Electronic Journal of Qualitative Theory of Differential Equations
2023, No. **14**, 1–32; https://doi.org/10.14232/ejqtde.2023.1.14 www.math.u-szeged.hu/ejqtde/

Concentration of solutions for an (N, q)-Laplacian equation with Trudinger–Moser nonlinearity

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Received 9 October 2022, appeared 4 May 2023
Communicated by Roberto Livrea

Abstract. In this article, we consider the concentration of positive solutions for the following equation with Trudinger–Moser nonlinearity:

$$\begin{cases} -\Delta_N u - \Delta_q u + V(\varepsilon x)(|u|^{N-2}u + |u|^{q-2}u) = f(u), & x \in \mathbb{R}^N, \\ u \in W^{1,N}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), & x \in \mathbb{R}^N, \end{cases}$$

where V is a positive continuous function and has a local minimum, $\varepsilon > 0$ is a small parameter, $2 \le N < q < +\infty$, f is C^1 with subcritical growth. When V and f satisfy some appropriate assumptions, we construct the solution u_ε that concentrates around any given isolated local minimum of V by applying the penalization method for the above equation.

Keywords: (N, q)-Laplacian equation, penalization method, variational methods.

2020 Mathematics Subject Classification: 35A15, 35B38, 35J60.

1 Introduction and main result

In this article, we consider the concentration of positive solutions for an (N, q)-Laplacian equation with Trudinger–Moser nonlinearity:

$$\begin{cases} -\Delta_N u - \Delta_q u + V(\varepsilon x)(|u|^{N-2}u + |u|^{q-2}u) = f(u), & x \in \mathbb{R}^N, \\ u \in W^{1,N}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $V : \mathbb{R}^N \mapsto \mathbb{R}$ is a function that satisfies continuity and has a local minimum, $\varepsilon > 0$ is a small parameter, $2 \le N < q < +\infty$, $f \in C^1$ is subcritical.

We first introduce some background about (p, q)-Laplacian equation. As described in [14], problem (1.1) originates from the following reaction-diffusion equation:

$$u_t = C(x, u) + \operatorname{div}(D(u)\nabla u), \quad D(u) = |\nabla u|^{q-2} + |\nabla u|^{p-2}.$$

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It is widely used in physics or chemistry, such as solid state physics, chemical reaction design, biophysics and plasma physics. Note that, in general reaction-diffusion equation, the physical meaning of u is concentration, and the physical meaning of $\operatorname{div}(D(u)\nabla u)$ is the diffusion generated by $D(u)$. $C(x,u)$ is related to the source and loss process. Generally, $C(x,u)$ is a polynomial with variable coefficients related to u in chemical and biological applications.

When $p<q<N$, Zhang et al. in [36] studied the following double phase problem

$$\begin{cases}(-\Delta)_q^m u+(-\Delta)_p^m u+V(\varepsilon x)\left(|u|^{q-2}u+|u|^{p-2}u\right)=\lambda f(u)+|u|^{r-2}u, & x\in\mathbb{R}^N,\\ u\in W^{m,p}\left(\mathbb{R}^N\right)\cap W^{m,q}\left(\mathbb{R}^N\right), u>0, & x\in\mathbb{R}^N,\end{cases}$$

where ε is a parameter small enough but λ is required to be large enough, $0<m<1$, $r=q_m^*=Nq/(N-mq)$, $2\leqslant p<q<N/m$, $(-\Delta)_t^m$ is the fractional t-Laplace operator and the potential $V:\mathbb{R}^N\mapsto\mathbb{R}$ is a continuous function. The authors obtained the existence and concentration properties of multiple positive solutions to the above problem. Note that, [36] assumed that the nonlinearity satisfies the Ambrosetti–Rabinowitz condition, that is, for all $t>0$, there is $\theta\in(q,q_m^*)$ that satisfies $0<\theta F(t):=\theta\int_0^t f(\tau)\mathrm{d}\tau\leqslant f(t)t$. So the authors can get the existence and concentration properties of multiple positive solutions by using Nehari manifold.

When $1<q<N=p$, the authors in [12] investigated the existence of solutions for the (N,q)-Laplacian equation:

$$-\Delta_q u-\Delta_N u=f(u) \text{ in } \mathbb{R}^N, \tag{1.2}$$

where the nonlinear term $f(u)$ satisfies exponential critical growth in the sense of Trudinger–Moser. In order to detect the solution, they used a variational method related to the new Trudinger–Moser type inequality. Figueiredo and Nunes in [19] used Nehari manifold method to studied the existence of positive solutions for the following class of quasilinear problems

$$\begin{cases}-\operatorname{div}(a(|\nabla u|^p)|\nabla u|^{p-2}\nabla u)=f(u) \text{ in } \Omega,\\ u=0 \text{ on } \partial\Omega.\end{cases}$$

It is worth pointing out that Theorems 1.1 and 1.2 in [19] are valid for the problem (1.2) if $\mathbb{R}^N$ is replaced by Ω which is a smooth bounded domain. In [15], Costa and Figueiredo studied a class of quasilinear equation with exponential critical growth. They used variational methods and del Pino and Felmer's technique (del Pino and Felmer 1996) in order to overcome the lack of compactness, and got the existence of a family nodal solutions, which concentrate on the minimum points set of the potential function, changes sign exactly once in $\mathbb{R}^N$.

When $p=N/m<q$, Nguyen in [29] studied the following Schrödinger equation involving the fractional (N,q)-Laplace operator and Trudinger–Moser nonlinear term

$$(-\Delta)_{N/m}^m u+(-\Delta)_q^m u+V(\varepsilon x)\left(|u|^{\frac{N}{m}-2}u+|u|^{q-2}u\right)=f(u) \quad \text{in } \mathbb{R}^N,$$

where $\varepsilon>0$ is a parameter small enough, $m\in(0,1)$, $N=pm$, $2\leq p=N/m<q$, the potential $V:\mathbb{R}^N\mapsto\mathbb{R}$ is a continuous function that satisfies some suitable conditions. The nonlinear term $f(u)$ satisfies exponential growth. In order to obtain existence and concentration properties of nontrivial nonnegative solutions, the author in [29] used the Ljusternik–Schnirelmann theory and Nehari manifold.

It is worth mentioning that both the nonlinearities of [12] and [29] satisfy the Ambrosetti–Rabinowitz condition. Inspired by the above works, it seems quite natural to ask if $f(u)$ does not satisfy the Ambrosetti–Rabinowitz condition but satisfies Beresticky–Lions type assumptions, do the same results hold for (N, q)-Laplacian problem? In this paper, we give a positive answer.

In the present paper, we assume that the potential $V : \mathbb{R}^N \mapsto \mathbb{R}$ is a continuous function satisfying the following conditions which are always called del Pino–Felmer type conditions (cf. [16]).

(V_1) $V \in C(\mathbb{R}^N, \mathbb{R})$ such that $\inf_{x\in\mathbb{R}^N} V(x) = V_0 > 0$.

(V_2) There exists a bounded domain $\Lambda \subset \mathbb{R}^N$ satisfies

$$m := \inf_{x\in\Lambda} V(x) < \min_{x\in\partial\Lambda} V(x).$$

Moreover, we can assume $0 \in \mathcal{M} := \{x \in \Lambda : V(x) = m\}$.

The nonlinear term $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. Moreover, for $t \leq 0$, we assume that $f(t) = 0$. Furthermore, $f(t)$ satisfies the following hypotheses:

(f_1) $\lim_{t\to 0} \frac{f(t)}{t^{q-1}} = 0$;

(f_2) $\forall \alpha > 0$, for $t \geq 0$, there is a $C_\alpha > 0$ satisfies $|f(t)| \leq C_\alpha e^{\alpha t^{\frac{N}{N-1}}}$;

(f_3) there is $T > 0$ satisfies $F(T) > \frac{m}{N} T^N + \frac{m}{q} T^q$.

