\pi_{i_3} = j$ is a subsequence of type 231. $\square$

From the first part of the proof, it is clear how the diagram of a 231-avoiding Schröder permutation has to look.

Corollary 3.12 *The diagram of a Schröder permutation satisfies the conditions of* Proposition 3.11 *if and only if it is of the following shape:*



where the diagram components E_0 *and* E_1 *are Young diagrams whose row lengths are each distinct. (It may be that* E_0 *and/or* E_1 *are empty.)*

As an immediate consequence we can characterize 231-avoiding Schröder permutations from their descent set.

Corollary 3.13 *Let $\pi \in \mathcal{S}_n$ be a permutation, and $\mathsf{D}(\pi)$ the set of its descents. Then π avoids the patterns 1243, 2143, and 231 if and only if one of the following conditions is satisfied:*

(i) *$s > d$ and $\mathsf{D}(\pi) = \{1, 2, \ldots, d\}$;*

(ii) *$s \leq d$ and $\mathsf{D}(\pi) = \{1, 2, \ldots, s-1, s+1, s+2, \ldots, d+1\}$, and $\pi_{s-1} > \pi_{s+1}$ if $1 < s < n$*

where $\pi_s = 1$ and $d = \mathsf{des}(\pi)$.

Remark 3.14 We can decide by these conditions whether a 231-avoiding Schröder permutation avoids the pattern 132 in addition or not. The first condition describes all permutations in $\mathcal{S}_n(132, 231)$.

Corollary 3.12 yields the answer to the third question asked by Egge and Mansour.

Corollary 3.15 *For $n \geq 2$ we have $|\mathcal{S}_n(1243, 2143, 231)| = (n+2)2^{n-3}$.*

Proof. Let $\pi \in \mathcal{S}_n(1243, 2143, 231)$. Consider the partition $\lambda(\pi)$ whose parts are just the lengths of the diagram rows where the length of the first row containing squares of rank 1 is listed twice. (By this information $D(\pi)$ and hence π is completely described.) Adding some zeros (if necessary), we may assume that $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ is of length $n-1$. By the previous discussion, we have

$$n > \lambda_1 > \lambda_2 > \ldots > \lambda_{i-1} \geq \lambda_i > \lambda_{i+1} > \ldots > \lambda_l > \lambda_{l+1} = \lambda_{l+2} = \ldots = \lambda_{n-1} = 0$$

with $l \in \{0, \ldots, n-1\}$, and $\lambda \subseteq (n-1, n-1, n-2, \ldots, 2)$ (where $\subseteq$ means the containment of the corresponding diagrams). Furthermore we have $\lambda_j \neq 1$ for all j if any positive part occurs twice.

The number of partitions λ whose positive parts are all distinct is $\sum_{l=0}^{n-1} \binom{n-1}{l} = 2^{n-1}$ ways. (This case corresponds to 132-avoiding permutations; hence $|\mathcal{S}_n(132, 231)| = 2^{n-1}$, see also [9, Prop. 9].) On the other hand the number of partitions which have a repeated positive part is

$$\sum_{l=1}^{n-2} \binom{n-2}{l} \binom{l}{1} = (n-2) \sum_{l=0}^{n-3} \binom{n-3}{l} = (n-2)2^{n-3}.$$

Note that 1 cannot be part of λ in this case. Consequently, there are $2^{n-1}+(n-2)2^{n-3} = (n+2)2^{n-3}$ Schröder permutations in $\mathcal{S}_n$ which avoid 231. $\square$

4 A correspondence to lattice paths

It is well-known that the nth Schröder number r_n counts the number of all lattice paths from the origin to (n,n), with steps $[1,0]$ (called East steps), $[0,1]$ (called North steps), and $[1,1]$ (called Diagonal steps), that never pass below the line $y=x$. Such paths we call *Schröder paths*. (See [10, Exc. 6.39] for further combinatorial interpretations of the Schröder numbers.)

