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Global dynamics of a class of age-infection structured cholera model with immigration

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Abstract. This paper is concerned with a class of age-structured cholera model with general infection rates. We first explore the existence and uniqueness, dissipativeness and persistence of the solutions, and the existence of the global attractor by verifying the asymptotical smoothness of the orbits. We then give mathematical analysis on the existence and local stability of the positive equilibrium. Based on the preparation, we further investigate the global behavior of the cholera infection model. Corresponding numerical simulations have been presented. Our results improve and generalize some known results on cholera models.

Keywords: cholera, age-structured, nonlinear incidence, global dynamics, Lyapunov functional.

2020 Mathematics Subject Classification: 92D30, 34K20, 35B35.

1 Introduction

Cholera is an acute water-borne infectious disease caused by *Vibrio cholerae*, with an estimated disease burden of 1.3 to 4.0 million cases and 21 000 to 143 000 deaths every year worldwide, which still affects at least 47 countries around the globe [2]. At present, there are 139 serogroups of *Vibrio cholerae*, of which O1 and O139 can cause cholera outbreaks. The disease peaks in the summer and it can be transmitted to humans by pathogen in the contaminated water and by person-to-person contact [20, 37]. Clinically, cholera can cause severe diarrhea, and the infected person will die of dehydration within a few days without prompt treatment [12]. In 1855, the British scholar John Snow found that the sewage in the city was the source of the spread of cholera epidemic [36], which was a major historical event in public hygiene. In the history of human epidemiology, cholera broke out many times in different countries and regions. In recent years, cholera outbreaks are mainly concentrated in developing countries with low medical and health level and lack of safe and hygienic drinking water sources. For example, cholera broke out in Haiti in 2010, leading to more than 665000 confirmed cases and 8183 deaths [10], and one of the causal factors for this outbreak is the transmission of local

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water source Artibonite river. The incidence rate of cholera will decrease in the future due to the global economic development and the reduction in global poverty [44], however, may increase in the next few decades due to the climate change and ocean changes caused by the extreme weather [1]. Therefore, it is of great theoretical and practical significance to study the transmission mechanism and development trend of cholera.

Recently, the mathematical model of cholera transmission has attracted widespread attention since the earlier study [9] on cholera modeling for the outbreak in the European Mediterranean region. In the aspect of mathematical modeling, Tien and Earn [37] introduced a water compartment into classical SIR model and established a water-borne infectious disease model with multiple transmission routes described by ordinary differential equations. In [37], the susceptible individual can not only be infected by the infected individual, but also be infected by indirect intake of contaminated water from the environment, which could be used to describe the transmission dynamics of cholera. By constructing an appropriate Lyapunov function, the global asymptotical stability of the equilibria of the system was obtained. Considering the hyperinfectious state of vibrio cholerae, Hartley et al. [20] extended the model proposed in [37] and studied the impact of hyperinfectious state on limiting the spread of cholera. Eisenberg et al. [15] aimed to evaluate the effects of patch structure on cholera spread and the type/target reproduction numbers were derived to quantify the strategies of cholera prevention. Some models involving different factors of cholera can be found in [3,33,38,40,41] and the references therein.

In modeling of epidemics, the age structure of individuals and pathogen is a significant characteristic [4, 8, 13, 27, 42]. In [7], Brauer et al. proposed an age-structured cholera model with multiple transmission pathways, which is

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_s - \mu S(t) - \beta_i S(t) \displaystyle\int_0^\infty q(b)p(t,b)\mathrm{d}b - \beta_d S(t) \int_0^\infty k(a)i(t,a)\mathrm{d}a, \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\delta(a)i(t,a), \\ \dfrac{\partial p(t,b)}{\partial t} + \dfrac{\partial p(t,b)}{\partial b} = -\gamma(b)p(t,b), \end{cases} \tag{1.1}$$

with initial condition

$$S(0) = S_0, \quad i(0,\cdot) = i_0(\cdot) \in \mathcal{L}^1_+(0,+\infty), \quad p(0,\cdot) = p_0(\cdot) \in \mathcal{L}^1_+(0,+\infty), \tag{1.2}$$

and boundary condition

$$\begin{cases} i(t,0) = \beta_i S(t) \displaystyle\int_0^\infty q(b)p(t,b)\mathrm{d}b + \beta_d S(t) \int_0^\infty k(a)i(t,a)\mathrm{d}a, \\ p(t,0) = \displaystyle\int_0^\infty \xi(a)i(t,a)\mathrm{d}a. \end{cases} \tag{1.3}$$

where $S(t)$ is the density of the susceptible population at time t, $i(t,a)$ and $p(t,b)$ are the densities of the infectious population and the pathogen at time t with age a and b, respectively. The parameters of model (1.1) are explained in Table 1.1. Brauer et al. successfully obtained the global dynamics of model (1.1) by using the method of Lyapunov functional. Moreover, some other results for model (1.1) can be found in the studies [7, 42], such as relative compactness of orbits and uniform persistence.

More and more studies showed that immigration of populations has a significant impact on the spread of cholera. Due to the drought, refugees from Mozambique poured into Zimbabwe at the end of 1992, making Zimbabwe face the first cholera epidemic since 1985, which

Parameter	Interpretation
Λ_s	Constant recruitment of susceptible individuals
μ	Natural death rate of susceptible individuals
β_d	Direct transmission coefficient of cholera
β_i	Indirect transmission coefficient of cholera
$\delta(a)$	Age specific removal rate of the infected individuals
$\gamma(b)$	Age specific removal rate of the pathogen
$\xi(a)$	Age specific shedding rate of an infected individual
$k(a)$	Measure the Infectivity of infected individuals
$q(b)$	Measure the Infectivity of pathogen

Table 1.1: Parameters and their biological meaning in model (1.1).

spread to 7 provinces (Zimbabwe has 8 provinces in total) within five months [5]. Research shows that cholera has a history of outbreak through the immigration caused by international flights [14] and international conferences [32]. By analyzing 26 strains isolated from the cholera outbreak in Haiti in 2010, Frerichs et al. [17] believes that this wave of cholera outbreak was caused by the spread of Vibrio cholerae to the local drinking water source by the UN peacekeeping force dispatched by Nepal to Haiti. In fact, for developed countries with safe and hygienic water resources, cholera can also enter through immigration. According to the Centers for Disease Control and Prevention of USA, there was an increase in cholera cases reported in the United States during cholera outbreaks in Latin America in the 1990s and countries close to the United States such as Haiti in 2010 [11]. Five European Union countries reported 26 confirmed cholera cases in 2018, of which 22 were immigrated from India, Pakistan, Thailand, Bangladesh, Myanmar and Tunisia [16]. Therefore, it is urgent to explore the impact of immigration on the development and evolution of cholera infectious disease, which is also one of the important topics in the study of infectious disease dynamics.

From the view of mathematical modeling of infectious disease, immigration of population was always supposed to be of constant recruitment rate in each compartment. Brauer and van den Driessche [6] studied the threshold-like results for disease transmission model with immigration of the infective. By using Lyapunov function, Sigdel and McCluskey [34] investigated the global stability for an SEI model with immigration. More specifically, the endemic equilibrium for the model proposed in [34] is globally asymptotically stable. Considering the vaccination effect in the modeling of infectious diseases, Henshaw and McCluskey [21] presented the results on the global stability of a vaccination model with immigration, by virtue of the key method of constructing appropriate Lyapunov function. Meanwhile, age-dependent immigration rate seems more realistic in the real world and it is meaningful to investigate the age-structured models with immigration. In [30], McCluskey introduced an age-structured epidemic model with immigration. With an ingenious Lyapunov functional, the stability of endemic equilibrium for the SEI model with immigration was proved successfully. Zhang and Liu [46] further extended the study in [30] by introducing general nonlinear incidence. More recently, McCluskey [31] proved a general result for a Lyapunov calculation for the model with immigration and applied the results to a multi-group SIR model.

In (1.1), the incidence rates are assumed to be bilinear. Actually, nonlinear incidence rates are critical for accounting for a variety of nonlinear features of the corresponding biological phenomena. For example, Beddington–DeAngelis [23], Holling type II [24], Crowley–Martin

[45] and general incidence [18,26]. Motivated by the above studies, in this paper, we shall consider a generalization of model (1.1) by taking general incidence rates into account. However, to our best knowledge, there is no study on the age-structured cholera model with immigration. Based on model (1.1), we further introduce the immigration of infectious individuals and pathogen into the cholera model. Let $\Lambda_i(a)$ and $\Lambda_p(b)$ represent the recruitment through immigration into the infectious group and the pathogen group. Let

$$Q(t) = \int_0^\infty q(b)p(t,b)\mathrm{d}b \quad \text{and} \quad J(t) = \int_0^\infty k(a)i(t,a)\mathrm{d}a$$

represent the infectivity of infected individuals with infection age a and the total infectivity of pathogen with pathogen age b. In the current paper, we focus on the following age-structured cholera model with immigration

$$\begin{cases} \dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = \Lambda_s - \mu S(t) - S(t)f(J(t)) - S(t)g(Q(t)), \\ \dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = \Lambda_i(a) - \delta(a)i(t,a), \\ \dfrac{\partial p(t,b)}{\partial t} + \dfrac{\partial p(t,b)}{\partial b} = \Lambda_p(b) - \gamma(b)p(t,b), \end{cases} \tag{1.4}$$

with boundary condition

$$\begin{cases} i(t,0) = S(t)f(J(t)) + S(t)g(Q(t)), \quad t > 0, \\ p(t,0) = P(t) := \displaystyle\int_0^\infty \xi(a)i(t,a)\mathrm{d}a, \quad t > 0, \end{cases} \tag{1.5}$$

and initial condition

$$X_0 := (S(0), i(0,\cdot), p(0,\cdot)) = (S_0, i_0(\cdot), p_0(\cdot)) \in \mathfrak{y}_+, \tag{1.6}$$

where $\mathfrak{y} := \mathbb{R} \times \mathcal{L}^1(0,\infty) \times \mathcal{L}^1(0,\infty)$ with norm

$$\|(\phi,\psi,\varphi)\|_{\mathfrak{y}} := |\phi| + \int_0^\infty |\psi(a)|\mathrm{d}a + \int_0^\infty |\varphi(b)|\mathrm{d}b, \quad \phi \in \mathbb{R}, \quad \psi,\varphi \in \mathcal{L}^1(0,\infty)$$

and $\mathfrak{y}_+ := \mathbb{R}_+ \times \mathcal{L}^1_+(0,\infty) \times \mathcal{L}^1_+(0,\infty)$ is the positive cone of $\mathfrak{y}$. Here $\mathcal{L}^1(0,\infty)$ denotes the space of $\mathcal{L}^1$-integrable functions from the interval $(0,\infty)$ to itself. All the other coefficients in system (1.4)–(1.6) and the corresponding biological interpretation are the same as those in (1.1)–(1.3).

In the current paper, we study the global asymptotical stability of the unique positive equilibrium, which need to construct suitable Lyapunov functional. Mathematically, the age-based immigration rate and the indirect/direct transmission route of cholera generate a huge difficulty in constructing the proper Lyapunov functional. Moreover, the general incidence will bring great trouble to the calculation of the derivative of Lyapunov functional. For the well-posedness of the Lyapunov functional, we also need verify the uniform persistence of the system. The theoretical analysis shows that there exists a unique globally asymptotically stable endemic equilibrium, and the disease persists at the endemic level. The results in present paper not only serve as a supplement and generalization of the works in F. Brauer et al. [7], but also deal with some other new epidemic models with multiple transmission routes and immigration.

