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EXISTENCE OF SOLUTIONS FOR P-KIRCHHOFF TYPE PROBLEMS WITH CRITICAL EXPONENT

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Abstract. We study the existence of solutions for the p-Kirchhoff type problem involving the critical Sobolev exponent,

$$-\Big[g\Big(\int_\Omega |\nabla u|^p dx\Big)\Big]\Delta_p u = \lambda f(x,u) + |u|^{p^\star-2}u \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $1 < p < N$, $p^\star = Np/(N-p)$ is the critical Sobolev exponent, λ is a positive parameter, f and g are continuous functions. The main results of this paper establish, via the variational method. The concentration-compactness principle allows to prove that the Palais-Smale condition is satisfied below a certain level.

1. Introduction and main results

We are concerned with the existence of solutions for the p-Kirchhoff type problem

$$\begin{gathered} -\Big[g\Big(\int_\Omega |\nabla u|^p dx\Big)\Big]\Delta_p u = \lambda f(x,u) + |u|^{p^\star-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $1 < p < N$, $p^\star = Np/(N-p)$ is the critical Sobolev exponent, and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions that satisfy the following conditions:

- (F1) $f(x,t) = o(|t|^{p-1})$ as $t \to 0$, uniformly for $x \in \Omega$;
- (F2) There exists $q \in (p, p^\star)$ such that
$$\lim_{|t|\to+\infty} \frac{f(x,t)}{|t|^{q-2}t} = 0, \quad \text{uniformly for} x \in \Omega.$$
- (F3) There exists $\theta \in (p/\sigma, p^\star)$ such that $0 < \theta F(x,t) \le tf(x,t)$ for all $x \in \Omega$ and $t \neq 0$, where $F(x,t) = \int_0^t f(x,s)ds$ and σ is given by (G2) below.
- (G1) There exists $\alpha_0 > 0$ such that $g(t) \ge \alpha_0$ for all $t \ge 0$;
- (G2) There exists $\sigma > p/p^\star$ such that $G(t) \ge \sigma g(t)t$ for all $t \ge 0$, where $G(t) = \int_0^t g(s)ds$;

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Much interest has grown on problems involving critical exponents, starting from the celebrated paper by Brezis and Nirenberg [5], where the case $p = 2$ is considered. We refer the reader to [1, 9, 10] and reference therein for the study of problems with critical exponent.

Problem (1.1) is a general version of a model presented by Kirchhoff [11]. More precisely, Kirchhoff introduced a model

$$\rho\frac{\partial^2 u}{\partial t^2} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |\frac{\partial u}{\partial x}|^2 dx\Big)\frac{\partial^2 u}{\partial x^2} = 0, \tag{1.2}$$

where ρ, ρ_0, h, E, L are constants, which extends the classical D'Alembert's wave equation by considering the effects of the changes in the length of the strings during the vibrations. The problem

$$\begin{gathered} -\Big(a + b\int_\Omega |\nabla u|^2 dx\Big)\Delta u = f(x,u) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered} \tag{1.3}$$

received much attention, mainly after the article by Lions [12]. Problems like (1.3) are also introduced as models for other physical phenomena as, for example, biological systems where u describes a process which depends on the average of itself (for example, population density). See [3] and its references therein. For a more detailed reference on this subject we refer the interested reader to [4, 6, 7, 8, 14, 15].

Motivated by the ideas in [2], our approach for studying problem (1.1) is variational and uses minimax critical point theorems. The difficulty is due to the lack of compactness of the imbedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$ and the Palais-Smale condition for the corresponding energy functional could not be checked directly. So the concentration-compact principle of Lions [13] is applied to deal with this difficulty.

The main result of this paper is the following theorem.

Theorem 1.1. *Suppose that* (G1)–(G2), (F1)–(F3) *hold. Then, there exists* $\lambda_* > 0$, *such that* (1.1) *has a nontrivial solution for all* $\lambda \geq \lambda_*$.

