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EXPLICIT SOLUTIONS FOR A SYSTEM OF FIRST-ORDER PARTIAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper we construct explicit weak solutions of a system of two partial differential equations in the quarter plane $\{(x,t) : x > 0, t > 0\}$ with initial conditions at $t = 0$ and a weak form of Dirichlet boundary conditions at $x = 0$. This system was first studied by LeFloch [9], where he constructed explicit formula for the weak solution of pure initial value problem.

1. Introduction

LeFloch [9] constructed an explicit formula for the solution to initial-value problem

$$\begin{aligned} u_t + f(u)_x &= 0, \\ v_t + f'(u)v_x &= 0, \end{aligned} \tag{1.1}$$

with initial conditions

$$\begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix}, \tag{1.2}$$

in the domain $\{(x,t) : -\infty < x < \infty, t > 0\}$, where $f(u)$ is strictly convex. The first equation is a convex conservation law and the Lax formula [8] gives the entropy weak solution $u(x,t)$ when the initial data $u(x,0) = u_0(x)$ is in the space of bounded measurable functions. The solution $u(x,t)$ remains in the space bounded functions and is locally a BV function for $t > 0$. Then the second equation for v is a nonconservative scalar equation with bounded and BV_{loc} function $f'(u)$ as coefficient and LeFloch [9] gave an explicit formula for the solution $v(x,t)$ satisfying initial data $v(x,0) = v_0(x)$, when v_0 is Lipschitz continuous. To justify the nonconservative product which appear in the second equation Volpert product [11] was used and the second equation was interpreted in the sense of measures.

In this paper we study (1.1) in the quarter plane $\{(x,t) : x > 0, t > 0\}$, supplemented with an initial condition at $t = 0$

$$\begin{pmatrix} u(x,0) \\ v(x,0) \end{pmatrix} = \begin{pmatrix} u_0(x) \\ v_0(x) \end{pmatrix} \tag{1.3}$$

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and a weak form of the Dirichlet boundary condition,

$$\begin{pmatrix} u(0,t) \\ v(0,t) \end{pmatrix} = \begin{pmatrix} u_b(t) \\ v_b(t) \end{pmatrix} \tag{1.4}$$

where $u_0(x)$ is bounded measurable and $v_0(x)$ are Lipschitz continuous functions of x and $u_b(t)$ and $v_b(t)$ are Lipschitz continuous functions of t. Indeed with strong form of Dirichlet boundary conditions (1.4), there is neither existence nor uniqueness as the speed of propagation $\lambda = f'(u)$ depends on the unknown variable u and does not have a definite sign at the boundary $x = 0$. We note that the speed is completely determined by the first equation. We use the Bardos Leroux and Nedelec [1] formulation of the boundary condition for the u component which for our case is equivalent to the following condition (see LeFloch [10]):

$$\begin{gathered} \text{either } u(0+,t) = u_b^+(t) \\ \text{or } f'(u(0+,t)) \leq 0 \text{ and } f(u(0+,t)) \geq f(u_b^+(t)). \end{gathered} \tag{1.5}$$

Here $u_b^+(t) = \max\{u_b(t), \lambda\}$ where λ is the unique point where $f'(u)$ changes sign. Because of convexity of f, $f(\lambda) = \inf f(u)$. There are explicit representations of the entropy weak solution of of the first component u of (1.1) with initial condition $u(x,0) = u_0(x)$ and the boundary condition (1.5) by Joseph and Gowda [5] and LeFloch [10]. We use the formula in [5] for u which involve a minimization of functionals on certain class of paths and generalized characteristics. Once u is obtained, the equation for v is linear equation with a discontinuous coefficient $f'(u(x,t))$. Now $v(0+,t) = v_b(t)$ is prescribed only if the characteristics at $(0,t)$ has positive speed, ie $f'(u(0+,t)) > 0$. So the weak form of boundary conditions for v component is

$$\text{if } f'(u(0+,t)) > 0, \text{ then } v(0+,t) = v_b(t). \tag{1.6}$$

The aim of this paper is to construct explicit formula for (1.1), with initial condition (1.3) and boundary conditions (1.5) and (1.6). We also indicate some generalizations to some other systems. The question of uniqueness is under investigation.