The plan of this article is as follows. In Section 2, we make some preliminaries for the system. In Section 3, we explore the asymptotical smoothness and global attractor. In Section 4, we explore the existence and local stability of the positive equilibrium. In Section 5, we construct a Lyapunov functional to discuss the global stability of the equilibrium. Numerical simulation and a brief conclusion will be given in section 6.

2 Preliminaries

Firstly, for system (1.4)–(1.6), we give the following assumptions.

Assumption 1. Constants $\Lambda_s,\ \mu \in \mathbb{R}_+$. For functions $\xi(\cdot),\ k(\cdot),\ q(\cdot),\ \delta(\cdot),\ \gamma(\cdot) \in \mathcal{L}_+^\infty(0,+\infty)$, $\Lambda_i(\cdot),\ \Lambda_p(\cdot) \in \mathcal{L}_+^1(0,+\infty)$, we make the following assumptions.

(I) For $\delta(\cdot)$, denote $\tilde{\delta} := \int_0^\infty \delta(\tau)\mathrm{d}\tau$, and denote $\bar{\delta}$ and $\underline{\delta}$ as the essential supremum and essential infimum of $\delta(\cdot)$, so do $\xi(\cdot),\ k(\cdot),\ q(\cdot),\ \gamma(\cdot),\ \Lambda_i(\cdot)$ and $\Lambda_p(\cdot)$;

(II) $\xi(\cdot),\ k(\cdot),\ q(\cdot),\ \delta(\cdot)$ and $\gamma(\cdot)$ are Lipschitz continuous.

Assumption 2. For functions $f(\cdot),\ g(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, we introduce the following assumptions.

(I) $f(\cdot)$ and $g(\cdot)$ are Lipschitz continuous on $\mathbb{R}_+$ with $f(0) = g(0) = 0$;

(II) $\frac{f(z)}{z} \geqslant f'(z) \geqslant 0,\ \frac{g(z)}{z} \geqslant g'(z) \geq 0$ and $f''(z) \leqslant 0, g''(z) \leqslant 0$, for $z \in \mathbb{R}_+$.

2.1 Existence of unique solution

Denote the following spaces

$$\begin{aligned}
\mathcal{X} &= \mathbb{R} \times \mathbb{R} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}),\\
\mathcal{X}_0 &= \mathbb{R} \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}) \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}),\\
\mathcal{X}_+ &= \mathbb{R}_+ \times \mathbb{R}_+ \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}) \times \mathbb{R}_+ \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}),\\
\mathcal{X}_{0+} &= \mathcal{X}_+ \cap \mathcal{X}_0 = \mathbb{R}_+ \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}) \times \{0\} \times \mathcal{L}^1(\mathbb{R}_+,\mathbb{R}),
\end{aligned}$$

One defines a linear operator $\mathbf{A} : \mathrm{Dom}(\mathbf{A}) \subset \mathcal{X} \to \mathcal{X}$ as follows,

$$\mathbf{A}\begin{pmatrix} \phi_1 \\ \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} -\mu\phi_1 \\ \begin{pmatrix} -\varphi_1(0) \\ -\delta(\cdot)\varphi_1 - \varphi_1' \end{pmatrix} \\ \begin{pmatrix} -\varphi_2(0) \\ -\gamma(\cdot)\varphi_2 - \varphi_2' \end{pmatrix} \end{pmatrix}$$

with $\mathrm{Dom}(\mathbf{A}) = \mathbb{R} \times \{0\} \times \mathcal{W}^{1,1}(0,\infty) \times \{0\} \times \mathcal{W}^{1,1}(0,\infty)$, where $\mathcal{W}^{1,1}(0,\infty)$ denotes the Sobolev space of locally summable functions $y : \mathbb{R}^+ \to \mathbb{R}$ such that for every multi-index α with $\|\alpha\| \leqslant 1$, the weak derivative $D^\alpha y \in \mathcal{L}^1(0,\infty)$ exists. Moreover, define a nonlinear operator $\mathbf{F} : \overline{\mathrm{Dom}(\mathbf{A})} \subset \mathcal{X} \to \mathcal{X}$ as

$$\mathbf{F}\begin{pmatrix} \phi_1 \\ \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \Lambda - \phi_1 f\left(\int_0^\infty k(a)\varphi_1(a)da\right) - \phi_1 g\left(\int_0^\infty q(b)\varphi_2(b)db\right) \\ \begin{pmatrix} \phi_1 f\left(\int_0^\infty k(a)\varphi_1(a)da\right) + \phi_1 g\left(\int_0^\infty q(b)\varphi_2(b)db\right) \\ \Lambda_i(a) \end{pmatrix} \\ \begin{pmatrix} \int_0^\infty \xi(a)\varphi_1(a)da \\ \Lambda_p(b) \end{pmatrix} \end{pmatrix}.$$

Let $u(t) = (S(t), (0, i(t,\cdot))^{\mathrm{T}}, (0, p(t,\cdot))^{\mathrm{T}})^{\mathrm{T}} \in \mathcal{X}_{0+}$, then we can write system (1.4) as the following abstract Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}u(t)}{\mathrm{d}t} = \mathbf{A}u(t) + \mathbf{F}(u(t)), \quad \forall t \geqslant 0, \\ u(0) = u_0 \in \mathcal{X}_0 \cap \mathcal{X}_{0+}. \end{cases} \tag{2.1}$$

To show the existence of unique solutions for system (1.4), we need to prove the operator $\mathbf{A}$ as a Hille–Yosida operator.

Theorem 2.1. *The operator* $\mathbf{A}$ *is a Hille–Yosida operator.*

Proof. In order to apply Hille–Yosida theorem [29], we need find $\zeta = (\hat{\phi}_1, \hat{\varphi}_{10}, \hat{\varphi}_1, \hat{\varphi}_{20}, \hat{\varphi}_2) \in \mathcal{X}$, such that for $(\phi_1, 0, \varphi_1, 0, \varphi_2) \in \mathrm{Dom}(\mathbf{A})$, there holds

$$(\lambda \mathbf{I} - \mathbf{A})^{-1} \begin{pmatrix} \hat{\phi}_1 \\ \begin{pmatrix} \hat{\varphi}_{10} \\ \hat{\varphi}_1 \end{pmatrix} \\ \begin{pmatrix} \hat{\varphi}_{20} \\ \hat{\varphi}_2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \phi_1 \\ \begin{pmatrix} 0 \\ \varphi_1 \end{pmatrix} \\ \begin{pmatrix} 0 \\ \varphi_2 \end{pmatrix} \end{pmatrix}.$$

From above equation, we then yield

$$\hat{\phi}_1 = (\lambda \mathbf{I} - \mathbf{A})\phi_1 = \lambda\phi_1 - \mathbf{A}\phi_1 = (\lambda + \mu)\phi_1.$$

Thus, $\phi_1 = \frac{\hat{\phi}_1}{\lambda+\mu}$. Besides, we obtain

$$\hat{\varphi}_1 = (\lambda \mathbf{I} - \mathbf{A})\varphi_1 = \lambda\varphi_1 - \mathbf{A}\varphi_1 = \lambda\varphi_1 + \delta(\cdot)\varphi_1 + \varphi_1' = (\lambda + \delta(\cdot))\varphi_1 + \varphi_1'.$$

Thus, there holds

$$\varphi_1' = \hat{\varphi}_1 - (\lambda + \delta(\cdot))\varphi_1,$$

and we have

$$\varphi_1(a) = \hat{\varphi}_{10} e^{-\int_0^a (\lambda+\delta(s))\mathrm{d}s} + \int_0^a \hat{\varphi}_1(\tau) e^{-\int_\tau^a (\lambda+\delta(s))\mathrm{d}s} \mathrm{d}\tau.$$

Similarly, we get that

$$\varphi_2(b) = \hat{\varphi}_{20} e^{-\int_0^b (\lambda+\gamma(s))\mathrm{d}s} + \int_0^b \hat{\varphi}_2(\tau) e^{-\int_\tau^b (\lambda+\gamma(s))\mathrm{d}s} \mathrm{d}\tau.$$

Further, there holds

$$\begin{aligned} \|\varphi_1\|_{\mathcal{L}_1} &= \int_0^\infty |\varphi_1(a)| \mathrm{d}a \\ &= \int_0^\infty \left| \hat{\varphi}_{10} e^{-\int_0^a (\lambda+\delta(s))\mathrm{d}s} + \int_0^a \hat{\varphi}_1(\tau) e^{-\int_\tau^a (\lambda+\delta(s))\mathrm{d}s} d\tau \right| \mathrm{d}a \\ &\leqslant |\hat{\varphi}_{10}| \int_0^\infty e^{-\int_0^a (\lambda+\delta(s))\mathrm{d}s} \mathrm{d}a + \int_0^\infty \int_0^a |\hat{\varphi}_1(\tau)| e^{-\int_\tau^a (\lambda+\delta(s))\mathrm{d}s} \mathrm{d}\tau \mathrm{d}a \\ &\leqslant |\hat{\varphi}_{10}| \frac{1}{\lambda + \underline{\delta}} + \int_0^\infty \int_\tau^\infty |\hat{\varphi}_1(\tau)| e^{-(a-\tau)(\lambda+\underline{\delta})} \mathrm{d}a \mathrm{d}\tau \\ &= |\hat{\varphi}_{10}| \frac{1}{\lambda + \underline{\delta}} + \int_0^\infty |\hat{\varphi}_1(\tau)| e^{\tau(\lambda+\underline{\delta})} \int_\tau^\infty e^{-a(\lambda+\underline{\delta})} \mathrm{d}a \mathrm{d}\tau \\ &= |\hat{\varphi}_{10}| \frac{1}{\lambda + \underline{\delta}} + \int_0^\infty \frac{|\hat{\varphi}_1(\tau)|}{\lambda + \underline{\delta}} \mathrm{d}\tau \\ &= \frac{|\hat{\varphi}_{10}|}{\lambda + \underline{\delta}} + \frac{\|\hat{\varphi}_1\|_{\mathcal{L}_1}}{\lambda + \underline{\delta}}. \end{aligned}$$

Similarly, we can obtain that

$$\|\varphi_2\|_{\mathcal{L}_1} \leqslant \frac{|\hat{\varphi}_{20}|}{\lambda+\underline{\gamma}} + \frac{\|\hat{\varphi}_2\|_{\mathcal{L}_1}}{\lambda+\underline{\gamma}}.$$

Let $\eta = \min\{\underline{\delta}, \underline{\gamma}, \mu\}$. For any $\zeta = (\hat{\phi}_1, \hat{\varphi}_{10}, \hat{\varphi}_1, \hat{\varphi}_{20}, \hat{\varphi}_2) \in \mathcal{X}$, we have

$$\begin{aligned}\|(\lambda\mathbf{I}-\mathbf{A})^{-1}\zeta\| &= |\phi_1| + |0| + \|\varphi_1\|_{\mathcal{L}_1} + |0| + \|\varphi_2\|_{\mathcal{L}_1} \\ &\leqslant \frac{|\hat{\phi}_1|}{\lambda+\mu} + \frac{|\hat{\varphi}_{10}|}{\lambda+\underline{\delta}} + \frac{\|\hat{\varphi}_1\|_{\mathcal{L}_1}}{\lambda+\underline{\delta}} + \frac{|\hat{\varphi}_{20}|}{\lambda+\underline{\gamma}} + \frac{\|\hat{\varphi}_2\|_{\mathcal{L}_1}}{\lambda+\underline{\gamma}} \\ &\leqslant \frac{\|\zeta\|}{\lambda+\eta}.\end{aligned}$$

Thus, the linear operator $\mathbf{A}$ is a Hille–Yosida operator due to [29]. □

Let $X_0 = (S(t), (0, i(t,\cdot))^{\mathrm{T}}, (0, p(t,\cdot))^{\mathrm{T}})^{\mathrm{T}} \in \mathcal{X}_{0+}$, thanks to [29, Theorem 5.2.7], we have the following theorem.