2. Preliminary results

We consider the energy functional $I : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$I(u) = \frac{1}{p}G(\|u\|^p) - \lambda\int_\Omega F(x,u)dx - \frac{1}{p^\star}\int_\Omega |u|^{p^\star} dx, \tag{2.1}$$

where $W_0^{1,p}(\Omega)$ is the Sobolev space endowed with the norm $\|u\|^p = \int_\Omega |\nabla u|^p dx$. It is well known that a critical point of I is a weak solution of problem (1.1).

To use variational methods, we give some results related to the Palais-Smale compactness condition. Recall that a sequence (u_n) is a Palais-Smale sequence of I at the level c, if $I(u_n) \to c$ and $I'(u_n) \to 0$.

In the sequel, we show that the functional I has the mountain pass geometry. This purpose is proved in the next lemmas.

Lemma 2.1. *Suppose that* (F1), (F2), (G1) *hold. Then, there exist* $r, \rho > 0$ *such that* $\inf_{\|u\|=r} I(u) \geq \rho > 0$.

Proof. It follows from (F1) and (F2) that for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$.

$$F(x,t) \leq \frac{1}{p}\varepsilon|t|^p + C(\varepsilon)|t|^q \quad \text{for all } t. \tag{2.2}$$

By (G1) and the Sobolev embdding, we have

$$\begin{aligned} I(u) &\geq \frac{\alpha_0}{p}\|u\|^p - \lambda C_1\varepsilon\|u\|^p - \lambda C_2(\varepsilon)\|u\|^q - C_3\|u\|^{p^\star} \\ &= \|u\|\Big[\big(\frac{\alpha_0}{p} - \lambda C_1\varepsilon\big)\|u\|^{p-1} - \lambda C_2(\varepsilon)\|u\|^{q-1} - C_3\|u\|^{p^\star-1}\Big]. \end{aligned} \tag{2.3}$$

Taking $\varepsilon = \alpha_0/(2p\lambda C_1)$ and setting

$$\xi(t) = \frac{\alpha_0}{2p}t^{p-1} - \lambda C_2 t^{q-1} - C_3 t^{p^\star-1}.$$

Since $p < q < p^\star$, we see that there exist $r > 0$ such that $\max_{t\geq 0}\xi(t) = \xi(r)$. Then, by (2.3), there exists $\rho > 0$ such that $I(u) \geq \rho$ for all $\|u\| = r$. ☐

Lemma 2.2. *Suppose that* (G2), (F3) *hold. Then for all* $\lambda > 0$*, there exists a nonnegative function* $e \in W_0^{1,p}(\Omega)$ *independent of* λ*, such that* $\|e\| > r$ *and* $I(e) < 0$.

Proof. Choose a nonnegative function $\phi_0 \in C_0^\infty(\Omega)$ with $\|\phi_0\| = 1$. By integrating (G2), we obtain

$$G(t) \leq \frac{G(t_0)}{t_0^{1/\sigma}}t^{1/\sigma} = C_0 t^{1/\sigma} \quad \text{for all } t \geq t_0 > 0. \tag{2.4}$$

By (F3), $\int_\Omega F(x, t\phi_0)dx \geq 0$. Hence

$$I(t\phi_0) \leq \frac{C_0}{p}t^{p/\sigma} - \frac{t^{p^\star}}{p^\star}\int_\Omega \phi_0^{p^\star}\,dx \quad \text{for all } t \geq t_0.$$

Since $p/\sigma < p^\star$, the lemma is proved by choosing $e = t_*\phi_0$ with $t_* > 0$ large enough. ☐

In view of Lemmas 2.1 and 2.2, we may apply a version of the Mountain Pass theorem without Palais-Smale condition to obtain a sequence $(u_n) \subset W_0^{1,p}(\Omega)$ such that

$$I(u_n) \to c_* \quad \text{and} \quad I'(u_n) \to 0,$$

where

$$c_* = \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I(\gamma(t)) > 0, \tag{2.5}$$

with

$$\Gamma = \big\{\gamma \in C([0,1], W_0^{1,p}(\Omega)) : \gamma(0) = 0,\, I(\gamma(1)) < 0\big\}.$$