2. A formula for the solution

In this section, using the explicit formula derived in [3, 5] for the scalar convex conservation laws with initial condition and Bardos Leroux and Nedelec boundary condition (1.6), we construct a solution for the problem stated in the introduction. To be more precise, We assume $f(u)$ satisfies the following conditions

$$f''(u) > 0, \quad \lim_{u\to\infty} \frac{f(u)}{u} = \infty, \tag{2.1}$$

and let $f^*(u)$ be the convex dual of $f(u)$ namely, $f^*(u) = \max_\theta[\theta u - f(\theta)]$.

For each fixed $(x,y,t), x > 0, y \geq 0, t > 0$, $C(x,y,t)$ denotes the following class of paths β in the quarter plane $D = \{(z,s) : z \geq 0, s \geq 0\}$. Each path is connected from the initial point $(y,0)$ to (x,t) and is of the form $z = \beta(s)$, where β is a piecewise linear function of maximum three lines and always linear in the interior of D. Thus for $x > 0$ and $y > 0$, the curves are either a straight line or have exactly three straight lines with one lying on the boundary $x = 0$. For $y = 0$ the curves are made up of one straight line or two straight lines with one piece lying on the boundary $x = 0$. Associated with the flux $f(u)$ and boundary data $u_b(t)$, we define the functional $J(\beta)$ on $C(x, y, t)$

$$J(\beta) = -\int_{\{s:\beta(s)=0\}} f(u_B(s)^+)ds + \int_{\{s:\beta(s)\neq 0\}} f^*(\frac{d\beta(s)}{ds})ds. \tag{2.2}$$

We call β_0 is straight line path connecting $(y, 0)$ and (x, t) which does not touch the boundary $x = 0$, $\{(0, t), t > 0\}$, then let

$$A(x, y, t) = J(\beta_0) = tf * (\frac{x - y}{t}). \tag{2.3}$$

For any $\beta \in C^*(x, y, t) = C(x, y, t) - \beta_0$, that is made up of three straight lines connecting $(y, 0)$ to $(0, t_1)$ in the interior and $(0, t_1)$ to $(0, t_2)$ on the boundary and $(0, t_2)$ to (x, t) in the interior, it can be easily seen from (2.2) that

$$J(\beta) = J(x, y, t, t_1, t_2) = -\int_{t_1}^{t_2} f(u_B(s)^+)ds + t_1 f^*(\frac{y}{-t_1}) + (t - t_2)f^*(\frac{x}{t - t_2}). \tag{2.4}$$

For the curves made up two straight lines with one piece lying on the boundary $x = 0$ which connects $(0, 0)$ and $(0, t_2)$ and the other connecting $(0, t_2)$ to (x, t).

$$J(\beta) = J(x, y, t, t_1 = 0, t_2) = -\int_0^{t_2} f(u_B(s)^+)ds + (t - t_2)f^*(\frac{x}{t - t_2}).$$

It was proved in [3, 5], that there exists a $\beta^* \in C^*(x, y, t)$ or correspondingly $t_1(x, y, t)$, $t_2(x, y, t)$ so that

$$\begin{aligned} B(x, y, t) &= J(\beta^*) \\ &= \min\{J(\beta) : \beta \in C^*(x, y, t)\} \\ &= \min\{J(x, y, t, t_1, t_2) : 0 \le t_1 < t_2 < t\} \\ &= J(x, y, t, t_1(x, y, t), t_2(x, y, t)) \end{aligned} \tag{2.5}$$

is a Lipschitz continuous so that

$$\begin{aligned} Q(x, y, t) &= \min\{J(\beta) : \beta \in C(x, y, t)\} \\ &= \min\{A(x, y, t), B(x, y, t)\}, \end{aligned} \tag{2.6}$$