Theorem 2.2. *There exists a unique determined semiflow $\{\mathfrak{U}(t)\}_{t\geqslant 0}$ on $\mathcal{X}_{0+}$ such that for any X_0, a unique continuous map $\mathfrak{U} \in C([0,\infty], \mathcal{X}_{0+})$ exists as an integrated solution of the Cauchy problem (2.1), that is,*

$$\begin{cases} \int_0^t \mathfrak{U}(s)X_0 \mathrm{d}s \in \mathrm{Dom}(\mathbf{A}), \\ \mathfrak{U}(t)X_0 = X_0 + \mathbf{A}\int_0^t \mathfrak{U}(s)X_0\mathrm{d}s + \int_0^\infty \mathbf{F}(\mathfrak{U}(s)X_0)\mathrm{d}s, \end{cases}$$

for all $t \geqslant 0$.

2.2 Dissipativeness and persistence

Combining equations (1.5) and (1.6), integrating the last two equations of (1.4) along the characteristic lines yields

$$i(t,a) = \begin{cases} i(t-a,0)\sigma_1(a) + \int_0^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant a \leqslant t, \\ i(0,a-t)\frac{\sigma_1(a)}{\sigma_1(a-t)} + \int_{a-t}^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant t \leqslant a, \end{cases} \tag{2.2}$$

$$p(t,b) = \begin{cases} p(t-b,0)\sigma_2(b) + \int_0^b \Lambda_p(\epsilon)\frac{\sigma_2(b)}{\sigma_2(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant b \leqslant t, \\ p(0,b-t)\frac{\sigma_2(b)}{\sigma_2(b-t)} + \int_{b-t}^b \Lambda_p(\epsilon)\frac{\sigma_2(b)}{\sigma_2(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant t \leqslant b, \end{cases} \tag{2.3}$$

where

$$\sigma_1(a) = e^{-\int_0^a \delta(\tau)\mathrm{d}\tau} \quad \text{and} \quad \sigma_2(b) = e^{-\int_0^b \gamma(\tau)\mathrm{d}\tau}. \tag{2.4}$$

Now, we are concerned with the boundedness of solutions. Let $\mathrm{Y}_1 := \frac{\Lambda_s+\tilde{\Lambda}_i}{\eta}$, $\mathrm{Y}_2 := \frac{\bar{\xi}\mathrm{Y}_1+\tilde{\Lambda}_p}{\eta}$ and

$$\Pi := \left\{ (S(t), i(t,a), p(t,b)) \in \mathcal{X}_{0+} \;\middle|\; S(t) + \int_0^\infty i(t,a)\mathrm{d}a + \int_0^\infty p(t,b)\mathrm{d}b \leqslant \mathrm{Y}_1 + \mathrm{Y}_2 \right\}.$$

We arrive at the following theorem.

Theorem 2.3. *For* (1.4), $\mathfrak{U}$ *is point dissipative, which means there is a bounded set* Π *that attracts all points in* $\mathcal{X}_+$.

Proof. Note that

$$\begin{aligned}\int_0^\infty i(t,a)\mathrm{d}a &= \int_0^t i(t,a)\mathrm{d}a + \int_t^\infty i(t,a)\mathrm{d}a\\ &= \int_0^t i(t-a,0)\sigma_1(a)\mathrm{d}a + \int_0^t\int_0^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a\\ &\quad + \int_t^\infty i(0,a-t)\frac{\sigma_1(a)}{\sigma_1(a-t)}\mathrm{d}a + \int_t^\infty\int_{a-t}^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a.\end{aligned}$$

Interchanging the order of integration for the double integrals and making change of integration variable for the two single integrals gives

$$\int_0^\infty i(t,a)\mathrm{d}a = \int_0^t i(\tau,0)\sigma_1(t-\tau)\mathrm{d}\tau + \int_0^\infty i(0,\tau)\frac{\sigma_1(t+\tau)}{\sigma_1(\tau)}\mathrm{d}\tau + \int_0^\infty\int_\epsilon^{\epsilon+t}\Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}a\mathrm{d}\epsilon.$$

Thus, there holds

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a &= \sigma_1(0)i(t,0) + \int_0^t i(\tau,0)\frac{\mathrm{d}}{\mathrm{d}t}\sigma_1(t-\tau)\mathrm{d}\tau\\ &\quad + \int_0^\infty i(0,\tau)\frac{\frac{\mathrm{d}}{\mathrm{d}t}\sigma_1(t+\tau)}{\sigma_1(\tau)}\mathrm{d}\tau + \int_0^\infty \Lambda_i(\epsilon)\frac{\sigma_1(\epsilon+t)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\\ &= i(t,0) - \int_0^t i(\tau,0)\delta(t-\tau)\sigma_1(t-\tau)\mathrm{d}\tau\\ &\quad - \int_0^\infty i(0,\tau)\frac{\delta(t+\tau)\sigma_1(t+\tau)}{\sigma_1(\tau)}\mathrm{d}\tau + \int_0^\infty \Lambda_i(\epsilon)\frac{\sigma_1(\epsilon+t)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\\ &= i(t,0) - \int_0^t i(t-a,0)\delta(a)\sigma_1(a)\mathrm{d}a\\ &\quad - \int_t^\infty i(0,a-t)\frac{\delta(a)\sigma_1(a)}{\sigma_1(a-t)}\mathrm{d}a + \int_0^\infty \Lambda_i(\epsilon)\frac{\sigma_1(\epsilon+t)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon.\end{aligned}$$

Note that

$$\begin{aligned}\int_0^t\int_0^a \Lambda_i(\epsilon)\frac{\sigma_1'(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a + \int_t^\infty\int_{a-t}^a \Lambda_i(\epsilon)\frac{\sigma_1'(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a &= \int_0^\infty\int_\epsilon^{\epsilon+t}\Lambda_i(\epsilon)\frac{\sigma_1'(a)}{\sigma_1(\epsilon)}\mathrm{d}a\mathrm{d}\epsilon\\ &= \int_0^\infty \Lambda_i(\epsilon)\frac{\sigma_1(\epsilon+t)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon - \int_0^\infty \Lambda_i(\epsilon)\mathrm{d}\epsilon,\end{aligned}$$

we thus have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^\infty i(t,a)\mathrm{d}a &= i(t,0) - \int_0^t i(t-a,0)\delta(a)\sigma_1(a)\mathrm{d}a - \int_t^\infty i(0,a-t)\frac{\delta(a)\sigma_1(a)}{\sigma_1(a-t)}\mathrm{d}a\\ &\quad + \int_0^t\int_0^a \Lambda_i(\epsilon)\frac{\sigma_1'(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a + \int_t^\infty\int_{a-t}^a \Lambda_i(\epsilon)\frac{\sigma_1'(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon\mathrm{d}a + \int_0^\infty \Lambda_i(\epsilon)\mathrm{d}\epsilon.\\ &= i(t,0) - \int_0^\infty \delta(a)i(t,a)\mathrm{d}a + \tilde{\Lambda}_i.\end{aligned}$$

Together with the first equation of (1.4), one has

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(S(t) + \int_0^\infty i(t,a)\mathrm{d}a\right) \leqslant (\Lambda_s + \tilde{\Lambda}_i) - \eta\left(S(t) + \int_0^\infty i(t,a)\mathrm{d}a\right).$$

Hence,

$$S(t) + \int_0^\infty i(t,a)\mathrm{d}a \leqslant \Upsilon_1 - e^{-\eta t}\left\{\Upsilon_1 - (S(0) + \int_0^\infty i(0,a)\mathrm{d}a)\right\}, \tag{2.5}$$

for any $X_0 \in \Pi$. Similarly, we can derive that

$$\int_0^\infty p(t,b)\mathrm{d}b \leqslant \Upsilon_2 - e^{-\eta t}\left\{\Upsilon_2 - \int_0^\infty p(0,b)\mathrm{d}b\right\}, \tag{2.6}$$

for any $X_0 \in \Pi$. Hence, combining (2.5) and (2.6) yields

$$\|\mathfrak{U}(t,X_0)\|_{\mathcal{X}} \leqslant \Upsilon_1 + \Upsilon_2 - e^{-\eta t}\left\{\Upsilon_1 + \Upsilon_2 - (S(0) + \int_0^\infty i(0,a)\mathrm{d}a + \int_0^\infty p(0,b)\mathrm{d}b)\right\}.$$

This implies that $\|\mathfrak{U}(t,X_0)\|_{\mathcal{X}} \leqslant \Upsilon_1 + \Upsilon_2$ for $X_0 \in \Pi$, and the proof is complete. □

From Theorem 2.3, we obtain the following result.

Corollary 2.4. *If $X_0 \in \mathcal{X}_+$ and $\|X_0\|_{\mathcal{X}} \leqslant B$ with some constant $B \geqslant \Upsilon_1 + \Upsilon_2$, then for $t \in \mathbb{R}_+$, we have the following statements*

(i) $0 \leqslant S(t)$, $\int_0^\infty i(t,a)\mathrm{d}a \leqslant B$ *and* $\int_0^\infty p(t,b)\mathrm{d}b \leqslant B$;

(ii) $i(t,0) \leqslant (\bar{k}f'(0) + \bar{q}g'(0))B^2$ *and* $p(t,0) \leqslant \bar{\xi}B$.

The following corollary generates a positive asymptotical lower bound of $S(t)$.

Corollary 2.5. *If $X_0 \in \mathcal{X}_+$, then*

$$\liminf_{t\to\infty} S(t) \geqslant \frac{\Lambda_s}{\mu + f'(0)\bar{k}B + g'(0)\bar{q}B}.$$

Proof. For any $\epsilon > 0$, there exists a $t_0 \in \mathbb{R}_+$ such that

$$\int_0^\infty i(t,a)\mathrm{d}a \leqslant B + \varepsilon \quad \text{and} \quad \int_0^\infty p(t,b)\mathrm{d}b \leqslant B + \varepsilon$$

for $t \geqslant t_0$. Then, for $t \geqslant t_0$,

$$\begin{aligned}\frac{\mathrm{dS(t)}}{\mathrm{dt}} &= \Lambda_s - S(t)(\mu + f(J(t)) + g(Q(t))) \\ &\geqslant \Lambda_s - S(t)(\mu + f'(0)\bar{k}(B+\varepsilon) + g'(0)\bar{q}(B+\varepsilon)).\end{aligned}$$

This implies that

$$\liminf_{t\to\infty} S(t) \geqslant \frac{\Lambda_s}{\mu + f'(0)\bar{k}(B+\varepsilon) + g'(0)\bar{q}(B+\varepsilon)}.$$

Letting ϵ tend to 0 gives the required result. □

Then by similar verification in [30], we obtain the following proposition.