Next, we state the main conclusion as follows:

Theorem 1.1. *If (V_1)–(V_2) and (f_1)–(f_3) are true, for small $\varepsilon > 0$, equation (1.1) has a positive solution u_ε which has a maximum point x_ε satisfying*

$$\lim_{\varepsilon\to 0} \operatorname{dist}(x_\varepsilon, \mathcal{M}) = 0.$$

Moreover, for any x_ε, as $\varepsilon \to 0$ (up to a subsequence), $v_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_\varepsilon)$ converges uniformly to a least energy solution of the following equation:

$$\begin{cases} -\Delta_q u - \Delta_N u + m(|u|^{q-2}u + |u|^{N-2}u) = f(u), & x \in \mathbb{R}^N, \\ u \in W^{1,q}(\mathbb{R}^N) \cap W^{1,N}(\mathbb{R}^N), & x \in \mathbb{R}^N. \end{cases} \tag{1.3}$$

Furthermore, we have

$$u_\varepsilon(x) \leq C_1 e^{-C_2|x-x_\varepsilon|}, \quad \forall x \in \mathbb{R}^N,\ C_1,\ C_2 > 0.$$

Remark 1.2. Without loss of generality, it can be assumed that $V_0 = 1$.

As far as we know, there is no result on the concentration of positive solutions for (N, q)-Laplacian problems with Berestycki–Lions nonlinearity.

Finally, we point out that Theorem 1.1 is proved by variational method, and there are four main difficulties we encounter during the preparation of manuscript:

(1) The nonlinear term $f(u)$ does not satisfy the Ambrosetti–Rabinowitz condition, and for $u > 0$, the function $\frac{f(u)}{u^{q-1}}$ is not increasing. They both prevent us from getting the boundedness of Palais–Smale sequence and using the Nehari manifold. Moreover, we can not apply the method in [16].

(2) Since $\mathbb{R}^N$ is unbounded, it will lead to the loss of compactness. In the later proof, we will find that this difficulty will prevent us from directly using the variational method.

(3) When $N > 2$, the working space X_ε is no longer a Hilbert space. This makes it more complicated to prove the following formula in Lemma 3.11:

$$J_\varepsilon\left(u_\varepsilon\right) \geq J_\varepsilon\left(u_\varepsilon^1\right) + J_\varepsilon\left(u_\varepsilon^2\right) + o(1)$$

as $\varepsilon \to 0$.

(4) Due to $N = p < q$, we can not use the method of [2] to obtain that $b_m \geq c_m$ in Lemma 3.6.

In order to overcome the above difficulties, inspired by [8,18,22,25], we recover the compactness by penalization method described in [10].

The plan of this paper is as follows. In Section 2, we give some definitions of function spaces and lemmas to be used later. In Section 3, we give the proof of Theorem 1.1.

2 Preliminary

In this section, we will give some definitions of symbols, and review some existing results that need to be used in the future.

Let $u : \mathbb{R}^N \mapsto \mathbb{R}$. For $2 \leq N < q < +\infty$, let us define $D^{1,N}(\mathbb{R}^N) = \overline{C^\infty(\mathbb{R}^N)}^{|\nabla\cdot|_N}$. We denote the following fractional Sobolev space

$$W^{1,N}(\mathbb{R}^N) = \{u : |\nabla u|_N < +\infty,\ |u|_N < +\infty\}$$

equipped with the natural norm

$$\|u\|_{W^{1,N}(\mathbb{R}^N)} = \Big(\ |\nabla u|_N^N + |u|_N^N\Big)^{1/N},$$

where $|\cdot|_N^N := \int_{\mathbb{R}^N} |\cdot|^N \mathrm{d}x$.

For all $u, v \in W^{1,N}(\mathbb{R}^N)$, we define

$$\langle u, v\rangle_{W^{1,N}(\mathbb{R}^N)} = \int_{\mathbb{R}^N} (|\nabla u|^{N-2}\nabla u \nabla v + |u|^{N-2}uv)\mathrm{d}x.$$

In this article, we need to introduce a work space

$$X = W^{1,N}(\mathbb{R}^N) \cap W^{1,q}(\mathbb{R}^N)$$

whose norm is defined as

$$\|u\|_X := \|u\|_{W^{1,q}(\mathbb{R}^N)} + \|u\|_{W^{1,N}(\mathbb{R}^N)}.$$

When $V(x) = V_0$, we define space

$$X_0 := \left\{u \in X : \int_{\mathbb{R}^N} V_0(|u|^q + |u|^N)\mathrm{d}x < +\infty\right\}$$

equipped with the norm as

$$\|u\|_{X_0} = \|u\|_{V_0,q} + \|u\|_{V_0,N},$$

where $\|u\|_{V_0,r}^r = \int_{\mathbb{R}^N} (|\nabla u|^r + V_0|u|^r)\,dx$, $\forall r \in \{N, q\}$. It should be noted that X_0 is a separable reflexive Banach space. Due to the Theorem 6.9 in [28], for any $\nu \in [N, +\infty)$, it is easy to see that the embedding from X_0 into $L^\nu(\mathbb{R}^N)$ is continuous. Then for all $\nu \in [N, +\infty)$, there exists $A_{\nu,m} > 0$ satisfies

$$A_{\nu,m} = \inf_{u \neq 0, u \in X_0} \frac{\|u\|_{X_0}}{\|u\|_{L^\nu(\mathbb{R}^N)}}.$$

This implies

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq A_{\nu,m}^{-1}\|u\|_{X_0} \quad \text{for all } \; u \in X_0. \tag{2.1}$$

Fix $\varepsilon \geq 0$, we also need to introduce the following space

$$X_\varepsilon := \left\{ u \in X : \int_{\mathbb{R}^N} V(\varepsilon x)(|u|^q + |u|^N)dx < +\infty \right\}$$

whose norm is defined as

$$\|u\|_{X_\varepsilon} := \|u\|_{V_\varepsilon,q} + \|u\|_{V_\varepsilon,N},$$

where $\|u\|_{V_\varepsilon,r}^{1,r} = \int_{\mathbb{R}^N} (|\nabla u|^r + V(\varepsilon x)|u|^r)\,dx$, $\forall r \in \{N, q\}$. According to Lemma 10 in [31], we obtain that X_ε is uniformly convex Banach space. Moreover, for any $\nu \in [N, +\infty)$, the embedding

$$X_\varepsilon \hookrightarrow L^\nu(\mathbb{R}^N)$$

is continuous. Then for all $\nu \in [N, +\infty)$, there is $S_{\nu,\varepsilon} > 0$ satisfies:

$$S_{\nu,\varepsilon} = \inf_{u \neq 0, u \in X_\varepsilon} \frac{\|u\|_{X_\varepsilon}}{\|u\|_{L^\nu(\mathbb{R}^N)}}.$$

It can be seen that

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq S_{\nu,\varepsilon}^{-1}\|u\|_{X_\varepsilon}, \quad \forall u \in X_\varepsilon. \tag{2.2}$$

Finally, we consider

$$X_{\text{rad}} := \{u \in X : u(x) = u(|x|)\}.$$

Lemma 2.1 (see [34, Theorem 2.8]). *Assume that X is a Banach space, M_0 is a closed subspace of the metric space M, $\Gamma_0 \subset \mathcal{C}(M_0, X)$. Consider*

$$\Gamma := \{\gamma \in \mathcal{C}(M, X) : \gamma|_{M_0} \in \Gamma_0\}.$$

Assume $\varphi \in C^1(X, \mathbb{R})$ satisfies

$$\infty > c := \inf_{\gamma \in \Gamma} \sup_{u \in M} \varphi(\gamma(u)) > a := \sup_{\gamma_0 \in \Gamma_0} \sup_{u \in M_0} \varphi(\gamma_0(u)).$$

For any $\varepsilon \in (0, (c-a)/2), \delta > 0$ and $\gamma \in \Gamma$ such that $\sup_M \varphi \circ \gamma \leq c + \varepsilon$, there is $u \in X$ satisfies

(a) $c - 2\varepsilon \leq \varphi(u) \leq c + 2\varepsilon$;

(b) $\text{dist}(u, \gamma(M)) \leq 2\delta$;

(c) $\|\varphi'(u)\| \leq \frac{8\varepsilon}{\delta}$.