Egge and Mansour have given a bijection Ψ_{EM} between these paths and Schröder permutations in $\mathcal{S}_{n+1}$, see [2, Sect. 4]. Its essential property is: the number of subsequences $12\cdots k$ occuring in $\pi \in \mathcal{S}_{n+1}(1243, 2143)$ can read be off (more or less) directly from the path $\Psi_{EM}(\pi)$.
This bijection can be understood as the analogue of Krattenthaler's correspondence Ψ_K between 132-avoiding permutations in $\mathcal{S}_n$ and lattice paths from $(0,0)$ to (n,n) without diagonal steps, never passing below the line $y=x$. The map Ψ_K encodes the number of increasing subsequences of prescribed length in the same way, see [5, (3.2)].

We pointed out in [8] that the path $\Psi_K(\pi)$ and the diagram of a 132-avoiding permutation π are closely related to each other. Considering the diagram of $\pi \in \mathcal{S}_n(132)$ as being contained in an $n \times n$-rectangle, $\Psi_K(\pi)$ is the lattice path which goes from the upper right-hand corner to the lower left-hand corner of the rectangle, and travels along the diagram boundary.
For $\pi = 6\,4\,5\,3\,2\,7\,1 \in \mathcal{S}_7(132)$, for example, $\Psi_K(\pi)$ is the lattice path displayed in bold:

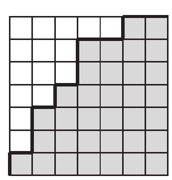


Figure 7 Lattice path $\Psi_K(6453271)$.

We can construct the path Ψ_{EM} just as simply from the permutation diagram.
Note again that each (Schröder) permutation is uniquely determined by its ranked essential set. Consequently, the position and rank of the diagram corners are all that we have to transfer to a path corresponding to the permutation. Since a permutation

$\pi \in \mathcal{S}_{n+1}(1243, 2143)$ should correspond to a Schröder path from $(0,0)$ to (n,n), we cannot use $D(\pi)$ itself but the diagram of $\phi(\pi)$ is suitable. Recall that the diagram of $\phi(\pi)$ is obtained from that of π by "moving" each diagram square of rank 1 northwestwards. Hence it is a Young diagram which is contained in $(n-1, n-2, \ldots, 1)$. Labeling all corners with their original rank we have all the information needed to recover π.
Now the Schröder path corresponding to π is constructed as follows: let $D(\phi(\pi))$ be embedded in an $(n-1) \times (n-1)$-rectangle. Analogously to the construction of Ψ_K, the lattice path is defined to go from the upper right-hand corner to the lower left-hand corner of the rectangle, along the diagram boundary, where every step sequence NE representing a corner labeled with 0 is replaced by a step D. Finally we convert the path into the form used in [2]. To this end, the rectangle is reflected so that the origin is placed at the bottom left instead of at the top right.

Example 4.1 Let $\pi = 4\,7\,5\,2\,6\,3\,1 \in \mathcal{S}_7(1243, 2143)$. Using the diagram of $\phi(\pi)$ we can immediately determine the Schröder path corresponding to π (displayed in bold again):

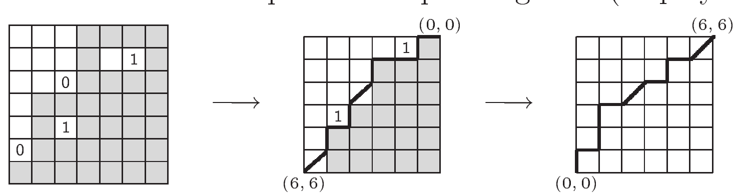


Figure 8 Construction of the path $\Psi_{EM}(\pi)$: On the left the diagram of π; in the centre the diagram of $\phi(\pi)$ with plotted path; on the right the converted path.