Theorem 2.6. *There exist $\tilde{t} > 0$ and $\epsilon > 0$ such that $i(t,0) > \epsilon$ and $p(t,0) > \epsilon$ for all $t \geqslant \tilde{t}$.*

3 Asymptotical smoothness and global attractor

For the existence of an attractor, the asymptotical smoothness of the semiflow $\mathfrak{U}$ is necessary. For this, by the similar argument in [30, Proposition 6], we claim that $J(t)$, $Q(t)$ and $P(t)$ are Lipschitz continuous with Lipschitz coefficients L_J, L_Q and L_P. Then we introduce the following lemma for the asymptotical smoothness of the semiflow.

Lemma 3.1 ([35]). *The semiflow $\mathfrak{U} : \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+$ is asymptotically smooth if there exist maps $\mathfrak{U}_1, \mathfrak{U}_2 : \mathbb{R}_+ \times \mathcal{X}_+ \to \mathcal{X}_+$ satisfying $\mathfrak{U}(t,x) = \mathfrak{U}_1(t,x) + \mathfrak{U}_2(t,x)$, and for any bounded closed set $\mathbb{B} \subset \mathcal{X}_+$, which is forward invariant under $\mathfrak{U}$, there holds:*

(i) $\lim_{t\to\infty} \operatorname{diam} \mathfrak{U}_2(t,\mathbb{B}) = 0$;

(ii) *There exists $t_{\mathbb{B}} \geqslant 0$ such that $\mathfrak{U}_1(t,\mathbb{B})$ has compact closure for $t \geqslant t_{\mathbb{B}}$.*

For Lemma 3.1 (ii), we utilize the following lemma.

Lemma 3.2 ([35]). *A set $\mathbb{B} \in \mathcal{L}^1_+(0,\infty)$ has compact closure iff the following conditions hold:*

(i) $\sup_{f\in\mathbb{B}} \int_0^\infty f(z)\mathrm{d}z < \infty$;

(ii) $\lim_{r\to\infty} \int_r^\infty f(z)\mathrm{d}z \to 0$ *uniformly in* $f \in \mathbb{B}$;

(iii) $\lim_{h\to 0^+} \int_0^\infty |f(z+h) - f(z)|\mathrm{d}z \to 0$ *uniformly in* $f \in \mathbb{B}$;

(iv) $\lim_{h\to 0^+} \int_0^h f(z)\mathrm{d}z \to 0$ *uniformly in* $f \in \mathbb{B}$.

Based on above lemmas, we can obtain the following result.

Theorem 3.3. *The semiflow $\mathfrak{U}$ generated by (1.4) is asymptotically smooth.*

Proof. Define maps $\mathfrak{U}_1$ and $\mathfrak{U}_2$ such that $\mathfrak{U} = \mathfrak{U}_1 + \mathfrak{U}_2$, satisfying

$$\begin{cases} \mathfrak{U}_1(t,x_0) = (S(t), \grave{i}(t,\cdot), \grave{p}(t,\cdot)), \\ \mathfrak{U}_2(t,x_0) = (0, \grave{\varphi}_i(t,\cdot), \grave{\varphi}_p(t,\cdot)), \end{cases}$$

where

$$\grave{i}(t,a) = \begin{cases} i(t-a,0)\sigma_1(a), & 0 \leqslant a \leqslant t, \\ 0, & 0 \leqslant t \leqslant a, \end{cases}$$

$$\grave{p}(t,b) = \begin{cases} p(t-b,0)\sigma_2(b), & 0 \leqslant b \leqslant t, \\ 0, & 0 \leqslant t \leqslant b, \end{cases}$$

$$\grave{\varphi}_i(t,a) = \begin{cases} \displaystyle\int_0^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant a \leqslant t, \\ \displaystyle i(0,a-t)\frac{\sigma_1(a)}{\sigma_1(a-t)} + \int_{a-t}^a \Lambda_i(\epsilon)\frac{\sigma_1(a)}{\sigma_1(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant t \leqslant a, \end{cases}$$

and

$$\grave{\varphi}_p(t,b) = \begin{cases} \displaystyle\int_0^b \Lambda_p(\epsilon)\frac{\sigma_2(b)}{\sigma_2(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant b \leqslant t, \\ \displaystyle p(0,b-t)\frac{\sigma_2(b)}{\sigma_2(b-t)} + \int_{b-t}^b \Lambda_p(\epsilon)\frac{\sigma_2(b)}{\sigma_2(\epsilon)}\mathrm{d}\epsilon, & 0 \leqslant t \leqslant b. \end{cases}$$

Firstly, we show that $\mathfrak{U}_2$ satisfies Lemma 3.1(i). For $X_0^1, X_0^2 \in \Pi$, letting $\varepsilon_1 = a - t$, we obtain

$$\begin{aligned}
\|\grave{\varphi}_i^1(t,\cdot) - \grave{\varphi}_i^2(t,\cdot)\|_{\mathcal{L}_1} &= \int_t^\infty |i^1(0,a-t) - i^2(0,a-t)| \frac{\sigma_1(a)}{\sigma_1(a-t)} da \\
&= \int_0^\infty |i^1(0,\varepsilon_1) - i^2(0,\varepsilon_1)| \frac{\sigma_1(t+\varepsilon_1)}{\sigma_1(\varepsilon_1)} d\varepsilon_1 \\
&\leqslant \int_0^\infty e^{-\underline{\delta} t} |i^1(0,\varepsilon_1) - i^2(0,\varepsilon_1)| d\varepsilon_1 \\
&\leqslant 2Be^{-\underline{\delta} t}.
\end{aligned}$$

Similarly, we have $\|\grave{\varphi}_p^1(t,\cdot) - \grave{\varphi}_p^2(t,\cdot)\|_{\mathcal{L}_1} \leqslant 2Be^{-\underline{\gamma} t}$. And thus, we have

$$\left\| \mathfrak{U}_2(t, X_0^1) - \mathfrak{U}_2(t, X_0^2) \right\|_{\mathcal{L}_1} \leqslant 2B(e^{-\underline{\delta} t} + e^{-\underline{\gamma} t}).$$

Hence, as $t \to \infty$, we have that $\operatorname{diam}\|\mathfrak{U}_2(t, X_0)\|_{\mathcal{X}} \to 0$. This accomplishes the verification of Lemma 3.1(i). Subsequently, we focus on the proof of Lemma 3.1(ii) by virtue of Lemma 3.2. By Proposition 2.4, we claim that conditions (i), (ii) and (iv) of Lemma 3.2 are satisfied since $0 \leqslant \grave{i}(t,a) = i(t-a,0)\sigma_1(a) \leqslant [f'(0)\bar{k} + g'(0)\bar{q}]B^2 e^{-\underline{\delta} a}$. It suffices to verify the condition of Lemma 3.2 (iii). Choosing $h \in (0,t)$ small enough, one has

$$\begin{aligned}
&\int_0^\infty |\grave{i}(a,t) - \grave{i}(a+h,t)| \mathrm{d}a \\
&\quad \leqslant \int_0^{t-h} |S(t-a-h)(f(J(t-a-h)) + g(Q(t-a-h)))(\sigma_1(a+h) - \sigma_1(a))| \mathrm{d}a \\
&\qquad + \int_0^{t-h} S(t-a-h)(|f(J(t-a-h)) - f(J(t-a))| \\
&\qquad\qquad + |g(Q(t-a-h)) - g(Q(t-a))|)\sigma_1(a) \mathrm{d}a \\
&\qquad + \int_0^{t-h} |S(t-a-h) - S(t-a)|(f(J(t-a)) + g(Q(t-a)))\sigma_1(a) \mathrm{d}a \\
&\qquad + f'(0) \int_{t-h}^t |S(t-a)J(t-a)\sigma_1(a)| \mathrm{d}a \\
&\qquad + g'(0) \int_{t-h}^t |S(t-a)Q(t-a)\sigma_1(a)| \mathrm{d}a \\
&\quad \leqslant (f'(0)\bar{k} + g'(0)\bar{q})B^2 \int_0^{t-h} |\sigma_1(a+h) - \sigma_1(a)| \mathrm{d}a \\
&\qquad + f'(0) \int_0^{t-h} S(t-a-h)|J(t-a-h) - J(t-a)|\sigma_1(a) \mathrm{d}a \\
&\qquad + g'(0) \int_0^{t-h} S(t-a-h)|Q(t-a-h) - Q(t-a)|\sigma_1(a) \mathrm{d}a \\
&\qquad + \int_0^{t-h} (f'(0)J(t-a) + g'(0)Q(t-a))|S(t-a-h) - S(t-a)|\sigma_1(a) \mathrm{d}a \\
&\qquad + (f'(0)\bar{k} + g'(0)\bar{q})B^2 h.
\end{aligned} \tag{3.1}$$

From (2.4), we have

$$0 \leqslant \int_0^{t-h} |\sigma_1(a+h) - \sigma_1(a)| \mathrm{d}a = \int_0^h \sigma_1(a) \mathrm{d}a - \int_{t-h}^t \sigma_1(a) \mathrm{d}a \leqslant h. \tag{3.2}$$

Note that

$$\left|\frac{\mathrm{d}S(t)}{\mathrm{d}t}\right| \leqslant \Lambda_S + \mu B + (\bar{k}f'(0) + \bar{q}g'(0))B^2,$$

which means that $S(t)$ is Lipschitz continuous. Then

$$\int_0^{t-h} S(t-a-h)|J(t-a-h) - J(t-a)|\sigma_1(a)\mathrm{d}a \leqslant \frac{1}{\underline{\delta}}BL_J h, \tag{3.3}$$

and

$$\int_0^{t-h} S(t-a-h)|Q(t-a-h) - Q(t-a)|\sigma_1(a)\mathrm{d}a \leqslant \frac{1}{\underline{\delta}}BL_Q h. \tag{3.4}$$

Moreover, one has that

$$\begin{aligned}&\int_0^{t-h}(f'(0)J(t-a) + g'(0)Q(t-a))|S(t-a-h) - S(t-a)|\sigma_1(a)\mathrm{d}a\\ &\qquad\leqslant \frac{1}{\underline{\delta}}(f'(0)\bar{k} + g'(0)\bar{q})BL_s h,\end{aligned} \tag{3.5}$$

where $L_S := \Lambda_S + \mu B + (\bar{k}f'(0) + \bar{q}g'(0))B^2$. Substituting equations (3.2)–(3.5) into (3.1), one has that

$$\begin{aligned}&\int_0^{\infty} |\grave{i}(a,t) - \grave{i}(a+h,t)|\mathrm{d}a\\ &\qquad\leqslant 2(f'(0)\bar{k} + g'(0)\bar{q})B^2 h + \frac{1}{\underline{\delta}}(f'(0)L_J + g'(0)L_Q)Bh + \frac{1}{\underline{\delta}}(f'(0)\bar{k} + g'(0)\bar{q})BL_s h.\end{aligned} \tag{3.6}$$

Thus, Lemma 3.2 holds. Hence, $\grave{i}(t,a)$ remains in a pre-compact subset in $\mathcal{L}^1_+(0,\infty)$. The same arguments can be derived on $\grave{p}(t,b)$ and this completes the proof. □

According to [19], a global attractor exists since the semiflow $\mathfrak{U}$ is asymptotically smooth.