Now, we recall follow Lemma 2.2 from J. M. do Ó [17] (or see [11]). The Lemma 2.3 follows from Adachi and Tanaka [1].

Lemma 2.2 (see [17])**.** *Assume* $N \geq 2$, $u \in W^{1,N}\left(\mathbb{R}^N\right)$ *and* $\alpha > 0$*, we have*

$$\int_{\mathbb{R}^N} \left(\exp\left(\alpha|u|^{N/(N-1)}\right) - S_{N-2}(\alpha, u)\right) \mathrm{d}x < \infty,$$

where

$$S_{N-2}(\alpha, u) = \sum_{k=0}^{N-2} \frac{\alpha^k}{k!} |u|^{\frac{kN}{(N-1)}}.$$

In addition, when $\alpha < \alpha_N$*, for* $\forall M > 0$*, there is* $C = C(\alpha, N, M)$ *satisfies*

$$\int_{\mathbb{R}^N} \left(\exp\left(\alpha|u|^{N/(N-1)}\right) - S_{N-2}(\alpha, u)\right) \mathrm{d}x \leq C, \quad \forall u \in W^{1,N}(\mathbb{R}^N).$$

We also have $\|u\|_N \leq M$ *and* $\|\nabla u\|_N \leq 1$*.*

Lemma 2.3 (see [1])**.** *Assume* $N \geq 2$*,* $\alpha \in (0, \alpha_N)$*, there is a constant* $C_\alpha > 0$ *that satisfies*

$$\|\nabla u\|_N^N \int_{\mathbb{R}^N} \Psi_N\left(\frac{u}{\|\nabla u\|_N}\right) \mathrm{d}x \leq C_\alpha \|u\|_N^N, \quad \forall u \in W^{1,N}(\mathbb{R}^N) \setminus \{0\}.$$

Here $\Psi_N(t) = e^{\alpha|t|^{N/(N-1)}} - S_{N-2}(\alpha, t)$*.*

3 Proof of Theorem 1.1

For $\forall B \subset \mathbb{R}^N$, $\varepsilon > 0$, B_ε can be define as $B_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon x \in B\}$. Next, we will use the method in [16,21] to modify f. According to (f_1), there exists $a > 0$ such that

$$f(t) \leq \frac{t^{N-1}}{2}, \ \forall t \in (0, a).$$

For $t \in \mathbb{R}, x \in \mathbb{R}^N$, assume that

$$g(x,t) = (1 - \chi_\Lambda(x)) \widetilde{f}(t) + \chi_\Lambda(x) f(t),$$

where

$$\widetilde{f}(t) = \begin{cases} f(t), & t \leq a, \\ \min\left\{f(t), \frac{1}{2}t^{N-1}\right\}, & t > a \end{cases}$$

and

$$\chi_\Lambda(x) = \begin{cases} 1, & x \in \Lambda, \\ 0, & x \notin \Lambda. \end{cases}$$

Obviously, $\forall x \in \mathbb{R}^N, t \in [0,a]$, we have $g(x,t) = f(t)$. Moreover, for $\forall x \in \mathbb{R}^N, t \geq 0$, we also obtain that $g(x,t) \leq f(t)$. Now, considering the modified problem

$$\begin{cases} -\Delta_N u - \Delta_q u + V_\varepsilon(|u|^{N-2}u + |u|^{q-2}u) = g(\varepsilon x, u), & x \in \mathbb{R}^N, \\ u \in X_\varepsilon, \ u > 0, & x \in \mathbb{R}^N, \end{cases} \tag{3.1}$$

where $g(\varepsilon x, t) = (1 - \chi_{\Lambda_\varepsilon}(x)) \widetilde{f}(t) + \chi_{\Lambda_\varepsilon}(x) f(t)$. Clearly, for $x \in \mathbb{R}^N \backslash \Lambda_\varepsilon$, if u_ε satisfies $u_\varepsilon(x) \leq a$ and it is a solution of (3.1), we know that u_ε is the solution of the original problem (1.1).

As to $u \in X_\varepsilon$, we assume that

$$I_\varepsilon(u) = \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + V_\varepsilon |u|^q)\,\mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N}\left(|\nabla u|^N + V_\varepsilon|u|^N\right)\mathrm{d}x - \int_{\mathbb{R}^N} G(\varepsilon x, u)\mathrm{d}x,$$

where $G(x,t) = \int_0^t g(x,\varrho)\mathrm{d}\varrho$. For $\forall \mu > 0$, define

$$\chi_\varepsilon(x) = \begin{cases} \varepsilon^{-\mu}, & x \in \mathbb{R}^N \backslash \Lambda_\varepsilon, \\ 0, & x \in \Lambda_\varepsilon, \end{cases}$$

$$Q_\varepsilon(u) = \left(\int_{\mathbb{R}^N} \chi_\varepsilon |u|^N \mathrm{d}x - 1\right)_+^2.$$

This penalization first appeared in [10] (or see [8]). It has the advantage that it can make the concentration phenomena to occur in Λ. Now, we define $J_\varepsilon : X_\varepsilon \to \mathbb{R}$ as follows:

$$J_\varepsilon(u) = Q_\varepsilon(u) + I_\varepsilon(u).$$

Clearly, $J_\varepsilon \in C^1(X_\varepsilon)$. Next, to find the solutions of equation (3.1) concentrated around the local minimum of potential function as $\varepsilon \to 0$, we will find the critical points of J_ε which make Q_ε zero.

3.1 Limit problem

First, considering the limit problem, i.e.

$$\begin{cases} -\Delta_q u - \Delta_N u + m(|u|^{q-2}u + |u|^{N-2}u) = f(u), & x \in \mathbb{R}^N, \\ u \in X, & x \in \mathbb{R}^N. \end{cases} \tag{3.2}$$

The energy functional corresponding to (3.2) is defined as follows

$$I_m(u) = \frac{1}{N}\int_{\mathbb{R}^N}\left(|\nabla u|^N + m|u|^N\right)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N}(|\nabla u|^q + m|u|^q)\,\mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x.$$

In view of [30], assuming that $u \in X_0$ is the weak solution of problem (3.2), it is easy to get the Pohozǎev identity:

$$P_m(u) = \frac{N-q}{q}\int_{\mathbb{R}^N}|\nabla u|^q \mathrm{d}x + m\int_{\mathbb{R}^N}|u|^N\mathrm{d}x + \frac{Nm}{q}\int_{\mathbb{R}^N}|u|^q\mathrm{d}x - N\int_{\mathbb{R}^N}F(u)\mathrm{d}x.$$

Lemma 3.1. *I_m has the Mountain-Pass geometry.*

Proof. According to (f_1), $\forall |t| \le \delta$, $\exists \varepsilon > 0$ and $\delta > 0$ such that

$$|f(t)| \le \varepsilon |t|^{q-1}.$$

In addition, by using the condition (f_1) and f is a function that satisfies continuity, $\forall \tau > q$, $\forall |t| \ge \delta$, it is easy to find a constant $C = C(\tau,\delta) > 0$ satisfies

$$|f(t)| \le C|t|^{\tau-1}\Psi_N(t).$$

Combining the above two formulas, we get

$$|f(t)| \le \varepsilon|t|^{q-1} + C|t|^{\tau-1}\Psi_N(t), \quad \forall t \ge 0.$$