and

$$U(x, t) = \min\{Q(x, y, t) + U_0(z),\ 0 \le y < \infty\} \tag{2.7}$$

are Lipschitz continuous functions in their variables, where $U_0(y) = \int_0^y u_0(z)dz$. Further minimum in (2.7) is attained at some value of $y \ge 0$ which depends on (x, t), we call it $y(x, t)$. If $A(x, y(x, t), t) \le B(x, y(x, t), t)$

$$U(x, t) = tf^*(\frac{x - y(x, t)}{t}) + U_0(y), \tag{2.8}$$

and if $A(x, y(x, t), t) > B(x, y(x, t), t)$

$$U(x, t) = J(x, y(x, t), t, t_1(x, y(x, t), t), t_2(x, y(x, t), t)) + U_0(y). \tag{2.9}$$

Here and hence forth $y(x, t)$ is a minimizer in (2.7) and in the case of (2.9), $t_2(x, t) = t_2(x, y(x, t), t)$ and $t_1(x, t) = t_1(x, y(x, t), t)$.

Theorem 2.1. *For every (x,t) minimum in (2.7) is achieved by some $y(x,t)$, and $U(x,t)$ is a Lipschitz continuous and for almost every (x,t) there is only one minimizer $y(x,t)$.*

For every points (x,t) satisfying $U(x,t) = A(x,y(x,t),t) \leq B(x,y(x,t),t)$, define

$$\begin{aligned} u(x,t) &= (f^*)'(\frac{x-y(x,t)}{t}) \\ v(x,t) &= v_0(y(x,t)). \end{aligned} \tag{2.10}$$

and for the points (x,t) where $B(x,y(x,t),t) < A(x,y(x,t),t)$, define

$$\begin{aligned} u(x,t) &= (f^*)'(\frac{x}{t-t_2(x,t)}) \\ v(x,t) &= v_b(t_2(x,t)). \end{aligned} \tag{2.11}$$

Then the function $(u(x,t),v(x,t))$ is a weak solution of (1.1), satisfying the initial condition (1.3) and boundary conditions (1.5) and (1.6)

Proof. First we recall from [3, 5] some properties of minimizers $y(x,t)$ in (2.7) and corresponding $t_2(x,t)$ and $t_1(x,t)$ that are required for our analysis. These minimizers $y(x,t)$ may not be unique for every (x,t). Let $y^-(x,t)$ and $y^+(x,t)$ are the smallest and the largest of the minimizers in (2.7), for each $t > 0$, they are nondecreasing function of x and hence except for a countable number of points they are equal. Corresponding $t_2^-(x,t)$ and $t_2^+(x,t)$ have the following properties. They are nondecreasing function of x, for each fixed t and except for a countable number of points x they are equal and nondecreasing function of t, for each fixed x and except for a countable number of points t they are equal.

Further if $A(x,y(x,t),t) < B(x,y(x,t),t)$, for some $x = x_0$ then this continues to be so for all $x < x_0$ and if $A(x,y(x,t),t) > B(x,y(x,t),t)$, for some $x = x_0$ then this continues to be so for all $x > x_0$.

It was proved in [5], that $u(x,t) = Q_1(x,y(x,t)) = \partial_x U(x,t)$ where $Q_1(x,y,t) = \partial_x Q(x,y,t)$, is the weak solution of

$$u_t + f(u)_x = 0 \tag{2.12}$$

satisfying the initial condition $u(x,0) = u_0(x)$ and weak form of boundary condition (1.5). To show that v satisfies the second equation, we follow LeFloch [9] and use the nonconservative product of Volpert [11] in sense of measures. Since u is a function of bounded variation, we write

$$[0,\infty) \times [0,\infty) = S_c \cup S_j \cup S_n$$

where S_c and S_j are points of approximate continuity of u and points of approximate jump of u and S_n is a set of one dimensional Hausdorff-measure zero. At any point $(x,t) \in S_j$, $u(x-0,t)$ and $u(x+0,t)$ denote the left and right values of $u(x,t)$. For any continuous function $g : R^1 \to R^1$, the Volpert product $g(u)v_x$ is defined as a Borel measure in the following manner. Consider the averaged superposition of $g(u)$ (see Volpert [11])