By Lemma 2.2a, each element $(i,j) \in \mathcal{E}_1(\pi)$ satisfies $i + j \leq n + 1$. Thus, for every corner (i', j') of $D(\phi(\pi))$ labeled with 1 we have $i' + j' \leq n - 1$. Therefore, and since $D(\phi(\pi))$ is contained in $(n-1, n-2, \ldots, 1)$, this construction indeed yields a Schröder path.
It is not difficult to see that the path obtained in this way is just $\Psi_{EM}(\pi)$ for $\pi \in \mathcal{S}_n(1243, 2143)$ but the construction via diagram requires less effort.

In [2], the path statistic τ_k corresponding to the number of subsequences of type $12 \cdots k$ in $\pi \in \mathcal{S}_n(1243, 2143)$ is defined for $k \geq 2$ by

$$\sum_{s \in \{\mathsf{E},\mathsf{D}\}} \binom{h(s)}{k-1}$$

where $h(s)$ denotes the height of the starting point of step s. (The *height* of a point (x,y) in the plane is defined to be the difference $y - x$.)

Example 4.2 Consider the Schröder path NENNEDENED appearing in the previous example. For each east and diagonal step the height is given in the picture.

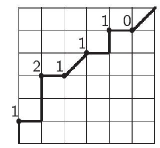


Figure 9 Schröder path with step heights.

Thus there occur six subsequences of type 12 (noninversions), one of type 123, and none of type $12\cdots k$ for $k \geq 4$ in the corresponding permutation $\pi = 4\,7\,5\,2\,6\,3\,1 \in \mathcal{S}_7(1243, 2143)$

Remarks 4.3

a) In particular, a permutation $\pi \in \mathcal{S}_n(1243, 2143)$ avoids $12\cdots k$ if and only if the path $\Psi_{EM}(\pi)$ has no step of height at least $k-1$. This result is equivalent to Theorem 3.1.

b) Combining $\Psi_{EM} : \mathcal{S}_n(1243, 2143) \to S_{n-1}$ with the bijection ω stated in Corollary 3.4 yields the answer to the first part of the first problem raised in [2]. (By S_{n-1} the set of Schröder paths from $(0,0)$ to $(n-1, n-1)$ is denoted.) The map $\omega \circ \Psi_{EM}^{-1}$: $S_{n-1} \to \mathcal{S}_n(1243, 2143)$ takes every Schröder path whose maximum step height is at most $k-2$ to a $213\cdots k$-avoiding Schröder permutation, and is bijective, of course.

c) Obviously, the path $\Psi_{EM}(\pi)$ contains no diagonal step if and only if $\mathcal{E}_0(\pi) = \emptyset$. As already noted, then $\pi_1 = 1$ and $\pi' := (\pi_2 - 1)(\pi_3 - 1)\cdots(\pi_n - 1)$ belongs to $\mathcal{S}_{n-1}(132)$. In particular, we have $\Psi_{EM}(\pi) = \Psi_K(\pi')$ in this case.

5 Perspectives

As already observed by Egge and Mansour in [2], the investigation of 132-avoiding permutations and $\{1243, 2143\}$-avoiding ones, respectively, can be continued in a canonical way. For $m \geq 3$ let T_m be the set of permutations in $\mathcal{S}_m$ for which $\pi_{m-1} = m$ and $\pi_m = m-1$. For example, $T_3 = \{132\}$ and $T_4 = \{1243, 2143\}$. Some of what we have done for 132-avoiding permutations in [8], and for T_4-avoiding permutations in this paper can be generalized for an arbitrary integer m.