Then

$$|F(t)| \leq \varepsilon |t|^q + C|t|^\tau \Psi_N(t).$$

So, for $2 \leq N < q < q^*$,

$$\begin{aligned} I_m(u) &= \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla u|^q + m|u|^q)\,\mathrm{d}x + \frac{1}{N} \int_{\mathbb{R}^N} \left(|\nabla u|^N + m|u|^N \right) \mathrm{d}x - \int_{\mathbb{R}^N} F(u)\mathrm{d}x \\ &\geq \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u|^N + m|u|^N)\mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla u|^q + m|u|^q)\mathrm{d}x - \varepsilon |u|_q^q \\ &\quad - C \int_{\mathbb{R}^N} |t|^\tau \Psi_N(u)\mathrm{d}x. \end{aligned}$$

Using Hölder's inequality, we have

$$\int_{\mathbb{R}^N} \Psi_N(u)\,|u|^\tau \mathrm{d}x \leq \|u\|^\tau_{L^{\tau t'}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} (\Phi_N(u))^t\,\mathrm{d}x \right)^{\frac{1}{t}},$$

where $\frac{1}{t} + \frac{1}{t'} = 1 (t' > 1,\ t > 1)$. Due to Lemma 2.3, we may find a constant $D > 0$ satisfies

$$\left(\int_{\mathbb{R}^N} (\Phi_N(u))^t\,\mathrm{d}x \right)^{\frac{1}{t}} \leq D.$$

By using (2.1), we obtain that

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq A^{-1}_{\nu,m} \|u\|_{X_0} \quad \text{for all } \ u \in X_0.$$

Hence, when $\|u\|_{X_0}$ is small enough, we obtain that

$$\begin{aligned} I_m(u) &\geq \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla u|^q + m|u|^q)\mathrm{d}x + \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla u|^N + m|u|^N)\mathrm{d}x \\ &\quad - C \int_{\mathbb{R}^N} |t|^\tau \Psi_N(u)\mathrm{d}x - \varepsilon |u|_q^q \\ &\geq \frac{1}{q \cdot 2^{q-1}} \|u\|^q_{X_0} - \varepsilon A^{-q}_{q,m} \|u\|^q_{X_0} - CDA^{-\tau}_{\tau t',m} \|u\|^\tau_{X_0} \\ &= \|u\|^q_{X_0} \left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m} - CDA^{-\tau}_{\tau t',m} \|u\|^{\tau-q}_{X_0} \right). \end{aligned}$$

From which we deduce that $\frac{1}{q \cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m} > 0$ for ε small enough. Let

$$h(t) = \frac{1}{q \cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m} - CDA^{-\tau}_{\tau t',m} t^{\tau - q}, \quad t \geq 0.$$

Next, we will prove there is $t_0 > 0$ small enough such that $\frac{1}{2}\left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m}\right) \leq h(t_0)$. Obviously, if $t \in [0, +\infty)$, h is a continuous function. Note that $\lim_{t \to 0^+} h(t) = \frac{1}{q \cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m}$, then we can find t_0 that satisfies $h(t) \geq \frac{1}{q \cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m} - \varepsilon_1$, $\forall t \in (0, t_0)$, t_0 is small enough. Choosing $\varepsilon_1 = \frac{1}{2}\left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m}\right)$, we have

$$h(t) \geq \frac{1}{2} \left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon A^{-q}_{q,m} \right)$$

for all $0 \leq t \leq t_0$. In particularly,

$$h(t_0) \geq \frac{1}{2}\left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon A_{q,m}^{-q}\right).$$

So, for $\|u\|_{X_0} = t_0$, we get

$$I_m(u) \geq \frac{t_0^q}{2} \cdot \left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon A_{q,m}^{-q}\right) = \rho_0 > 0.$$

Now, $\forall R > 0$, define $w_R(x,y)$ as follows:

$$w_R(x,y) := \begin{cases} T, & x \in B_R^+(0), \\ 0, & x \in \mathbb{R}_+^N \backslash B_{R+1}^+(0), \\ T\left(R+1-\sqrt{|x|}\right), & x \in B_{R+1}^+(0) \backslash B_R^+(0). \end{cases}$$

It is easy to get that $w_R \in X_{\mathrm{rad}}\left(\mathbb{R}^N\right)$. It is worth noting that, for $R > 0$ large enough, according to (f_3), we have that

$$\int_{\mathbb{R}^N} \left[F\left(w_R(x)\right) - \frac{m}{N} w_R^N(x) - \frac{m}{q} w_R^q(x)\right] \mathrm{d}x \geq 0.$$

Next, consider $w_{R,\theta}(x) := w_R\left(\frac{x}{e^\theta}\right)$. Fix $R > 0$, then we have

$$\begin{aligned} I_m\left(w_{R,\theta}\right) = & \frac{1}{q} e^{(N-q)\theta} \int_{\mathbb{R}_+^N} |\nabla u|^q \mathrm{d}x - e^{N\theta} \int_{\mathbb{R}^N} \left[F\left(w_R(x)\right) - \frac{m}{N} w_R^N(x) - \frac{m}{q} w_R^q(x)\right] \mathrm{d}x \\ & \to -\infty \quad \text{as } \theta \to \infty. \end{aligned}$$

This ends the proof. □

Therefore, according to Lemma 3.1, we may define c_m as follows:

$$c_m := \inf_{\gamma \in \Gamma_m} \sup_{t \in [0,1]} I_m(\gamma(t)). \tag{3.3}$$

Here Γ_m is defined by

$$\Gamma_m := \{\gamma \in C([0,1], X_0) : \gamma(0) = 0 \text{ and } I_m(\gamma(1)) < 0\}. \tag{3.4}$$

Clearly, $c_m > 0$. Moreover, similar to [2], we note that

$$c_m = c_{m,\mathrm{rad}},$$

where

$$c_{m,\mathrm{rad}} := \inf_{\gamma \in \Gamma_{m,\mathrm{rad}}} \max_{t \in [0,1]} I_m(\gamma(t))$$

and

$$\Gamma_{m,\mathrm{rad}} := \left\{\gamma \in C\left([0,1], X_{\mathrm{rad}}(\mathbb{R}^N)\right) : I_m(\gamma(1)) < 0, \gamma(0) = 0\right\}.$$

Next, we will construct a (PS) sequence $\{w_n\}_{n=1}^\infty$ for I_m at the level c_m that satisfies $I_m'(w_n) \to 0$ as $n \to \infty$, that is

Proposition 3.2. *There exists a sequence* $\{w_n\}_{n=1}^{\infty}$ *in* X_0 *that satisfies, as* $n \to \infty$,

$$I_m(w_n) \to c_m, \quad I'_m(w_n) \to 0, \quad P_m(w_n) \to 0. \tag{3.5}$$

Proof. For $(\theta, u) \in \mathbb{R} \times X_{\text{rad}}(\mathbb{R}^N)$, define $\widetilde{I}_m(\theta, u) := (I_m \circ \Phi)(\theta, u)$, where $\Phi(\theta, u) := u(\frac{x}{e^\theta})$. The standard norm of $\mathbb{R} \times X_{\text{rad}}(\mathbb{R}^N)$ is defined as

$$\|(\theta, u)\|_{\mathbb{R}\times X_0} = \left(\|u\|^2_{X_0} + |\theta|^2\right)^{\frac{1}{2}}.$$

According to Lemma 3.1, $\widetilde{I}_m$ has a mountain pass geometry, so we can define $\tilde{c}_m$ as follows:

$$\tilde{c}_m = \inf_{\widetilde{\gamma}\in\widetilde{\Gamma}_m} \max_{t\in[0,1]} \widetilde{I}_m(\widetilde{\gamma}(t)),$$

where

$$\widetilde{\Gamma}_m = \left\{\widetilde{\gamma} \in C\left([0,1], \mathbb{R} \times X_{\text{rad}}(\mathbb{R}^N)\right) : \widetilde{I}_m(\widetilde{\gamma}(1)) < 0, \widetilde{\gamma}(0) = (0)\right\}.$$

It is easy to prove that $\widetilde{c}_m = c_m$(see [3,23]). Then according to Lemma 2.1, we obtain that there exists a sequence $(\theta_n, u_n) \subset \mathbb{R} \times X_{\text{rad}}(\mathbb{R}^N)$ such that, as $n \to \infty$,

(i) $(I_m \circ \Phi)(\theta_n, u_n) \to c_m$,

(ii) $(I_m \circ \Phi)'(\theta_n, u_n) \to 0$,

(iii) $\theta_n \to 0$.