Denoted by S_* the best positive constant of the Sobolev embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^\star}(\Omega)$ given by

$$S_* = \inf\Big\{\int_\Omega |\nabla u|^p dx : u \in W_0^{1,p}(\Omega),\ \int_\Omega |u|^{p^\star}dx = 1\Big\}. \tag{2.6}$$

Lemma 2.3. *Suppose that* (G1)–(G2), (F1)–(F3) *hold. Then there exists* $\lambda_* > 0$ *such that* $c_* \in \big(0, (\frac{1}{\theta} - \frac{1}{p^\star})(\alpha_0 S_*)^{\frac{N}{p}}\big)$ *for all* $\lambda \geq \lambda_*$*, where* c_* *is given by* (2.5).

Proof. For e given by Lemma 2.2, we have $\lim_{t\to+\infty} I(te) = -\infty$, then there exists $t_\lambda > 0$ such that $I(t_\lambda e) = \max_{t\geq 0} I(te)$. Therefore,

$$t_\lambda^{p-1} g(\|t_\lambda e\|^p)\|e\|^p = \lambda\int_\Omega f(x, t_\lambda e)e\,dx + t_\lambda^{p^\star-1}\int_\Omega e^{p^\star}dx;$$

thus

$$g(\|t_\lambda e\|^p)\|t_\lambda e\|^p = \lambda t_\lambda \int_\Omega f(x, t_\lambda e)e\,dx + t_\lambda^{p^\star}\int_\Omega e^{p^\star}\,dx. \tag{2.7}$$

By (2.4), it follows that

$$\frac{C_0}{\sigma}\|e\|^{p/\sigma} t_\lambda^{p/\sigma} \geq t_\lambda^{p^\star}\int_\Omega e^{p^\star}\,dx, \quad \text{with } t_0 < t_\lambda.$$

Since $p/\sigma < p^\star$, (t_λ) is bounded. So, there exists a sequence $\lambda_n \to +\infty$ and $s_0 \geq 0$ such that $t_{\lambda_n} \to s_0$ as $n \to \infty$. Hence, there exists $C > 0$ such that

$$g(\|t_{\lambda_n} e\|^p)\|t_{\lambda_n} e\|^p \leq C \quad \text{for all } n;$$

that is,

$$\lambda_n t_{\lambda_n}\int_\Omega f(x, t_{\lambda_n} e)e\,dx + t_{\lambda_n}^{p^\star}\int_\Omega e^{p^\star}\,dx \leq C \quad \text{for all } n.$$

If $s_0 > 0$, the above inequality implies that

$$\lambda_n t_{\lambda_n}\int_\Omega f(x, t_{\lambda_n} e)e\,dx + t_{\lambda_n}^{p^\star}\int_\Omega e^{p^\star}\,dx \to +\infty \leq C, \quad \text{as } n \to \infty,$$

which is impossible, and consequently $s_0 = 0$. Let $\gamma_*(t) = te$. Clearly $\gamma_* \in \Gamma$, thus

$$0 < c_* \leq \max_{t\geq 0} I(\gamma_*(t)) = I(t_\lambda e) \leq \frac{1}{p}G(\|t_\lambda e\|^p).$$

Since $t_{\lambda_n} \to 0$ and $(\frac{1}{\theta} - \frac{1}{p^\star})(\alpha_0 S_*)^{N/p} > 0$, for $\lambda > 0$ sufficiently large, we have

$$\frac{1}{p}G(\|t_\lambda e\|^p) < \big(\frac{1}{\theta} - \frac{1}{p^\star}\big)(\alpha_0 S_*)^{N/p},$$

and hence

$$0 < c_* < \big(\frac{1}{\theta} - \frac{1}{p^\star}\big)(\alpha_0 S_*)^{N/p}.$$