$$\overline{g(u)}(x,t) = \begin{cases} g(u(x,t)), & \text{if } (x,t) \in S_c, \\ \int_0^1 g(1-\alpha)(u(x-,t) + \alpha u(x+,t)) d\alpha, & \text{if } (x,t) \in S_j \end{cases} \tag{2.13}$$

and the associated measure

$$[g(u)v_x](A) = \int_A \overline{g(u)}(x,t) v_x \tag{2.14}$$

where A is a Borel measurable subset of S_c and

$$[g(u)v_x](\{(x,t)\}) = \overline{g(u)}(x,t)(v(x+0,t) - v(x-0,t)) \tag{2.15}$$

provided $(x,t) \in S_j$. The second equation in (1.1) is understood as

$$\mu = v_t + \overline{f'(u)}(u)v_x = 0 \tag{2.16}$$

in the sense of measures. Let $(x,t) \in S_c$ and $u = f^{*\prime}(\frac{x-y(x,t)}{t})$, since u satisfies (2.12), we have

$$f''(u)\{-\frac{(x-y(x,t))}{t^2} - \frac{\partial_t y(x,t)}{t} + f'(u)\frac{(1-\partial_x y(x,t))}{t}\} = 0.$$

This can be written as

$$f''(u)\{-\frac{1}{t}[(\frac{(x-y(x,t))}{t} - f'(u))] - \frac{1}{t}[\partial_t y(x,t) + f'(u)\partial_x y(x,t)]\} = 0. \tag{2.17}$$

Using $f''(u) > 0$ and $f'(u)$ and $(f^{*\prime})(u)$ are inverses of each other, it follows from (2.17) that

$$\partial_t y(x,t) + f'(u)\partial_x y(x,t) = 0. \tag{2.18}$$

Now

$$\partial_t v(x,t) + f'(u)\partial_x v(x,t) = (\frac{dv_0}{dx})(y(x,t)\{\partial_t y(x,t) + f'(u)\partial_x y(x,t)\}$$

and from (2.18), we get

$$\partial_t v(x,t) + f'(u)\partial_x v(x,t) = 0. \tag{2.19}$$

Similarly if $(x,t) \in S_c$ and $u(x,t) = f^{*\prime}(\frac{x}{t-t_2(x,y(x,t),t)})$, we can show that

$$\partial_t(t_2(x,y(x,t),t)) + f'(u(x,t))\partial_x(t_2(x,y(x,t),t) = 0$$

and hence

$$\partial_t v(x,t) + f'(u)\partial_x v(x,t) = 0, \tag{2.20}$$

So from (2.19) and (2.20), for any Borel subset A of S_c

$$\mu(A) = 0. \tag{2.21}$$

Now we consider a point $(s(t),t) \in S_j$, then

$$\frac{ds(t)}{dt} = \frac{f(u(s(t)+,t)) - f(u(s(t)-,t))}{u(s(t)+,t) - u(s(t)-,t)}$$

is the speed of propagation of the discontinuity at this point.

$$\begin{aligned}
&\mu\{(s(t),t)\} \\
&= -\frac{ds(t)}{dt}(v(s(t)+,t) - v(s(t)-,t)) \\
&\quad + \int_0^1 f'(u(s(t)-,t) + \alpha(u(s(t)+,t) - u(s(t)-,t))d\alpha(v(s(t)+,t) - v(s(t)-,t)) \\
&= [-\frac{ds(t)}{dt} + \frac{f(u(s(t)+,t)) - f(u(s(t)-,t))}{u(s(t)+,t) - u(s(t)-,t)}](v(s(t)+,t) - v(s(t)-,t)) \\
&= 0.
\end{aligned} \tag{2.22}$$

Form (2.21) and (2.22), (2.16) follows.