In fact, let $\delta = \delta_n = \frac{1}{n}, \varepsilon = \varepsilon_n = \frac{1}{n^2}$ in Lemma 2.1, by using (a) and (c) in Lemma 2.1, we can obtain (i) and (ii). Due to (3.3) and (3.4), for $\varepsilon = \varepsilon_n = \frac{1}{n^2}$, it is easy to find that $\gamma_n \in \Gamma_m$ such that $\sup_{t\in[0,1]} I_m(\gamma_n(t)) \leq c_m + \frac{1}{n^2}$. Now define $\widetilde{\gamma}_n(t) = (0, \gamma_n(t))$, we obtain

$$\sup_{t\in[0,1]} (I_m \circ \Phi)(\widetilde{\gamma}_n(t)) = \sup_{t\in[0,1]} I_m(\gamma_n(t)) \leq c_m + \frac{1}{n^2}.$$

According to (b) in Lemma 2.1, then there is $(\theta_n, u_n) \in \mathbb{R} \times X_0$ such that

$$\operatorname*{dist}_{\mathbb{R}\times X_0} \left((0, \gamma_n(t)), (\theta_n, u_n)\right) \leq \frac{2}{n},$$

so (iii) holds. Now, for $A \subset \mathbb{R} \times X_0$, define

$$\operatorname*{dist}_{\mathbb{R}\times X_0} ((\theta, u), A) = \inf_{(\tau,v)\in\mathbb{R}\times X_0} \left(\|u - v\|^2_{X_0} + |\theta - \tau|^2\right)^{\frac{1}{2}}.$$

So, for $(h, w) \in \mathbb{R} \times X_0$, we have

$$\left\langle (I_m \circ \Phi)' (\theta_n, u_n), (h, w)\right\rangle = P_m (\Phi (\theta_n, u_n)) h + \left\langle I'_m (\Phi (\theta_n, u_n)), \Phi' (\theta_n, w)\right\rangle. \tag{3.6}$$

Now, put $w = 0$ and $h = 1$, it is easy to get

$$P_m (\Phi (\theta_n, u_n)) \to 0.$$

Moreover, for all $v \in X_0$, we only take $h = 0$ and $w(x) = v\left(e^{\theta_n} x\right)$ in (3.6), by using (ii), (iii), we get

$$o(1)\|v\|_{X_0} = o(1) \left\|v\left(e^{\theta_n} x\right)\right\|_{X_0} = \left\langle I'_m (\Phi (\theta_n, u_n)), v\right\rangle.$$

Hence, $w_n = \Phi(\theta_n, u_n)$ is just the sequence we need. □

Lemma 3.3. *The sequence (w_n) that satisfies (3.5) is bounded in X_0.*

Proof. According to (3.5), we have

$$\begin{aligned} c_m + o_n(1) &= I_m(w_n) - \frac{1}{N} P_m(w_n) \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla w_n|^N \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N} m|w_n|^N \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} m|w_n|^q \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} F(w_n)\mathrm{d}x - \frac{1}{N}\left(\frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x + m\int_{\mathbb{R}^N} |w_n|^p \mathrm{d}x \right. \\ &\quad \left. + \frac{N}{q}\int_{\mathbb{R}^N} m|w_n|^q \mathrm{d}x - N\int_{\mathbb{R}^N} F(w_n)\mathrm{d}x\right) \\ &= \frac{1}{N}\left(\int_{\mathbb{R}^N} |\nabla w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x\right). \end{aligned}$$

Hence, we get that $\int_{\mathbb{R}^N} |\nabla w_n|^N \mathrm{d}x$ and $\int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x$ are bounded in $\mathbb{R}$. Moreover, $P_m(w_n) = o_n(1)$ and (f_1)–(f_2) show that

$$\begin{aligned} &\frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|w_n|^N \mathrm{d}x + \frac{N}{q}\int_{\mathbb{R}^N} m|w_n|^q \mathrm{d}x \\ &\quad = o_n(1) + N\int_{\mathbb{R}^N} F(w_n)\mathrm{d}x \\ &\quad \le o_n(1) + \varepsilon N |w_n|_q^q + NC\int_{\mathbb{R}^N} |w_n|^\tau \Psi_N(w_n)\mathrm{d}x. \end{aligned}$$

According to the boundedness of $\int_{\mathbb{R}^N} |w_n|^\tau \Psi_N(w_n)\mathrm{d}x$ and choosing $\varepsilon > 0$ small enough, we can deduce that $(|w_n|_N)$ and $(|w_n|_q)$ are bounded in $\mathbb{R}$. Therefore, (w_n) is bounded in X_0. □

According to the method in [33], we have:

Lemma 3.4 (see [33]). *Assume that (u_n) is a bounded sequence in X_0, if there exist for some $R > 0, t \ge N$ such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_R(y)} |u_n(x)|^t \, \mathrm{d}x = 0,$$

then for all $\xi \in (t, +\infty)$, $u_n \to 0$ in $L^\xi(\mathbb{R}^N)$.

Lemma 3.5. *Assume (w_n) satisfies Proposition 3.2, then there exist a sequence $(x_n) \subset \mathbb{R}^N$ and constants $R > 0, \beta > 0$ satisfy*

$$\int_{B_R(x_n)} w_n^q(x)\mathrm{d}x \ge \beta.$$

Proof. In fact, we assume that the conclusion is not true. According to Lemma 3.4, it is easy to get

$$w_n(\cdot) \to 0 \text{ in } L^\xi\left(\mathbb{R}^N\right), \quad \forall \xi \in (t, +\infty). \tag{3.7}$$

Therefore, due to (f_1) and (f_2), we obtain that

$$\int_{\mathbb{R}^N} f(w_n(x))\, w_n(x)\mathrm{d}x = o_n(1).$$

According to $\langle I_m'(w_n), w_n\rangle = o_n(1)$, we can obtain that

$$\int_{\mathbb{R}^N} |\nabla w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|w_n|^q \mathrm{d}x - \int_{\mathbb{R}^N} f(w_n)w_n \mathrm{d}x = o_n(1),$$

and so we deduce that $\|w_n\|_{X_0} \to 0$. Therefore, $I_m(w_n) \to 0$ and then we get contradiction since $c_m > 0$. □