This completes the proof. □

Proof of Theorem 1.1. From Lemmas 2.1, 2.2 and 2.3, there exists a sequence $(u_n) \subset W_0^{1,p}(\Omega)$ such that

$$I(u_n) \to c_* \quad \text{and} \quad I'(u_n) \to 0, \tag{2.8}$$

with $c_* \in \big(0, (\frac{1}{\theta} - \frac{1}{p^\star})(\alpha_0 S_*)^{N/p}\big)$ for $\lambda \geq \lambda_*$. Then, there exists $C > 0$ such that $|I(u_n)| \leq C$, and by (F3) for n large enough, it follows from (G1) and (G2) that

$$\begin{aligned} C + \|u_n\| &\geq I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n\rangle \\ &\geq \frac{1}{p}G(\|u\|^p) - \frac{1}{\theta}g(\|u_n\|^p)\|u_n\|^p \\ &\geq \big(\frac{\sigma}{p} - \frac{1}{\theta}\big)\alpha_0\|u_n\|^p. \end{aligned} \tag{2.9}$$

Since $\theta > p/\sigma$, (u_n) is bounded. Hence, up to a subsequence, we may assume that

$$\begin{gathered} u_n \rightharpoonup u \quad \text{weakly in } W_0^{1,p}(\Omega), \\ u_n \to u \quad \text{a.e. in } \Omega, \\ u_n \to u \quad \text{in } L^s(\Omega),\ 1 \leq s < p^\star, \\ |\nabla u_n|^p \rightharpoonup \mu \quad \text{(weak*-sense of measures)} \\ |u_n|^{p^\star} \rightharpoonup \nu \quad \text{(weak*-sense of measures),} \end{gathered} \tag{2.10}$$

where μ and ν are a nonnegative bounded measures on $\overline{\Omega}$. Then, by concentration-compactness principle due to Lions [13], there exists some at most countable index set J such that

$$\begin{gathered}\nu = |u|^{p^\star} + \sum_{j\in J} \nu_j \delta_{x_j}\,, \quad \nu_j > 0, \\ \mu \geq |\nabla u|^p + \sum_{j\in J} \mu_j \delta_{x_j}\,, \quad \mu_j > 0, \\ S_* \nu_j^{p/p^\star} \leq \mu_j, \end{gathered} \tag{2.11}$$

where δ_{x_j} is the Dirac measure mass at $x_j \in \overline{\Omega}$.

Let $\psi(x) \in C_0^\infty$ such that $0 \leq \psi \leq 1$,

$$\psi(x) = \begin{cases} 1 & \text{if } |x| < 1 \\ 0 & \text{if } |x| \geq 2 \end{cases} \tag{2.12}$$

and $|\nabla \psi|_\infty \leq 2$.

For $\varepsilon > 0$ and $j \in J$, denote $\psi_\varepsilon^j(x) = \psi((x - x_j)/\varepsilon)$. Since $I'(u_n) \to 0$ and $(\psi_\varepsilon^j u_n)$ is bounded, $\langle I'(u_n), \psi_\varepsilon^j u_n \rangle \to 0$ as $n \to \infty$; that is,

$$\begin{aligned} &g(\|u_n\|^p) \int_\Omega |\nabla u_n|^p \psi_\varepsilon^j dx \\ &= -g(\|u_n\|^p) \int_\Omega u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \psi_\varepsilon^j dx \\ &\quad + \lambda \int_\Omega f(x, u_n) u_n \psi_\varepsilon^j dx + \int_\Omega |u_n|^{p^\star} \psi_\varepsilon^j dx + o_n(1). \end{aligned} \tag{2.13}$$

By (2.10) and Vitali's theorem, we see that

$$\lim_{n\to\infty} \int_\Omega |u_n \nabla \psi_\varepsilon^j|^p dx = \int_\Omega |u \nabla \psi_\varepsilon^j|^p dx$$