To show that the solution satisfies the initial conditions, first we observe that given $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \geq \epsilon$, $t \leq \delta$,

$$u(x,t) = (f^*)'(\frac{x - y(x,t)}{t})$$

where $y(x,t)$ minimizes $\min_{y\geq 0}[U_0(y) + tf^*(\frac{x-y}{t})]$ see [5]. So u and v are given by the formula (2.10). Then Lax's argument [8], gives $\lim_{t\to 0} u(x,t) = u_0(x)$ a.e. $x \geq \epsilon$. Since $\epsilon > 0$ is arbitrary,

$$\lim_{t\to 0} u(x,t) = u_0(x), \;\; a.e.\ x.$$

Since f' and $f^{*\prime}$ are inverses of each other $y(x,t) - x = -tf'(u(x,t))$, then it follows that $y(x,t) \to x$ as $t \to 0$ a.e x. Since v_0 is continuous we get

$$\lim_{t\to 0} v(x,t) = \lim_{t\to 0} v_0(y(x,t)) = v_0(x), \;\; a.e.\ x.$$

Now we show the solution satisfies the boundary condition (1.5) and (1.6). That the u component satisfies the boundary condition (1.5) is proved in [5]. Further if $f'(u(0+,t)) > 0$ then $f'(u(x,t)) > 0$ for $0 < x \leq \epsilon$ for some sufficiently small ϵ and u and v are given by (2.11). Now

$$u(x,t) = (f^*)'(\frac{x}{t - t_2(x,t}).$$

so that $t - t_2(x,t) = x/f'(u(x,t))$, and it follows that $\lim_{x\to 0} t_2(x,t) = t$, since we assumed that $\lim_{x\to 0} f'(u(x,t)) = f(u(0+,t)) > 0$. So we have

$$\lim_{x\to 0} v(x,t) = \lim_{x\to 0} v_b(t_2(x,t)) = v_b(t).$$

as v_b is continuous. This proves v satisfies the boundary condition (1.6). The proof of the theorem is complete. □

3. Extensions to some other cases

Generalized Lax equation. The initial value problem for the system

$$\begin{gathered} u_t + (\log(ae^u + be^{-u}))_x = 0 \\ v_t + \frac{ae^u - be^{-u}}{ae^u + be^{-u}} v_x = 0 \end{gathered} \tag{3.1}$$

was studied and explicit solution was constructed by Joseph and Gowda [7] using a difference scheme of Lax [8]. This system of equations is of the form (1.1),with

$$f(u) = \log(ae^u + be^{-u}) \tag{3.2}$$

For the case $f(u)$ satisfying (2.1), f^* is defined everywhere. The flux $f(u)$ given by (3.2) is convex but does not satisfies (2.1) and f^* is not defined everywhere. Indeed f^* is defined only on $(-1, 1)$ and is given by

$$f^*(u) = \frac{1}{2}\log\big((1+u)^{1+u}(1-u)^{1-u}\big) - \frac{1}{2}\log\big(4a^{1+u}b^{1-u}\big) \tag{3.3}$$

and its derivative is

$$f^{*\prime}(u) = \frac{1}{2}\log(\frac{b}{a}\frac{1+u}{1-u}). \tag{3.4}$$

Explicit formula of the theorem (2.1) can be obtained for (3.1) on the domain $D = \{(x,t), x > 0, t > 0\}$ with initial condition (1.3) and boundary conditions (1.5) and (1.6) with minor modifications. Here we define $C(x,y,t)$, the set of curves β

as in section 2, but with a restriction on its slope $|\frac{d\beta(s)}{ds}| < 1$. Using the same notations as in theorem, and using the explicit form of $f^{*\prime}(u)$ given by (3.4), we have the following result.