Next, define

$$\mathcal{T}_m := \left\{ u \in X(\mathbb{R}^N)\backslash\{0\} : \max_{x\in\mathbb{R}^N} u(x) = u(0), I'_m(u) = 0 \right\},$$

$$b_m := \inf_{u\in\mathcal{T}_m} I_m(u),$$

and

$$\mathcal{S}_m := \{u \in \mathcal{T}_{V_0} : I_m(u) = b_m\}.$$

Lemma 3.6. *There exists $u \in \mathcal{S}_m$.*

Proof. Assume (w_n) satisfies Proposition 3.2. Let $\widetilde{w}_n(x) := w_n(x_n + x)$, here x_n comes from Lemma 3.5. According to Lemma 3.4, we can see that (w_n) is bounded in $X_{\rm rad}(\mathbb{R}^N)$, that is, for all $n \in \mathbb{N}$, we have $\|w_n\|_{X_{\rm rad}(\mathbb{R}^N)} \le C$. Going if necessary to a subsequence, for some $\widetilde{w} \in X_{\rm rad}(\mathbb{R}^N) \setminus \{0\}$, we assume that $\widetilde{w}_n \rightharpoonup \widetilde{w}$ in $X_{\rm rad}(\mathbb{R}^N)$, then

$$\widetilde{w}_n(x) \to \widetilde{w}(x) \quad \text{in } L^{\xi}(\mathbb{R}^N), \quad \forall \xi \in (N, +\infty).$$

So

$$\int_{\mathbb{R}^N} f(\widetilde{w}_n)\widetilde{w_n} \to \int_{\mathbb{R}^N} f(\widetilde{w})\widetilde{w}. \tag{3.8}$$

Moreover, $\widetilde{w}$ satisfies

$$(-\Delta)_N\widetilde{w} + (-\Delta)_q\widetilde{w} + m(|\widetilde{w}|^{N-2}\widetilde{w} + |\widetilde{w}|^{q-2}\widetilde{w}) = f(\widetilde{w}) \quad \text{in } \mathbb{R}^N. \tag{3.9}$$

From (3.8) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\nabla\widetilde{w}|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla\widetilde{w}|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}|^q \mathrm{d}x \\
&\le \liminf_{n\to\infty} \left[\int_{\mathbb{R}^N} |\nabla\widetilde{w}_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla\widetilde{w}_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}_n|^q \mathrm{d}x \right] \\
&\le \limsup_{n\to\infty} \left[\int_{\mathbb{R}^N} |\nabla\widetilde{w}_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla\widetilde{w}_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}_n|^q \mathrm{d}x \right] \\
&= \limsup_{n\to\infty} \left[\int_{\mathbb{R}^N} |\nabla w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} m|w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|w_n|^q \mathrm{d}x \right] \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^N} f(w_n)w_n \mathrm{d}x \\
&= \limsup_{n\to\infty} \int_{\mathbb{R}^N} f(\widetilde{w}_n)\widetilde{w}_n \mathrm{d}x \\
&= \int_{\mathbb{R}^N} f(\widetilde{w})\widetilde{w} \mathrm{d}x \\
&= \int_{\mathbb{R}^N} |\nabla\widetilde{w}|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla\widetilde{w}|^q \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}|^p \mathrm{d}x + \int_{\mathbb{R}^N} m|\widetilde{w}|^q \mathrm{d}x,
\end{aligned}$$

which implies that $\|\widetilde{w}_n\|_{X_0} \to \|\widetilde{w}\|_{X_0}$ and thus $\widetilde{w}_n \to \widetilde{w}$ in X_0. Therefore, by $I_m(w_n) = I_m(\widetilde{w}_n) \to c_m$ and $I'_m(w_n) = I'_m(\widetilde{w}_n) \to 0$, we obtain that $I_m(\widetilde{w}) = c_m$ and $I'_m(\widetilde{w}) = 0$. Due to $\widetilde{w} \neq 0$, we get that $c_m \ge b_m$.

Now, let $w \in X_0\backslash\{0\}$ be an arbitrary solution of (3.2). We define

$$w_t(x) := \begin{cases} w\left(\frac{x}{t}\right) & \text{for } t > 0, \\ 0 & \text{for } t = 0. \end{cases}$$

Next, choosing the real number $\theta_1 > t_1 > 1 > t_0 > 0$, we denote the curve γ consisting of three parts as follows:

$$\gamma(\theta)=\begin{cases}\theta w_{t_0}, & \theta\in[0,t_0]\,,\\ \theta w_{\theta}, & \theta\in[t_0,t_1]\,,\\ \theta w_{t_1}, & \theta\in[t_1,\theta_1]\,.\end{cases}$$

Due to w is a weak solution, then

$$\int_{\mathbb{R}^N} f(w)w\mathrm{d}x=\int_{\mathbb{R}^N}|\nabla w|^N\mathrm{d}x+\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x+\int_{\mathbb{R}^N}m|w|^N\mathrm{d}x+\int_{\mathbb{R}^N}m|w|^q\mathrm{d}x>0.$$

Hence, we can find $\theta_1>1$ such that

$$\int_{\mathbb{R}^N} f(\theta w)w\mathrm{d}x>0,\quad \forall\theta\in[1,\theta_1]\,.$$

Let $\varphi(s)=\frac{f(s)}{s^{q-1}}$. Due to (f_1), we know that $\varphi\in C(\mathbb{R},\mathbb{R})$. Hence, we have

$$\int_{\mathbb{R}^N}\varphi(\theta w)w^q\mathrm{d}x>0,\quad \forall\theta\in[1,\theta_1]\,. \tag{3.10}$$

Moreover,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}I_m(\theta w_t)&=\langle I'_m(\theta w_t),w_t\rangle\\
&=\theta^{N-1}\int_{\mathbb{R}^N}|\nabla w_t|^N\mathrm{d}x+\theta^{q-1}\int_{\mathbb{R}^N}|\nabla w_t|^q\mathrm{d}x+\theta^{N-1}\int_{\mathbb{R}^N}m|w_t|^N\mathrm{d}x\\
&\quad+\theta^{q-1}\int_{\mathbb{R}^N}m|w_t|^q\mathrm{d}x-\theta^{q-1}\int_{\mathbb{R}^N}\varphi(\theta w_t)w_t^q\mathrm{d}x\\
&=\theta^{N-1}\int_{\mathbb{R}^N}|\nabla w_t|^N\mathrm{d}x+\theta^{q-1}\int_{\mathbb{R}^N}|\nabla w_t|^q\mathrm{d}x+\theta^{N-1}\int_{\mathbb{R}^N}m|w_t|^N\mathrm{d}x\\
&\quad+\theta^{q-1}\int_{\mathbb{R}^N}m|w_t|^q\mathrm{d}x-\frac{\theta^{q-1}}{2}\int_{\mathbb{R}^N}\varphi(\theta w_t)w_t^q\mathrm{d}x-\frac{\theta^{q-1}}{2}\int_{\mathbb{R}^N}\varphi(\theta w_t)w_t^q\mathrm{d}x\\
&=\theta^{N-1}\left(\int_{\mathbb{R}^N}|\nabla w|^N\mathrm{d}x+t^N\int_{\mathbb{R}^N}m|w|^N\mathrm{d}x-\frac{\theta^{q-N}t^N}{2}\int_{\mathbb{R}^N}\varphi(\theta w)w^q\mathrm{d}x\right)\\
&\quad+\theta^{N-1}\cdot t^{N-q}\left(\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x+t^q\int_{\mathbb{R}^N}m|w|^q\mathrm{d}x-\frac{t^q}{2}\int_{\mathbb{R}^N}\varphi(\theta w)w^q\mathrm{d}x\right).\end{aligned}$$