Hence, by Hölder's inequality we obtain

$$\begin{aligned} &\limsup_{n\to\infty} \Big| \int_\Omega u_n |\nabla u_n|^{p-2} \nabla u_n \nabla \psi_\varepsilon^j dx \Big| \\ &\leq \limsup_{n\to\infty} \Big(\int_\Omega |\nabla u_n|^p dx \Big)^{(p-1)/p} \Big(\int_\Omega |u_n \nabla \psi_\varepsilon^j|^p dx \Big)^{1/p} \\ &\leq C_1 \Big(\int_{B(x_j, 2\varepsilon)} |u|^p |\nabla \psi_\varepsilon^j|^p dx \Big)^{1/p} \\ &\leq C_1 \Big(\int_{B(x_j, 2\varepsilon)} |\nabla \psi_\varepsilon^j|^N dx \Big)^{1/N} \Big(\int_{B(x_j, 2\varepsilon)} |u|^{p^\star} dx \Big)^{1/p^\star} \\ &\leq C_2 \Big(\int_{B(x_j, 2\varepsilon)} |u|^{p^\star} dx \Big)^{1/p^\star} \to 0 \quad \text{as } \varepsilon \to 0\,. \end{aligned} \tag{2.14}$$

On the other hand, from (2.10) we have

$$f(x, u_n) u_n \to f(x, u) u \quad \text{a.e. in } \Omega,$$

and $u_n \to u$ strongly in $L^p(\Omega)$ and in $L^q(\Omega)$. By (F1)–(F3), for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$|f(x, t)| \leq \varepsilon |t|^{p-1} + C_\varepsilon |t|^{q-1} \quad \text{for all } (x, t) \in \Omega \times \mathbb{R}; \tag{2.15}$$

thus

$$|f(x,u_n)u_n| \le \varepsilon |u_n|^p + C_\varepsilon |u_n|^q.$$

This is what we need to apply Vitali's theorem, which yields

$$\lim_{n\to\infty} \int_\Omega f(x,u_n)u_n dx = \int_\Omega f(x,u)u\, dx.$$

Since ψ_ε^j has compact support, letting $n \to \infty$ in (2.13) we deduce from (2.10) and (2.14) that

$$\alpha_0 \int_\Omega \psi_\varepsilon^j d\mu \le C_2 \Big(\int_{B(x_j,2\varepsilon)} |u|^{p^\star} dx \Big)^{1/p^\star} + \lambda \int_{B(x_j,2\varepsilon)} f(x,u)u dx + \int_\Omega \psi_\varepsilon^j d\nu.$$

Letting $\varepsilon \to 0$, we obtain $\alpha_0 \mu_j \le \nu_j$. Therefore,

$$(\alpha_0 S_*)^{N/p} \le \nu_j. \tag{2.16}$$

We will prove that this inequality is not possible. Let us assume that $(\alpha_0 S_*)^{N/p} \le \nu_{j_0}$ for some $j_0 \in J$. From (G2) we see that

$$\frac{1}{p} G(\|u_n\|^p) - \frac{1}{\theta} g(\|u_n\|^p)\|u_n\|^p \ge 0 \quad \text{for all } n.$$

Since

$$c_* = I(u_n) - \frac{1}{\theta}\langle I'(u_n), u_n\rangle + o_n(1),$$

it follows that

$$\begin{aligned} c_* &\ge \big(\frac{1}{\theta} - \frac{1}{p^\star}\big) \int_\Omega |u_n|^{p^\star} dx + o_n(1) \\ &\ge \big(\frac{1}{\theta} - \frac{1}{p^\star}\big) \int_\Omega \psi_\varepsilon^{j_0} |u_n|^{p^\star} dx + o_n(1) \end{aligned}$$