Theorem 3.4. *The semi-flow $\mathfrak{U}(t)$ has a global attractor in $\mathcal{X}_+$.*

4 Local stability of the infection equilibrium

Because the model introduces immigration terms, there exists no infection-free equilibrium for system (1.4). Assume $E^* = (S^*, i^*(a), p^*(b))$ be an equilibrium for system (1.4), then it satisfies the following equations.

$$\begin{cases}\Lambda_s = \mu S^* + S^* f(J^*) + S^* g(Q^*),\\ \dfrac{\mathrm{d}i^*(a)}{\mathrm{d}a} = \Lambda_i(a) - \delta(a)i^*(a),\\ \dfrac{\mathrm{d}p^*(b)}{\mathrm{d}b} = \Lambda_p(b) - \gamma(b)p^*(b),\\ i^*(0) = S^* f(J^*) + S^* g(Q^*),\\ p^*(0) = \displaystyle\int_0^{\infty} \xi(a)i^*(a)\mathrm{d}a,\end{cases} \tag{4.1}$$

where $Q^* = \int_0^\infty q(b)p^*(b)\mathrm{d}b$ and $J^* = \int_0^\infty k(a)i^*(a)\mathrm{d}a$. Denote

$$\begin{aligned}
\Xi_1 &= \int_0^\infty k(a)\sigma_1(a)\mathrm{d}a, & \Xi_4 &= \int_0^\infty k(a)\sigma_1(a)\int_0^a \frac{\Lambda_i(\tau)}{\sigma_1(\tau)}\mathrm{d}\tau\mathrm{d}a,\\
\Xi_2 &= \int_0^\infty q(b)\sigma_2(b)\mathrm{d}b, & \Xi_5 &= \int_0^\infty q(b)\sigma_2(b)\int_0^b \frac{\Lambda_p(\tau)}{\sigma_2(\tau)}\mathrm{d}\tau\mathrm{d}b,\\
\Xi_3 &= \int_0^\infty \xi(a)\sigma_1(a)\mathrm{d}a, & \Xi_6 &= \int_0^\infty \xi(a)\sigma_1(a)\int_0^a \frac{\Lambda_i(\tau)}{\sigma_1(\tau)}\mathrm{d}\tau\mathrm{d}a.
\end{aligned} \tag{4.2}$$

Owing to equations (4.1), we derive

$$i^*(0) = \Lambda_s - \mu S^* \tag{4.3}$$

and

$$\begin{aligned}
i^*(a) &= i^*(0)\sigma_1(a) + \int_0^a \Lambda_i(\tau)\frac{\sigma_1(a)}{\sigma_1(\tau)}\mathrm{d}\tau,\\
p^*(b) &= p^*(0)\sigma_2(b) + \int_0^b \Lambda_p(\tau)\frac{\sigma_2(b)}{\sigma_2(\tau)}\mathrm{d}\tau.
\end{aligned} \tag{4.4}$$

Then substituting (4.3) into the first equation of (4.4) yields

$$i^*(a) = (\Lambda_s - \mu S^*)\sigma_1(a) + \int_0^a \Lambda_i(\tau)\frac{\sigma_1(a)}{\sigma_1(\tau)}\mathrm{d}\tau. \tag{4.5}$$

Combining (4.5) and the last equation of (4.1) yields

$$p^*(0) = (\Lambda_s - \mu S^*)\Xi_3 + \Xi_6. \tag{4.6}$$

Further, substituting (4.6) into the second equation of (4.4), one has that

$$p^*(b) = (\Lambda_s - \mu S^*)\Xi_3\sigma_2(b) + \Xi_6\sigma_2(b) + \int_0^b \Lambda_p(\tau)\frac{\sigma_2(b)}{\sigma_2(\tau)}\mathrm{d}\tau. \tag{4.7}$$

Thus, in order to find E^*, inspired from the first equation of (4.1), we need to search for the zero of the following formula

$$h(S) = \Lambda_s - \mu S - Sf((\Lambda_s - \mu S)\Xi_1 + \Xi_4) - Sg((\Lambda_s - \mu S)\Xi_2\Xi_3 + \Xi_2\Xi_6 + \Xi_5).$$

Since $h(0) = \Lambda_s > 0$ and $h(\frac{\Lambda_s}{\mu}) < 0$, by the Intermediate Vale Theorem, $h(S)$ has one zero in $(0, \frac{\Lambda_s}{\mu})$. Thus, there exists at least one $S^* \in \left(0, \frac{\Lambda_s}{\mu}\right)$ and thus at least one positive equilibrium E^* exists.

In the following, we first focus on the local stability.

Theorem 4.1. *System (1.4) has one infection equilibrium E^*, which is locally asymptotically stable.*

Proof. The linearization of system (1.4)–(1.5) on $(S^*, i^*(a), p^*(b))$ is

$$\begin{cases}
\dfrac{\mathrm{d}S(t)}{\mathrm{d}t} = (-\mu - f(J^*) - g(Q^*))S(t) - S^*f'(J^*)J - S^*(t)g'(Q^*)Q,\\
\dfrac{\partial i(t,a)}{\partial t} + \dfrac{\partial i(t,a)}{\partial a} = -\delta(a)i(t,a),\\
\dfrac{\partial p(t,b)}{\partial t} + \dfrac{\partial p(t,b)}{\partial b} = -\gamma(b)p(t,b),\\
i(t,0) = (f(J^*) + g(Q^*))S + S^*f'(J^*)J + S^*g'(Q^*)Q,\\
p(t,0) = \displaystyle\int_0^\infty \xi(a)i(t,a)\mathrm{d}a.
\end{cases} \tag{4.8}$$

Substituting $(S(t), i(t,a), p(t,b)) = (\hat{S}, \hat{i}(a), \hat{p}(b))e^{\lambda t}$ into (4.8) and dropping the hat, we can obtain

$$\begin{cases} \lambda S = -(\mu + f(J^*) + g(Q^*))S - S^* f'(J^*) \int_0^\infty k(a)i(a)\mathrm{d}a - S^* g'(Q^*) \int_0^\infty q(b)p(b)\mathrm{d}b, \\ i(a) = i(0)e^{-\lambda a}\sigma_1(a), \\ p(b) = p(0)e^{-\lambda b}\sigma_2(b), \\ i(0) = (f(J^*) + g(Q^*))S + S^* g'(Q^*) \int_0^\infty q(b)p(b)\mathrm{d}b + S^* f'(J^*) \int_0^\infty k(a)i(a)\mathrm{d}a, \\ p(0) = \int_0^\infty \xi(a)i(a)\mathrm{d}a. \end{cases} \tag{4.9}$$

Denote

$$\Gamma_1(\lambda) = \int_0^\infty k(a)e^{-\lambda a}\sigma_1(a)\mathrm{d}a, \qquad \Gamma_2(\lambda) = \int_0^\infty q(b)e^{-\lambda b}\sigma_2(b)\mathrm{d}b,$$

and

$$\Gamma_3(\lambda) = \int_0^\infty \xi(a)e^{-\lambda a}\sigma_1(a)\mathrm{d}a.$$

It follows from (4.9) that

$$\begin{cases} (\lambda + \mu + f(J^*) + g(Q^*))S + [S^* f'(J^*)\Gamma_1(\lambda)]i(0) + [S^* g'(Q^*)\Gamma_2(\lambda)]p(0) = 0, \\ (f(J^*) + g(Q^*))S - [1 - S^* f'(J^*)\Gamma_1(\lambda)]i(0) + [S^* g'(Q^*)\Gamma_2(\lambda)]p(0) = 0, \\ \Gamma_3(\lambda)i(0) - p(0) = 0. \end{cases}$$

Thus, the corresponding characteristic equation of the linearization for system (1.4) at infection equilibrium $(S^*, i^*(a), p^*(b))$ is

$$\begin{vmatrix} \lambda + \mu + f(J^*) + g(Q^*) & S^* f'(J^*)\Gamma_1(\lambda) & S^* g'(Q^*)\Gamma_2(\lambda) \\ f(J^*) + g(Q^*) & S^* f'(J^*)\Gamma_1(\lambda) - 1 & S^* g'(Q^*)\Gamma_2(\lambda) \\ 0 & \Gamma_3(\lambda) & -1 \end{vmatrix} = 0.$$

Clearly, $\lambda = -\mu$ is not a root of the above equation, then

$$(\lambda + \mu + f(J^*) + g(Q^*))/(\lambda + \mu) = S^* f'(J^*)\Gamma_1(\lambda) + S^* g'(Q^*)\Gamma_2(\lambda)\Gamma_3(\lambda). \tag{4.10}$$

Assume that equation (4.10) has one root with positive real part. The module of the left side of the equation (4.10) is more than one. The module of the right side is

$$|S^* f'(J^*)\Gamma_1(\lambda) + S^* g'(Q^*)\Gamma_2(\lambda)\Gamma_3(\lambda)| \leqslant \left| S^* \frac{f(J^*)}{J^*}\Gamma_1(\lambda) + S^* \frac{g(Q^*)}{Q^*}\Gamma_2(\lambda)\Gamma_3(\lambda) \right|.$$

Since

$$Q^* = \int_0^\infty q(b)p^*(b)\mathrm{d}b = p^*(0)\Xi_2 + \Xi_5, \qquad J^* = \int_0^\infty k(a)i^*(a)\mathrm{d}a = i^*(0)\Xi_1 + \Xi_4,$$

and

$$\Xi_3 = \int_0^\infty \xi(a)\sigma_1(a)\mathrm{d}a \leqslant \int_0^\infty \xi(a)\frac{i^*(a)}{i^*(0)}\mathrm{d}a = \frac{p^*(0)}{i^*(0)},$$

we have

$$|S^* f'(J^*)\Gamma_1(\lambda) + S^* g'(Q^*)\Gamma_2(\lambda)\Gamma_3(\lambda)| \leqslant \left| \frac{S^* f(J^*)}{i^*(0)} + \frac{S^* g(Q^*)}{p^*(0)}\Xi_3 \right| = 1.$$

This is a contradiction and we finish the proof. □

5 Global asymptotic stability of the positive equilibrium

For the global asymptotic stability of the positive equilibrium, we apply Lyapunov functional method. For this, we introduce a function

$$\hbar(z) = z - 1 - \ln z, \qquad z \in \mathbb{R}_+. \tag{5.1}$$

In order to ensure $\hbar\left(\frac{i(t,a)}{i^*(a)}\right)$ and $\hbar\left(\frac{p(t,b)}{p^*(b)}\right)$ well-defined, we need to show that $\frac{i(t,a)}{i^*(a)}$ and $\frac{p(t,b)}{p^*(b)}$ are bounded by some positive constants through dissipativeness and persistence analysis in Section 2.

For the verification of Lyapunov functional, we need the following lemmas.