Selecting $t_0\in(0,1)$ small enough, we obtain

$$\int_{\mathbb{R}^N}|\nabla w|^N\mathrm{d}x+t_0^N\int_{\mathbb{R}^N}m|w|^N\mathrm{d}x-\frac{\theta^{q-N}t_0^N}{2}\int_{\mathbb{R}^N}\varphi(\theta w)w^q\mathrm{d}x>0\quad\text{for all }\theta\in[1,\theta_1] \tag{3.11}$$

and

$$\int_{\mathbb{R}^N}|\nabla w|^q\mathrm{d}x+t_0^q\int_{\mathbb{R}^N}m|w|^q\mathrm{d}x-\frac{t_0^q}{2}\int_{\mathbb{R}^N}\varphi(\theta w)w^q\mathrm{d}x>0\quad\text{for all }\theta\in[1,\theta_1]\,. \tag{3.12}$$

According to (3.10), for all $\theta\in[1,\theta_1]$, we select $t_1>1$ such that

$$\int_{\mathbb{R}^N}|\nabla w|^N\mathrm{d}x+t_1^N\int_{\mathbb{R}^N}m|w|^N\mathrm{d}x-\frac{\theta^{q-N}t_1^N}{2}\int_{\mathbb{R}^N}\varphi(\theta w)w^q\mathrm{d}x\le-\frac{N}{\theta_1^N-1}\int_{\mathbb{R}^N}|\nabla w|^N\mathrm{d}x, \tag{3.13}$$

and

$$\int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x + t_1^q \int_{\mathbb{R}^N} m|w|^q \mathrm{d}x - \frac{t_1^q}{2} \int_{\mathbb{R}^N} \varphi(\theta w) w^q \mathrm{d}x \leq -\frac{N t_1^{q-N}}{(\theta_1^N - 1)} \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x. \qquad (3.14)$$

Therefore, according to (3.11) and (3.12), we know $I(\gamma(\theta))$ increases at the interval $[0, t_0]$, then takes its maximum value at $\theta = 1$. According to the Pohozǎev identity:

$$P_m(u) = \frac{N-q}{q} \int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x + m \int_{\mathbb{R}^N} |u|^N \mathrm{d}x + \frac{Nm}{q} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x - N \int_{\mathbb{R}^N} F(u) \mathrm{d}x.$$

Consequently,

$$\begin{aligned}
I_m(w_{t_1}(x)) &\leq I_m(w(x)) \\
&= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x + \frac{m}{N} \int_{\mathbb{R}^N} |w|^N \mathrm{d}x + \frac{m}{q} \int_{\mathbb{R}^N} |w|^q \mathrm{d}x \\
&\quad - \frac{1}{N} \left(\frac{N-q}{q} \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x + m \int_{\mathbb{R}^N} |w|^N \mathrm{d}x + \frac{N}{q} \int_{\mathbb{R}^N} m|w|^q \mathrm{d}x \right) \\
&= \frac{1}{N} \left(\int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x \right).
\end{aligned}$$

Now by using (3.13) and (3.14), we have

$$\begin{aligned}
I_m(\theta_1 w_{t_1}) &= I_m(w_{t_1}) + \int_1^{\theta_1} \frac{\mathrm{d}}{\mathrm{d}\theta} I(\theta w_{t_1}) \mathrm{d}\theta \\
&\leq \frac{1}{N} \left(\int_{\mathbb{R}^N} |\nabla w_n|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x \right) - \frac{N}{\theta_1^N - 1} \int_{\mathbb{R}^N} |\nabla w|^N \mathrm{d}x \int_1^{\theta_1} \theta^{N-1} \mathrm{d}\theta \\
&\quad - \frac{N t_1^{q-N}}{(\theta_1^N - 1)} \int_{\mathbb{R}^N} |\nabla w|^q \mathrm{d}x \cdot t_1^{N-q} \int_1^{\theta_1} \theta^{N-1} \mathrm{d}\theta \\
&= \left(\frac{1}{N} - 1 \right) \int_{\mathbb{R}^N} |\nabla w_n|^N \mathrm{d}x + \left(\frac{1}{N} - 1 \right) \int_{\mathbb{R}^N} |\nabla w_n|^q \mathrm{d}x < 0.
\end{aligned}$$

So we know $\gamma(\theta) \in \Gamma_m$. According to the definition of c_m, we have $I_m(\gamma(\theta)) \geq c_m$. Due to w is arbitrary, we obtain that $b_m \geq c_m$ and this means $b_m = c_m$.

Selecting $w^- = \min\{w, 0\}$ as a test function of (3.2), we infer that $w \geq 0$ in $\mathbb{R}^N$. Using (f_1)–(f_2) and according to the Moser iteration (see [3, 13]), it is easy to obtain that $w \in L^\infty(\mathbb{R}^N)$. By means of Corollary 2.1 in [4], we can see that $w \in C^\sigma(\mathbb{R}^N)$ for some $\sigma \in (0,1)$. Similar to the proof of Theorem 1.1-(ii) in [24], we obtain that $w > 0$ in $\mathbb{R}^N$. □

Remark 3.7. As to $m > 0$, we define

$$I_{m'}(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla u|^q \mathrm{d}x + \frac{m'}{p} \int_{\mathbb{R}^N} |u|^p \mathrm{d}x + \frac{m'}{q} \int_{\mathbb{R}^N} |u|^q \mathrm{d}x - \int_{\mathbb{R}^N} F(u) \mathrm{d}x,$$

the mountain pass level is $c_{m'}$. By using standard method, we can prove that $c_{m'_1} > c_{m'_2}$ when $m'_1 > m'_2$.

In the following, we will prove that $\mathcal{S}_{V_0}$ is compact in X_0.

Lemma 3.8. *$\mathcal{S}_{V_0}$ is compact in X_0.*

Proof. For any $U \in \mathcal{S}_{V_0}$, we have that

$$\begin{aligned} c_m + o_n(1) &= I_m(U) - \frac{1}{N} P_m(U) \\ &= \frac{1}{N}\int_{\mathbb{R}^N} |\nabla U|^N \mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} |\nabla U|^q \mathrm{d}x + \frac{m}{N}\int_{\mathbb{R}^N} |U|^N \mathrm{d}x + \frac{m}{q}\int_{\mathbb{R}^N} |U|^q \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} F(U)\mathrm{d}x - \frac{1}{N}\left(\frac{N-q}{q}\int_{\mathbb{R}^N} |\nabla U|^q \mathrm{d}x + m\int_{\mathbb{R}^N} |U|^p \mathrm{d}x \right. \\ &\quad \left. + \frac{Nm}{q}\int_{\mathbb{R}^N} |U|^q \mathrm{d}x - N\int_{\mathbb{R}^N} F(U)\mathrm{d}x\right) \\ &= \frac{1}{N}\left(\int_{\mathbb{R}^N} |\nabla U|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla U|^q \mathrm{d}x\right). \end{aligned}$$

So $\mathcal{S}_m$ is bounded in X_0.