Theorem 3.1. *For every (x,t), $x > 0$, $t > 0$, let (u,v) be defined as follows: When $A(x,y(x,t),t) \le B(x,y(x,t),t)$, by*

$$u(x,t) = \frac{1}{2}\log[\frac{b}{a}\frac{t+x-y(x,t)}{t-x+y(x,t)}], \quad v(x,t) = v_0(y(x,t));$$

when $A(x,y(x,t),t) > B(x,y(x,t),t)$, by

$$u(x,t) = \frac{1}{2}\log[\frac{b}{a}\frac{t+x+t_2(x,t)}{t-x+t_2(x,t)}], \quad v(x,t) = v_b(t_2(x,t)).$$

Then (u,v) solves (3.1)*, satisfies the initial conditions* (1.3) *and the boundary conditions* (1.5) *and* (1.6).

Generalized Hopf equation. Solution for the initial-value problem for the non-conservative system for $u_j, j = 1,2,\dots,n$

$$(u_j)_t + (\sum_{k=1}^{n} c_k u_k)(u_j)_x = 0, \quad j = 1,2,\dots,n \tag{3.5}$$

was constructed by Joseph [4, 6] by a vanishing viscosity method and a generalization of Hopf-Cole transformation. Here we assume that at least one k, $c_k \neq 0$. When $n = 1, c_1 = 1$, (3.5) is the inviscid Burgers equation or the Hopf equation and Hopf [2] derived a formula for the entropy weak solution for the initial value problem and boundary case was treated in [3]. In the present discussion we consider (3.5) in $D = \{(x,t) : x > 0, t > 0\}$ with initial condition

$$u_j(x,0) = u_{0j}(x), \quad x > 0, \quad j = 1,2,\dots,n \tag{3.6}$$

and boundary conditions

$$u_j(0,t) = u_{bj}(t), \quad t > 0 \quad j = 1,2,\dots,n. \tag{3.7}$$

Here again a weak form of the boundary condition is required as characteristic speed of the system, $\sigma = \sum_{k=1}^{n} c_k u_k$ need not be positive at the boundary $x = 0$. First we note from (3.5) that u_j satisfies

$$(u_j)_t + \sigma(u_j)_x = 0, \quad j = 1,2,\dots,n \tag{3.8}$$

where σ satisfies

$$\sigma_t + (\frac{\sigma^2}{2})_x = 0. \tag{3.9}$$

Now (3.9) together with (3.8) is exactly the form of equation we have studied in section 1, with $f(u) = u^2/2$. Let σ is the entropy weak solution of (3.9) with the initial condition

$$\sigma(x,0) = \sigma_0(x) \tag{3.10}$$

and weak form of boundary condition

$$\begin{gathered} \text{either } \sigma(0+,t) = \sigma_b^+(t) \\ \text{or } \sigma(0+,t) \le 0 \text{ and } \frac{u(0+,t)}{2} \ge \frac{u_b^+(t)}{2}, \end{gathered} \tag{3.11}$$

with $\sigma_0(x) = \sum_{k=1}^{n} c_k u_{0k}(x)$ and $\sigma_b(t) = \sum_{k=1}^{n} c_k u_{bk}(t)$ constructed in [3, 5].

The analysis of section 1 then shows that with the formulation of boundary condition

$$\text{if } \sigma(0+,t) > 0, \text{ then } u_j(0+,t) = u_{bj}(t). \tag{3.12}$$

for u_j, Theorem (1.1) applies to the present case with $f(u) = \frac{u^2}{2}$. With the same notations as Theorem 1.1, we obtain the following theorem.

Theorem 3.2. *For $x > 0$, $t > 0$, let u_j be defined as follows:*
For points (x,t) where $U(x,t) = A(x, y(x,t), t) \leq B(x, y(x,t), t)$, define

$$u_j(x,t) = u_{0j}(y(x,t)),$$

and for the points (x,t) where $B(x, y(x,t), t) < A(x, y(x,t), t)$, define

$$u_j(x,t) = u_{bj}(t_2(x,t)).$$

Then $u_j(x,t)$, $j = 1, 2, \dots, n$ is a solution to (3.5) with initial condition (3.6) and boundary condition (3.12).

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