Theorem 5.1 *A permutation $\pi \in \mathcal{S}_n$ avoids each pattern in T_m if and only if every element of its essential set is of rank at most $m-3$.*

Proof. If there exists an element $(i,j) \in \mathcal{E}(\pi)$ with $\rho(i,j) \geq m-2$ then at least $m-2$ dots appear northwest of (i,j). Consequently, there are integers $i_1 < i_2 < \ldots < i_{m-2} < i$

for which $\pi_{i_1}, \ldots, \pi_{i_{m-2}} < j$. Furthermore, we have $i < i_{m-1}$ and $\pi_i > j$ where $\pi_{i_{m-1}} = j$. Thus the subsequence $\pi_{i_1} \cdots \pi_{i_{m-2}} \pi_i \pi_{i_{m-1}}$ forms a pattern belonging to T_m. (For a better understanding draw a picture similar to the one in the proof of Theorem 2.1.)
On the other hand, it is clear from the diagram construction that the occurrence of a pattern of T_m in a permutation yields a diagram corner of rank at least $m-2$. $\square$

By reasoning similar to the proof of Theorem 3.3, one can show that this theorem holds for each $m \geq 3$ if $k = 3$. In case $k \geq 4$ and $m \geq 5$, the condition $i + j \geq n + 3 - k + \rho(i,j)$ for all diagram corners (i,j) is only sufficient for avoiding $213 \cdots k$ and all the patterns of T_m. For example, the permutation $\pi = 5\,4\,7\,1\,3\,2\,6 \in \mathcal{S}_7(T_5)$ avoids 2134 but the square $(1,4)$ is a diagram corner of rank 0.

Suggested by computer tests, we conjecture the Wilf-equivalence of $T_m \cup \{12 \cdots k\}$ and $T_m \cup \{213 \cdots k\}$ for all $k \geq 1$ and $m \geq 3$.

Conjecture 5.2 *For $m \geq 3$, and all n and k we have*

$$|\mathcal{S}_n(T_m \cup \{12 \cdots k\})| = |\mathcal{S}_n(T_m \cup \{213 \cdots k\})|.$$

In order to study increasing subsequences in permutations which belong to $\mathcal{S}_n(T_m)$ it would be nice to have a surjection $\mathcal{S}_n(T_m) \rightarrow \mathcal{S}_n(T_{m-1})$ similar the map ϕ stated in Section 2. Then the problem could successively be reduced to the case $m = 3$.

The map ϕ defined in Section 2 can be extended to all vexillary permutations. This is because no diagram corner has another one to its northwest. For $\pi \in \mathcal{S}_n(2143)$, the set $\mathcal{E}^*(\pi)$ obtained from $\mathcal{E}(\pi)$ by replacing the element (i,j) with $(i - \rho(i,j), j - \rho(i,j))$ and defining it to be of rank 0 is the essential set of a 132-avoiding permutation. The extended surjection $\phi' : \mathcal{S}_n(2143) \rightarrow \mathcal{S}_n(132)$ preserves both the inversion number and the length of the longest increasing subsequence.

References

1. E. Deutsch, Dyck path enumeration, *Discrete Math.* **204** (1999), 167-202.
2. E. S. Egge and T. Mansour, Permutations which avoid 1243 and 2143, continued fractions, and Chebyshev polynomials, *Electronic J. Combinatorics* **9(2)** (2003), R7.
3. K. Eriksson and S. Linusson, Combinatorics of Fulton's essential set, *Duke Math. J.* **85** (1996), 61-80.
4. W. Fulton, Flags, Schubert polynomials, degeneracy loci, and determinantal formulas, *Duke Math. J.* **65** (1992), 381-420.
5. C. Krattenthaler, Permutations with restricted patterns and Dyck paths, *Adv. Appl. Math.* **27** (2001), 510-530.
6. D. Kremer, Permutations with forbidden subsequences and a generalized Schröder number, *Discrete Math.* **218** (2000), 121-130.
7. I. G. Macdonald, *Notes on Schubert Polynomials*, LaCIM, Université du Québec à Montréal, 1991.
8. A. Reifegerste, On the diagram of 132-avoiding permutations, preprint, 2002, available from arXiv:math.CO/0208006.
9. R. Simion and F. W. Schmidt, Restricted Permutations, *Europ. J. Combinatorics* **6** (1985), 383-406.
10. R. P. Stanley, *Enumerative Combinatorics Volume II*, Cambridge University Press, 1999.