Letting $n \to \infty$, we obtain

$$c_* \ge \big(\frac{1}{\theta} - \frac{1}{p^\star}\big) \sum_{j\in J} \psi_\varepsilon^{j_0}(x_j)\nu_j \ge \big(\frac{1}{\theta} - \frac{1}{p^\star}\big)(\alpha_0 S_*)^{N/p}.$$

This contradicts Lemma 2.3. Then $J = \emptyset$, and hence $u_n \to u$ in $L^{p^\star}(\Omega)$. By (2.15) we have

$$\begin{aligned} \int_\Omega |f(x,u_n)(u_n-u)| dx &\le \int_\Omega \big(\varepsilon |u_n|^{p-1} + C_\varepsilon |u_n|^{q-1}\big)|u_n - u| dx \\ &\le \varepsilon \Big(\int_\Omega |u_n|^p dx \Big)^{p-1)/p} \Big(\int_\Omega |u_n - u|^p dx \Big)^{1/p} \\ &\quad + C_\varepsilon \Big(\int_\Omega |u_n|^q dx \Big)^{(q-1)/q} \Big(\int_\Omega |u_n - u|^q dx \Big)^{1/q}. \end{aligned}$$

Then, using again (2.10), we obtain

$$\lim_{n\to\infty} \int_\Omega f(x,u_n)(u_n - u) dx = 0. \tag{2.17}$$

Since $u_n \to u$ in $L^{p^\star}(\Omega)$, we see that

$$\lim_{n\to\infty} \int_\Omega |u_n|^{p^\star - 2} u_n (u_n - u) dx = 0. \tag{2.18}$$

From $\langle I'(u_n), u_n - u\rangle = o_n(1)$, we deduce that

$$\langle I'(u_n), u_n - u\rangle = g(\|u_n\|^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx \\ - \lambda\int_\Omega f(x,u_n)(u_n-u)dx - \int_\Omega |u_n|^{p^\star-2}u_n(u_n-u)dx = o_n(1)$$

This, (2.17) and (2.18) imply

$$\lim_{n\to\infty} g(\|u_n\|^p)\int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx = 0.$$

Since u_n is bounded and g is continuous, up to subsequence, there is $t_0 \geq 0$ such that

$$g(\|u_n\|^p) \to g(t_0^p) \geq \alpha_0, \quad \text{as } n \to \infty,$$

and so

$$\lim_{n\to\infty} \int_\Omega |\nabla u_n|^{p-2}\nabla u_n \nabla(u_n - u)dx = 0.$$

Thus by the (S_+) property, $u_n \to u$ strongly in $W_0^{1,p}(\Omega)$, and hence $I'(u) = 0$. The proof is complete. □

3. A special case

We consider the problem

$$\begin{gathered} -\Big(\alpha + \beta\int_\Omega |\nabla u|^p dx\Big)\Delta_p u = \lambda f(x,u) + |u|^{p^\star-2}u \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{3.1}$$

where Ω is a bounded smooth domain of $\mathbb{R}^N$, $1 < p < N < 2p$, α and β are a positive constants.

Set $g(t) = \alpha + \beta t$. Then, $g(t) \geq \alpha$ and

$$G(t) = \int_0^1 g(s)ds = \alpha t + \frac{1}{2}\beta t^2 \geq \frac{1}{2}(\alpha + \beta t)t = \sigma g(t)t$$

where $\sigma = 1/2$. Hence the conditions (G1) and (G2) are satisfied.

For this case, a typical example of a function satisfying the conditions (F1)–(F3) is given by

$$f(x,t) = \sum_{i=1}^k a_i(x)|t|^{q_i-2}t,$$

where $k \geq 1$, $2p < q_i < p^\star$ and $a_i(x) \in C(\overline{\Omega})$. In view of Theorem 1.1, we have the following corollary.

Corollary 3.1. *Suppose that* (F1)–(F3) *hold. Then, there exists* $\lambda_* > 0$*, such that problem* (3.1) *has a nontrivial solution for all* $\lambda \geq \lambda_*$.

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