Lemma 5.1. $\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) i^*(a)\xi(a)\left[1 - \frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right]\mathrm{d}a = 0.$

Proof. Since $p(t,0) = \int_0^\infty \xi(a) i(t,a)\mathrm{d}a$ and $p^*(0) = \int_0^\infty \xi(a) i^*(a)\mathrm{d}a$, we have

$$\begin{aligned}
&\frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) i^*(a)\xi(a)\left[1 - \frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right]\mathrm{d}a \\
&= \frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*) i^*(a)\xi(a)\mathrm{d}a - \frac{1}{\Xi_3}\int_0^\infty S^* g(Q^*)\xi(a)\frac{i(t,a)p^*(0)}{p(t,0)}\mathrm{d}a \\
&= \frac{1}{\Xi_3} S^* g(Q^*)\int_0^\infty i^*(a)\xi(a)\mathrm{d}a - \frac{1}{\Xi_3} S^* g(Q^*) p^*(0)\frac{1}{p(t,0)}\int_0^\infty \xi(a) i(t,a)\mathrm{d}a \\
&= 0.
\end{aligned}$$

The proof is completed. □

Lemma 5.2. *Define a function h not depending on a and b. Then we have*

$$\frac{1}{\Xi_2}\int_0^\infty S^* g(Q^*) p^*(0)\sigma_2(b) q(b) h\mathrm{d}b = \int_0^\infty S^* g(Q^*) i^*(a)\xi(a) h\mathrm{d}a.$$

Proof. Since $\Xi_2 = \int_0^\infty q(b)\sigma_2(b)\mathrm{d}b$ and $p^*(0) = \int_0^\infty i^*(a)\xi(a)\mathrm{d}a$, we have

$$\frac{1}{\Xi_2}\int_0^\infty S^* g(Q^*) p^*(0)\sigma_2(b) q(b) h\mathrm{d}b = S^* g(Q^*) p^*(0) h = \int_0^\infty S^* g(Q^*) i^*(a)\xi(a) h\mathrm{d}a.$$

This completes the proof. □

Theorem 5.3. *The infection equilibrium E^* of system (1.4) is globally asymptotically stable.*

Proof. We define the Lyapunov function $\ell(t) = \ell_1(t) + \ell_2(t) + \ell_3(t)$ with

$$\ell_1(t) = S^*\hbar\left(\frac{S(t)}{S^*}\right) i^*(0), \qquad \ell_2(t) = \int_0^\infty \Phi(a) i^*(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\mathrm{d}a,$$

and

$$\ell_3(t) = \frac{1}{\Xi_3}\int_0^\infty \Psi(b) p^*(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b,$$

where

$$\Phi(a) = \frac{1}{\Xi_1}\int_a^\infty S^* f(J^*) k(u) e^{-\int_a^u \delta(\tau)\mathrm{d}\tau}\mathrm{d}u + \frac{1}{\Xi_3}\int_a^\infty \Psi(0)\xi(u) e^{-\int_a^u \delta(\tau)\mathrm{d}\tau}\mathrm{d}u,$$

and

$$\Psi(b) = \frac{1}{\Xi_2} \int_b^\infty S^* g(Q^*) q(v) e^{-\int_b^v \gamma(\tau)\mathrm{d}\tau} \mathrm{d}v.$$

Then, calculating the derivative of $\ell_1(t)$ along (1.4) yields

$$\frac{\mathrm{d}\ell_1(t)}{\mathrm{d}t} = \left(1 - \frac{S^*}{S}\right) [\Lambda - \mu S - S f(J) - S g(Q)] i^*(0).$$

Using the fact that $\Lambda = \mu S^* + S^* f(J^*) + S^* g(Q^*)$, one has that

$$\begin{aligned} \frac{\mathrm{d}\ell_1(t)}{\mathrm{d}t} &= \left(1 - \frac{S^*}{S}\right) [\mu S^* + S^* f(J^*) + S^* g(Q^*) - \mu S - S f(J) - S g(Q)] i^*(0) \\ &= \Big[-\frac{\mu}{S}(S - S^*)^2 + S^* f(J^*) + S^* g(Q^*) - S f(J) - S g(Q) \\ &\quad - \frac{S^*}{S} S^* f(J^*) - \frac{S^*}{S} S^* g(Q^*) + S^* f(J) + S^* g(Q) \Big] i^*(0). \end{aligned} \tag{5.2}$$

Define $H(a) = \int_0^a \frac{\Lambda_i(\epsilon)}{\sigma_1(\epsilon)} \mathrm{d}\epsilon$ and $K(b) = \int_0^b \frac{\Lambda_p(\epsilon)}{\sigma_2(\epsilon)} \mathrm{d}\epsilon$. In what follows, calculating the derivative of $\ell_2(t)$ along (1.4) and then letting $\tau = t - a$ gives

$$\begin{aligned} \frac{\mathrm{d}\ell_2(t)}{\mathrm{d}t} &= \frac{\mathrm{d}}{\mathrm{d}t} \int_0^\infty \Phi(a) i^*(a) \hbar \left(\frac{i(t,a)}{i^*(a)} \right) \mathrm{d}a \\ &= \frac{\mathrm{d}}{\mathrm{d}t} \int_{-\infty}^t \Phi(t-\tau) i^*(t-\tau) \hbar \left(\frac{i(\tau,0) + H(t-\tau)}{i^*(0) + H(t-\tau)} \right) \mathrm{d}\tau \\ &= \Phi(0) i^*(0) \hbar \left(\frac{i(t,0) + H(0)}{i^*(0) + H(0)} \right) \\ &\quad + \int_{-\infty}^t \frac{\mathrm{d}}{\mathrm{d}t} [\Phi(t-\tau) i^*(t-\tau)] \hbar \left(\frac{i(\tau,0) + H(t-\tau)}{i^*(0) + H(t-\tau)} \right) \mathrm{d}\tau \\ &\quad + \int_{-\infty}^t \Phi(t-\tau) i^*(t-\tau) \frac{\mathrm{d}}{\mathrm{d}t} \left[\hbar \left(\frac{i(\tau,0) + H(t-\tau)}{i^*(0) + H(t-\tau)} \right) \right] \mathrm{d}\tau. \end{aligned}$$

Letting $a = t - \tau$, we have

$$\begin{aligned} \frac{\mathrm{d}\ell_2(t)}{\mathrm{d}t} &= \Phi(0) i^*(0) \hbar \left(\frac{i(t,0)}{i^*(0)} \right) + \int_0^\infty \frac{\mathrm{d}}{\mathrm{d}a} [\Phi(a) i^*(a)] \hbar \left(\frac{i(t,a)}{i^*(a)} \right) \mathrm{d}a \\ &\quad + \int_0^\infty \Phi(a) i^*(a) \left(1 - \frac{i^*(a)}{i(t,a)}\right) \frac{H'(a)}{i^*(0) + H(a)} \left(1 - \frac{i(t,a)}{i^*(a)}\right) \mathrm{d}a \\ &= \Phi(0) i^*(0) \hbar \left(\frac{i(t,0)}{i^*(0)} \right) + \int_0^\infty [\Phi'(a) i^*(a) + \Phi(a) i^*_a(a)] \hbar \left(\frac{i(t,a)}{i^*(a)} \right) \mathrm{d}a \\ &\quad + \int_0^\infty \Phi(a) i^*(a) \left(1 - \frac{i^*(a)}{i(t,a)}\right) \left(1 - \frac{i(t,a)}{i^*(a)}\right) \frac{\Lambda_i(a)}{\sigma_1(a)(i^*(0) + H(a))} \mathrm{d}a \\ &= \Phi(0) i^*(0) \hbar \left(\frac{i(t,0)}{i^*(0)} \right) + \int_0^\infty [\Phi'(a) i^*(a) + \Phi(a) i^*_a(a)] \hbar \left(\frac{i(t,a)}{i^*(a)} \right) \mathrm{d}a \\ &\quad + \int_0^\infty \Phi(a) \Lambda_i(a) \left(1 - \frac{i^*(a)}{i(t,a)}\right) \left(1 - \frac{i(t,a)}{i^*(a)}\right) \mathrm{d}a. \end{aligned}$$

Since

$$\Phi'(a) = -\frac{1}{\Xi_1} S^* f(J^*) k(a) - \frac{1}{\Xi_3} \Psi(0) \xi(a) + \delta(a) \Phi(a) \quad \text{and} \quad i^*_a(a) = -i^*(a)\delta(a) + \Lambda_i(a),$$

we further derive

$$\frac{\mathrm{d}\ell_2(t)}{\mathrm{d}t} = \Phi(0)i^*(0)\hbar\left(\frac{i(t,0)}{i^*(0)}\right) - \int_0^\infty \Phi(a)\Lambda_i(a)\hbar\left(\frac{i^*(a)}{i(t,a)}\right)\mathrm{d}a - \int_0^\infty i^*(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\left[\frac{1}{\Xi_1}S^*f(J^*)k(a) + \frac{1}{\Xi_3}S^*g(Q^*)\xi(a)\right]\mathrm{d}a.$$

Since

$$\Phi(0)i^*(0) = \frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)k(a)i^*(0)\sigma_1(a)\mathrm{d}a + \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)\xi(a)i^*(0)\sigma_1(a)\mathrm{d}a,$$

we subsequently obtain

$$\begin{aligned}\frac{\mathrm{d}\ell_2(t)}{\mathrm{d}t} \leqslant{}& \frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)i^*(0)\sigma_1(a)k(a)\left[\hbar\left(\frac{i(t,0)}{i^*(0)}\right) - \hbar\left(\frac{i(t,a)}{i^*(a)}\right)\right]\mathrm{d}a\\ &+ \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[\hbar\left(\frac{i(t,0)}{i^*(0)}\right) - \hbar\left(\frac{i(t,a)}{i^*(a)}\right)\right]\mathrm{d}a\\ &- \int_0^\infty \Phi(a)\Lambda_i(a)\hbar\left(\frac{i^*(a)}{i(t,a)}\right)\mathrm{d}a\\ &- \int_0^\infty H(a)\sigma_1(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\left[\frac{1}{\Xi_1}S^*f(J^*)k(a) + \frac{1}{\Xi_3}S^*g(Q^*)\xi(a)\right]\mathrm{d}a.\end{aligned} \tag{5.3}$$

Similarly, we have

$$\begin{aligned}\frac{\mathrm{d}\ell_3(t)}{\mathrm{d}t} \leqslant{}& \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)\left[\hbar\left(\frac{p(t,0)}{p^*(0)}\right) - \hbar\left(\frac{p(t,b)}{p^*(b)}\right)\right]\mathrm{d}b\\ &- \frac{1}{\Xi_3}\int_0^\infty \Psi(b)\Lambda_p(b)\hbar\left(\frac{p^*(b)}{p(t,b)}\right)\mathrm{d}b\\ &- \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)q(b)\sigma_2(b)K(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b.\end{aligned} \tag{5.4}$$

From equations (5.2), (5.3) and (5.4), we yield

$$\begin{aligned}\frac{\mathrm{d}\ell(t)}{\mathrm{d}t} \leqslant{}& \Big[-\frac{\mu}{S}(S-S^*)^2 + S^*f(J^*) + S^*g(Q^*) - Sf(J) - Sg(Q)\\ &- \frac{S^*}{S}S^*f(J^*) - \frac{S^*}{S}S^*g(Q^*) + S^*f(J) + S^*g(Q)\Big]\, i^*(0)\\ &+ \frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)i^*(0)\sigma_1(a)k(a)\left[\frac{i(t,0)}{i^*(0)} - \frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)i^*(a)}{i^*(0)i(t,a)}\right]\mathrm{d}a\\ &+ \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[\frac{i(t,0)}{i^*(0)} - \frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)i^*(a)}{i^*(0)i(t,a)}\right]\mathrm{d}a\\ &+ \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)\left[\frac{p(t,0)}{p^*(0)} - \frac{p(t,b)}{p^*(b)} - \ln\frac{p(t,0)p^*(b)}{p^*(0)p(t,b)}\right]\mathrm{d}b\\ &- \int_0^\infty \Phi(a)\Lambda_i(a)\hbar\left(\frac{i^*(a)}{i(t,a)}\right)\mathrm{d}a - \frac{1}{\Xi_3}\int_0^\infty \Psi(b)\Lambda_p(b)\hbar\left(\frac{p^*(b)}{p(t,b)}\right)\mathrm{d}b\\ &- \int_0^\infty H(a)\sigma_1(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\left[\frac{1}{\Xi_1}S^*f(J^*)k(a) + \frac{1}{\Xi_3}S^*g(Q^*)\xi(a)\right]\mathrm{d}a\\ &- \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)q(b)\sigma_2(b)K(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b.\end{aligned}$$

Since $\int_0^\infty k(a)\sigma_1(a)\mathrm{d}a = \Xi_1$ and $\int_0^\infty \xi(a)\sigma_1(a)\mathrm{d}a = \Xi_3$, we have

$$\begin{aligned}
&\frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)k(a)i^*(0)\sigma_1(a)\left[1-\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}\right]\mathrm{d}a \\
&\qquad + \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)\xi(a)i^*(0)\sigma_1(a)\left[1-\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right]\mathrm{d}a \\
&\quad = S^*f(J^*)\frac{1}{\Xi_1}\int_0^\infty k(a)\sigma_1(a)i^*(0)\mathrm{d}a\left[1-\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}\right] \\
&\qquad + S^*g(Q^*)\frac{1}{\Xi_3}\int_0^\infty \xi(a)\sigma_1(a)i^*(0)\mathrm{d}a\left[1-\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right] \\
&\quad = \left(S^*f(J^*)\left[1-\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}\right] + S^*g(Q^*)\left[1-\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right]\right)i^*(0) \\
&\quad = \left(S^*f(J^*) - \frac{i^*(0)Sf(J)}{i(t,0)} + S^*g(Q^*) - \frac{i^*(0)Sg(Q)}{i(t,0)}\right)i^*(0) \\
&\quad = \left(i^*(0) - i(t,0)\frac{i^*(0)}{i(t,0)}\right)i^*(0) \\
&\quad = 0.
\end{aligned}$$

Then $\frac{\mathrm{d}\ell(t)}{\mathrm{d}t} \leqslant \sum_{i=1}^6 \Theta_i$, where

$$\begin{aligned}
\Theta_1 :=\ & \left[-\frac{\mu}{S}(S-S^*)^2 - Sf(J) - Sg(Q)\right]i^*(0) + \frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)i^*(0)\sigma_1(a)k(a)\frac{i(t,0)}{i^*(0)}\mathrm{d}a \\
& + \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\frac{i(t,0)}{i^*(0)}\mathrm{d}a, \\
\Theta_2 :=\ & -\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\frac{i(t,a)}{i^*(a)}\mathrm{d}a + \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)\frac{p(t,0)}{p^*(0)}\mathrm{d}b, \\
\Theta_3 :=\ & \left[S^*f(J^*) - \frac{S^*}{S}S^*f(J^*) + S^*f(J)\right]i^*(0) \\
& + \frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)i^*(0)\sigma_1(a)k(a)\left[-\frac{i(t,a)}{i^*(a)} - \ln\frac{i(t,0)}{i^*(0)} + \ln\frac{i(t,a)}{i^*(a)}\right]\mathrm{d}a \\
& + \frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)i^*(0)\sigma_1(a)k(a)\left[1-\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}\right]\mathrm{d}a, \\
\Theta_4 :=\ & \left[S^*g(Q^*) - \frac{S^*}{S}S^*g(Q^*)\right]i^*(0) + \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\ln\frac{i(t,a)}{i^*(a)}\mathrm{d}a \\
& - \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)\ln\frac{p(t,0)}{p^*(0)}\mathrm{d}b, \\
\Theta_5 :=\ & S^*g(Q)i^*(0) - \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\ln\frac{i(t,0)}{i^*(0)}\mathrm{d}a \\
& + \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)[-\frac{p(t,b)}{p^*(b)} + \ln\frac{p(t,b)}{p^*(b)}]\mathrm{d}b \\
& + \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[1-\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right]\mathrm{d}a, \\
\Theta_6 :=\ & -\int_0^\infty \Phi(a)\Lambda_i(a)\hbar\left(\frac{i^*(a)}{i(t,a)}\right)\mathrm{d}a - \frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)q(b)\sigma_2(b)K(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b \\
& - \int_0^\infty H(a)\sigma_1(a)\hbar\left(\frac{i(t,a)}{i^*(a)}\right)\left[\frac{1}{\Xi_1}S^*f(J^*)k(a) + \frac{1}{\Xi_3}S^*g(Q^*)\xi(a)\right]\mathrm{d}a.
\end{aligned}$$

Thanks to $Sf(J) + Sg(Q) = i(t,0)$, one has

$$
\begin{aligned}
\Theta_1 =& -\frac{\mu}{S}(S-S^*)^2 i^*(0) - i(t,0)i^*(0) + S^* f(J^*) i(t,0) + S^* g(Q^*) i(t,0) \\
=& -\frac{\mu}{S}(S-S^*)^2 i^*(0).
\end{aligned}
\tag{5.5}
$$

By virtue of $\int_0^\infty \xi(a) i(t,a)\mathrm{d}a = p(t,0)$, $\int_0^\infty \sigma_2(b) q(b)\mathrm{d}b = \Xi_2$ and the first equation of (4.4), we obtain that

$$
\begin{aligned}
\Theta_2 =& \frac{1}{\Xi_3} S^* g(Q^*) \left(\frac{1}{\Xi_2} p(t,0) \int_0^\infty \sigma_2(b) q(b)\mathrm{d}b - \int_0^\infty \xi(a) i(t,a)\mathrm{d}a \right) \\
&+ \frac{1}{\Xi_3} S^* g(Q^*) \int_0^\infty \xi(a) H(a) \sigma_1(a) \frac{i(t,a)}{i^*(a)} \mathrm{d}a \\
=& \frac{1}{\Xi_3} S^* g(Q^*) \int_0^\infty \xi(a) H(a) \sigma_1(a) \frac{i(t,a)}{i^*(a)} \mathrm{d}a.
\end{aligned}
\tag{5.6}
$$

It follows from $\Xi_1 = \int_0^\infty k(a)\sigma_1(a)\mathrm{d}a$ that

$$
\begin{aligned}
\Theta_3 =& S^* f(J^*) \frac{1}{\Xi_1} \int_0^\infty k(a) i^*(0) \sigma_1(a) \left[1 - \frac{S^*}{S} + \frac{f(J)}{f(J^*)} \right] \mathrm{d}a \\
&+ \frac{1}{\Xi_1} \int_0^\infty S^* f(J^*) k(a) i^*(0) \sigma_1(a) \left[-\frac{i(t,a)}{i^*(a)} - \ln \frac{i(t,0)}{i^*(0)} + \ln \frac{i(t,a)}{i^*(a)} \right] \mathrm{d}a \\
&+ \frac{1}{\Xi_1} \int_0^\infty S^* f(J^*) k(a) i^*(0) \sigma_1(a) \left[1 - \frac{i^*(0) S f(J)}{i(t,0) S^* f(J^*)} \right] \mathrm{d}a \\
=& \frac{1}{\Xi_1} \int_0^\infty S^* f(J^*) k(a) i^*(0) \sigma_1(a) \left[-\hbar\left(\frac{S^*}{S}\right) - \hbar\left(\frac{i^*(0) S f(J)}{i(t,0) S^* f(J^*)} \right) \right. \\
&\left. - \hbar\left(\frac{i(t,a)}{i^*(a)} \right) + \hbar\left(\frac{f(J)}{f(J^*)} \right) \right] \mathrm{d}a.
\end{aligned}
\tag{5.7}
$$

Due to

$$
\Xi_1 i^*(0) = \int_0^\infty k(a)\sigma_1(a) i^*(0)\mathrm{d}a \leqslant \int_0^\infty k(a) i^*(a)\mathrm{d}a = J^*
$$

and Jensen's inequality, we have

$$
\begin{aligned}
\frac{1}{\Xi_1} \int_0^\infty k(a) i^*(a) \hbar\left(\frac{i(t,a)}{i^*(a)} \right) \mathrm{d}a \geqslant& i^*(0) \int_0^\infty \frac{k(a) i^*(a)}{J^*} \hbar\left(\frac{i(t,a)}{i^*(a)} \right) \mathrm{d}a \\
\geqslant& i^*(0) \hbar\left(\int_0^\infty \frac{k(a) i^*(a)}{J^*} \frac{i(t,a)}{i^*(a)} \mathrm{d}a \right) \\
=& \frac{1}{\Xi_1} \int_0^\infty k(a) i^*(0) \sigma_1(a) \hbar\left(\frac{J}{J^*} \right) \mathrm{d}a \\
\geqslant& \frac{1}{\Xi_1} \int_0^\infty k(a) i^*(0) \sigma_1(a) \hbar\left(\frac{f(J)}{f(J^*)} \right) \mathrm{d}a.
\end{aligned}
\tag{5.8}
$$

Then, combining (5.7) and (5.8), we have

$$
\begin{aligned}
\Theta_3 \leqslant& \frac{1}{\Xi_1} \int_0^\infty S^* f(J^*) k(a) i^*(0) \sigma_1(a) \left[-\hbar\left(\frac{S^*}{S}\right) - \hbar\left(\frac{i^*(0) S f(J)}{i(t,0) S^* f(J^*)} \right) \right] \mathrm{d}a \\
&+ \frac{1}{\Xi_1} \int_0^\infty S^* f(J^*) k(a) H(a) \sigma_1(a) \hbar\left(\frac{i(t,a)}{i^*(a)} \right) \mathrm{d}a.
\end{aligned}
\tag{5.9}
$$

Recall that $\Xi_2 = \int_0^\infty \sigma_2(b)q(b)\mathrm{d}b$, $p^*(0) = \int_0^\infty \xi(a)i^*(a)\mathrm{d}a$ and Lemmas 5.1 and 5.2, we derive that

$$\begin{aligned}\Theta_4 &= \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[2-\frac{S^*}{S}+\ln\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right]\mathrm{d}a\\&\quad+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[1-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\ln\frac{p(t,0)}{p^*(0)}\right]\\&=\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{S^*}{S}\right)-\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)+\ln\frac{S}{S^*}\right]\mathrm{d}a\\&\quad+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[1-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\ln\frac{p(t,0)}{p^*(0)}\right].\end{aligned}$$

Thus, combining Θ_4 and Θ_5 gives

$$\begin{aligned}\Theta_4+\Theta_5 &\leqslant \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right)\right]\mathrm{d}a\\&\quad+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{S^*}{S}\right)-\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)\right]\mathrm{d}a\\&\quad+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[\frac{g(Q)}{g(Q^*)}-\ln\frac{g(Q)}{g(Q^*)}\right]\mathrm{d}a\\&\quad+\frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)\left[\ln\frac{p(t,b)}{p^*(b)}-\frac{p(t,b)}{p^*(b)}\right]\mathrm{d}b\\&\quad+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[1-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\ln\frac{p(t,0)}{p^*(0)}\right].\end{aligned}$$

Using Lemma 5.2, we further have

$$\begin{aligned}\Theta_4+\Theta_5 &\leqslant \frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right)\right]\mathrm{d}a\\&\quad+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{S^*}{S}\right)-\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)\right]\mathrm{d}a\\&\quad+\frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)p^*(0)\sigma_2(b)q(b)\left[\hbar\left(\frac{g(Q)}{g(Q^*)}\right)-\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\right]\mathrm{d}b\\&\quad-\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[\frac{g(Q)}{g(Q^*)}-\ln\frac{g(Q)}{g(Q^*)}\right]\mathrm{d}a\\&\quad+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[1-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\ln\frac{p(t,0)}{p^*(0)}\right].\end{aligned}$$

Since

$$\Xi_2p^*(0)=\int_0^\infty q(b)\sigma_2(b)p^*(0)\mathrm{d}b\leqslant\int_0^\infty q(b)p^*(b)\mathrm{d}b=Q^*$$

and Jensen's inequality, we have

$$\begin{aligned}\frac{1}{\Xi_2}\int_0^\infty q(b)p^*(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b &\geqslant p^*(0)\int_0^\infty \frac{q(b)p^*(b)}{Q^*}\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b\\&\geqslant p^*(0)\hbar\left(\int_0^\infty\frac{q(b)p^*(b)}{Q^*}\frac{p(t,b)}{p^*(b)}\mathrm{d}b\right)\\&=\frac{1}{\Xi_2}\int_0^\infty q(b)p^*(0)\sigma_2(b)\hbar\left(\frac{Q}{Q^*}\right)\mathrm{d}b\\&\geqslant\frac{1}{\Xi_2}\int_0^\infty q(b)p^*(0)\sigma_2(b)\hbar\left(\frac{g(Q)}{g(Q)^*}\right)\mathrm{d}b.\end{aligned}$$

Thus, we finally have

$$\begin{aligned}\Theta_4+\Theta_5\leqslant{}&\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right)\right]\mathrm{d}a\\&+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[-\hbar\left(\frac{S^*}{S}\right)-\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)\right]\mathrm{d}a\\&+\frac{1}{\Xi_2\Xi_3}\int_0^\infty S^*g(Q^*)K(b)\sigma_2(b)q(b)\hbar\left(\frac{p(t,b)}{p^*(b)}\right)\mathrm{d}b\\&-\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[\frac{g(Q)}{g(Q^*)}-\ln\frac{g(Q)}{g(Q^*)}\right]\mathrm{d}a\\&+\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[1-\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}-\ln\frac{p(t,0)}{p^*(0)}\right].\end{aligned}\tag{5.10}$$

Hence, combining (5.6), (5.9) and (5.10), we yield

$$\begin{aligned}\frac{\mathrm{d}\ell(t)}{\mathrm{d}t}\leqslant{}&-\frac{\mu}{S}(S-S^*)^2i^*(0)\\&-\int_0^\infty\Phi(a)\Lambda_i(a)\hbar\left(\frac{i^*(a)}{i(t,a)}\right)\mathrm{d}a-\frac{1}{\Xi_3}\int_0^\infty\Psi(b)\Lambda_p(b)\hbar\left(\frac{p^*(b)}{p(t,b)}\right)\mathrm{d}b\\&-\frac{1}{\Xi_1}\int_0^\infty S^*f(J^*)i^*(0)\sigma_1(a)k(a)\left[\hbar\left(\frac{S^*}{S}\right)+\hbar\left(\frac{i^*(0)Sf(J)}{i(t,0)S^*f(J^*)}\right)\right]\mathrm{d}a\\&-\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[\hbar\left(\frac{i^*(0)Sg(Q)}{i(t,0)S^*g(Q^*)}\right)\right]\mathrm{d}a\\&-\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)i^*(0)\sigma_1(a)\xi(a)\left[\hbar\left(\frac{S^*}{S}\right)+\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)\right]\mathrm{d}a\\&-\frac{1}{\Xi_3}\int_0^\infty S^*g(Q^*)H(a)\sigma_1(a)\xi(a)\left[\hbar\left(\frac{g(Q)}{g(Q^*)}\right)+\hbar\left(\frac{i(t,a)p^*(0)}{i^*(a)p(t,0)}\right)\right]\mathrm{d}a.\end{aligned}$$

Consequently, from above discussion, we assert that $\frac{\mathrm{d}\ell(t)}{\mathrm{d}t}\leqslant 0$ and the largest invariant subset of set $\{\frac{\mathrm{d}\ell(t)}{\mathrm{d}t}=0\}$ is E^*. Due to the invariance principle [39, Theorem 4.2], E^* is globally asymptotically stable. □

6 Numerical simulation and conclusion

In this section, we consider a special model with nonlinear functional responses:

$$\begin{cases}\dfrac{\mathrm{d}S(t)}{\mathrm{d}t}=\Lambda_s-\mu S(t)-\dfrac{S(t)\int_0^\infty k(a)i(t,a)\mathrm{d}a}{A\int_0^\infty k(a)i(t,a)\mathrm{d}a+1}-\dfrac{S(t)\int_0^\infty q(b)p(t,b)\mathrm{d}b}{A\int_0^\infty q(b)p(t,b)\mathrm{d}b+1},\\[2mm]\dfrac{\partial i(t,a)}{\partial t}+\dfrac{\partial i(t,a)}{\partial a}=\Lambda_i(a)-\delta(a)i(t,a),\\[2mm]\dfrac{\partial p(t,b)}{\partial t}+\dfrac{\partial p(t,b)}{\partial b}=\Lambda_p(b)-\gamma(b)p(t,b),\end{cases}\tag{6.1}$$

with initial condition (1.6) and boundary condition

$$\begin{aligned}&i(t,0)=\frac{S(t)\int_0^\infty k(a)i(t,a)\mathrm{d}a}{A\int_0^\infty k(a)i(t,a)\mathrm{d}a+1}+\frac{S(t)\int_0^\infty q(b)p(t,b)\mathrm{d}b}{A\int_0^\infty q(b)p(t,b)\mathrm{d}b+1},\quad t>0,\\&p(t,0)=\int_0^\infty\xi(a)i(t,a)\mathrm{d}a,\quad t>0.\end{aligned}\tag{6.2}$$

Then, from Theorem 5.3, we obtain the following corollary:

Corollary 6.1. *The infection equilibrium of system* (6.1) *is globally asymptotically stable.*

To verify the validity of the result, we perform numerical simulations. Let $\Lambda_s = 2000$, $\mu = \frac{1}{70}$ and

$$\begin{aligned} k(a) &= 0.003\left(1+\sin\frac{(a-5)\,\Xi}{10}\right), & q(b) &= 0.01\left(1+\sin\frac{(b-5)\,\Xi}{10}\right) \\ \Lambda_i(a) &= 5\left(1+\sin\frac{(a-5)\,\Xi}{10}\right), & \Lambda_p(b) &= 80\left(1+\sin\frac{(b-5)\,\Xi}{10}\right) \\ \delta(a) &= 0.18\left(1+\sin\frac{(a-5)\,\Xi}{10}\right), & \gamma(b) &= 2\left(1+\sin\frac{(b-5)\,\Xi}{10}\right), \\ \xi(a) &= \left(1+\sin\frac{(a-5)\,\Xi}{10}\right), \end{aligned}$$

for $0 \leqslant a, b \leqslant 10$. Clearly, as in Figure 6.1, all the solutions converge to the positive steady state. In Figure 6.2, we further show the distribution of $i(t,a)$ and $p(t,b)$ at age $a = b = 5$.

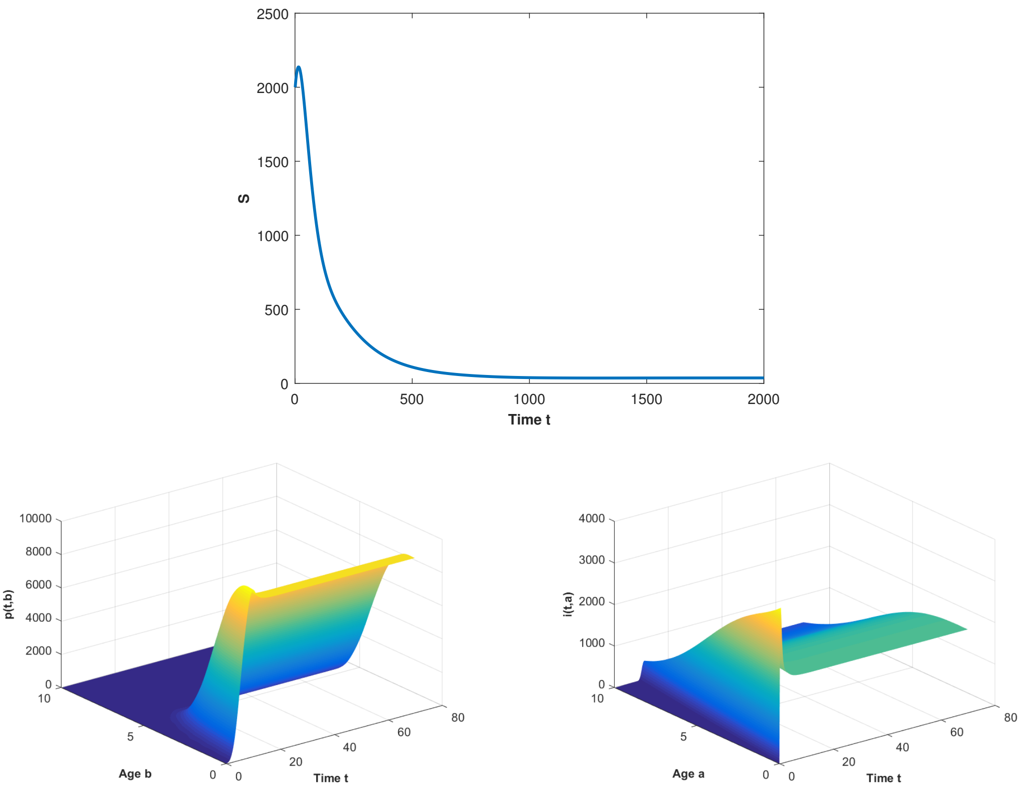
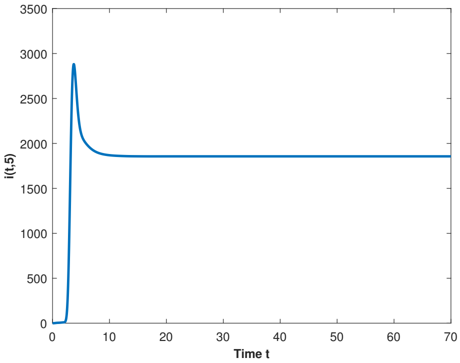


Figure 6.1: Long-time dynamical behavior of system (6.1)–(6.2).

Now, we finish this paper with a conclusion. In this paper, we considered an age-infection model of cholera with general infection rates. We focused on the global asymptotical stability of the unique positive equilibrium under some assumptions. For this, we directly used the Lyapunov functional method. It is necessarily pointed here that the uniform persistence and asymptotical smoothness play the key role for the construction of Lyapunov functional.



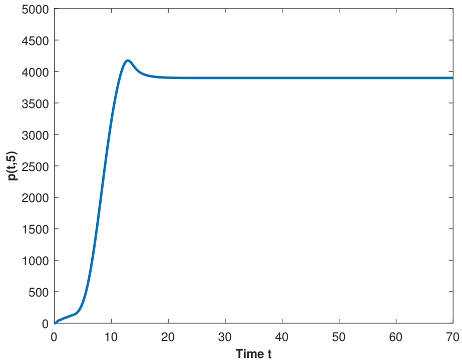


Figure 6.2: Long-time dynamical behavior of $i(t,a)$ and $p(t,b)$ for $a = b = 5$.

Finally, we performed numerical simulations. On account of the waterborne disease, we incorporated indirect pathogen-to-person transmission and direct person-to-person transmission. By taking general infection rates into account, we gain a unified theoretical framework to describe the cholera propagation process. In a recent paper [25], Liu et al. proposed an age-space structured cholera model, and studied the local stability of equilibria, disease persistence and global attractivity of equilibria for their model. How about introducing immigration into the age-space structured cholera model, which will be an interesting problem and we will leave it for the future work.

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