For any sequence $\{U_k\} \subset \mathcal{S}_{V_0}$, up to a subsequence, we can find a $U_0 \in X_0$ satisfies

$$U_k \rightharpoonup U_0 \quad \text{in } X_0 \tag{3.15}$$

and U_0 satisfies

$$-\Delta_N U_0 - \Delta_q U_0 + m(|U_0|^{N-2}U_0 + |U_0|^{q-2}U_0) = f(U_0), \quad \text{in } \mathbb{R}^N,\ U_0 \geq 0.$$

Next, we will prove that U_0 is nontrivial. Note that, up to a subsequence, we have

$$U_k \to U_0 \text{ in } L^t_{\mathrm{loc}}(\mathbb{R}^N), \quad t \in (N, +\infty). \tag{3.16}$$

By using (3.16), any bounded region in $\mathbb{R}^N$, (U_k^t) is uniformly integrable. According to Lemma 2.2 (i) in [22], $\|U_k\|_{L^\infty_{\mathrm{loc}}(\mathbb{R}^N)} \leq C$. In view of [26], there exists $\alpha \in (0,1)$ such that $\|U_k\|_{C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)} \leq C$. Due to $(U_k) \subset \mathcal{S}_{V_0}$, by Lemma 3.6, we have that $U_k > 0$. We can prove that $\liminf_{k\to\infty} \|U_k\|_\infty > 0$ because of $\lim_{t\to 0} \frac{f(t)}{t^{q-1}} = 0$. In fact, since U_k satisfies (3.1), we have that

$$-\Delta_N U_k - \Delta_q U_k + m(|U_k|^{N-2}U_k + |U_k|^{q-2}U_k) = f(U_k),$$

that is

$$\int_{\mathbb{R}^N} |\nabla U_k|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla U_k|^q \mathrm{d}x + m\int_{\mathbb{R}^N} |U_k|^N \mathrm{d}x + m\int_{\mathbb{R}^N} |U_k|^q \mathrm{d}x = \int_{\mathbb{R}^N} f(U_k)U_k \mathrm{d}x.$$

According to $\lim_{t\to 0} \frac{f(t)}{t^{q-1}} = 0$, $\forall \varepsilon > 0$, we can find $\delta > 0$ satisfies

$$f(t) < \varepsilon t^{q-1}, \quad |t| < \delta,$$

then $f(U_k)U_k < \varepsilon |U_k|^q$. Assume by contradiction, we have $\liminf_{k\to\infty} \|U_k\|_\infty = 0$, then for δ given above, we have $|U_k| < \delta$. Therefore,

$$\int_{\mathbb{R}^N} |\nabla U_k|^N \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla U_k|^q \mathrm{d}x = \int_{\mathbb{R}^N} f(U_k)U_k \mathrm{d}x - m\int_{\mathbb{R}^N} |U_k|^N \mathrm{d}x - m\int_{\mathbb{R}^N} |U_k|^q \mathrm{d}x < 0,$$

which leads to a contradiction. Noting that $U_k(0) = \|U_k\|_\infty$, we get that $U_0 \not\equiv 0$. Therefore, there exists $\exists C_0 > 0$ such that $U_k(0) \geq C_0 > 0$, then $U_0(0) \geq C_0 > 0$, this means that U_0 is nontrivial. Using the same method as Lemma 3.6, we get $I_m(U_0) = c_m$ and $U_k \to U_0$ in X_0. Therefore, $\mathcal{S}_m$ is compact in X_0. □

3.2 Proof of Theorem 1.1

This section will prove Theorem 1.1. For $U \in \mathcal{S}_m$, set $c_m = I_m(U)$ and $10\delta = \text{dist}\{\mathcal{M}, \mathbb{R}^N \backslash \Lambda\}$. Now, fix a $\beta \in (0, \delta)$ and a cut-off function $\varphi \in C_c^\infty(\mathbb{R}^N)$ satisfies

$$\varphi := \begin{cases} 1, & |x| \leq \beta, \\ 0, & |x| \geq 2\beta \end{cases}$$

and $|\nabla \varphi| \leq C/\beta$. Moreover, let $y \in \mathbb{R}^N$, $\varphi_\varepsilon(y) = \varphi(\varepsilon y)$. For $\varepsilon > 0$ small enough, we will look for solutions of (1.1) near the set

$$Y_\varepsilon := \left\{ \varphi(\varepsilon y - x) U\left(y - \frac{x}{\varepsilon}\right) : x \in \mathcal{M}^\beta, U \in \mathcal{S}_m \right\},$$

where $\mathcal{M}^\beta := \{y \in \mathbb{R}^N : \inf_{z \in \mathcal{M}} |z - y| \leq \beta\}$. Moreover, as to $A \subset X_\varepsilon$, define

$$A^a := \left\{ u \in X_\varepsilon : \inf_{v \in A} \|u - v\|_{X_\varepsilon} \leq a \right\}.$$

For any $U \in \mathcal{S}_m$, define $W_{\varepsilon,t}(x) := \varphi(\varepsilon x) U\left(\frac{x}{t}\right)$.

Next, we show that J_ε has the Mountain-Pass geometry. Let $U_t(x) := U(\frac{x}{t})$, by using the same proof as in Lemma 3.1, we have

$$\begin{aligned} I_m(U_t) &= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla U|^N \mathrm{d}x + \frac{t^N}{N} \int_{\mathbb{R}^N} m|U|^N \mathrm{d}x + \frac{t^{N-q}}{q} \int_{\mathbb{R}^N} |\nabla U|^q \mathrm{d}x \\ &\quad + \frac{t^N}{q} \int_{\mathbb{R}^N} m|U|^q \mathrm{d}x - t^N \int_{\mathbb{R}^N} F(U) \mathrm{d}x \\ &\to -\infty \quad \text{as } t \to \infty. \end{aligned}$$

So there exists $t_0 > 0$ such that $I_m(U_{t_0}) < -3$.

Clearly, $Q_\varepsilon(W_{\varepsilon,t_0}) = 0$. As to $\varepsilon > 0$ sufficiently small, by using the Dominated Convergence Theorem, one has

$$\begin{aligned} J_\varepsilon(W_{\varepsilon,t_0}) &= I_\varepsilon(W_{\varepsilon,t_0}) \\ &= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla W_{\varepsilon,t_0}|^N \mathrm{d}x + \frac{1}{q} \int_{\mathbb{R}^N} |\nabla W_{\varepsilon,t_0}|^q \mathrm{d}x + \frac{1}{N} \int_{\mathbb{R}^N} V(\varepsilon x) |W_{\varepsilon,t_0}|^p \mathrm{d}x \\ &\quad + \frac{1}{q} \int_{\mathbb{R}^N} V(\varepsilon x) |W_{\varepsilon,t_0}|^q \mathrm{d}x - \int_{\mathbb{R}^N} F(W_{\varepsilon,t_0}) \mathrm{d}x \\ &\overset{\tilde{x} = \frac{x}{t_0}}{=} \frac{1}{N} \int_{\mathbb{R}^N} \left| \varepsilon t_0^2 \nabla \varphi(\varepsilon t_0 \tilde{x}) U(\tilde{x}) + \varphi(\varepsilon \tilde{x}) \nabla U(\tilde{x}) \right|^N \mathrm{d}\tilde{x} \\ &\quad + \frac{t_0^{N-q}}{q} \int_{\mathbb{R}^N} \left| \varepsilon t_0^2 \nabla \varphi(\varepsilon t_0 \tilde{x}) U(\tilde{x}) + \varphi(\varepsilon t_0 \tilde{x}) \nabla U(\tilde{x}) \right|^q \mathrm{d}\tilde{x} \\ &\quad + \frac{t_0^N}{N} \int_{\mathbb{R}^N} V(\varepsilon t_0 \tilde{x}) |\varphi(\varepsilon t_0 \tilde{x}) U(\tilde{x})|^N \mathrm{d}\tilde{x} \\ &\quad + \frac{t_0^N}{q} \int_{\mathbb{R}^N} V(\varepsilon t_0 \tilde{x}) |\varphi(\varepsilon t_0 \tilde{x}) U(\tilde{x})|^q \mathrm{d}\tilde{x} \\ &\quad - t^N \int_{\mathbb{R}^N} F(\varphi(\varepsilon t_0 \tilde{x}) U(\tilde{x}) \mathrm{d}\tilde{x} \\ &= I_m(U_{t_0}) + o(1) < -2. \end{aligned} \tag{3.17}$$

According to (f_1) and (f_2), it is easy to see that

$$|F(t)| \leq \varepsilon |t|^q + C|t|^\tau \Psi_N(t).$$

So, for $2 \leq N < q < q^*$, we get

$$\begin{aligned} J_\varepsilon(u) &\geq I_\varepsilon(u) \\ &= \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + V_\varepsilon |u|^q)\, dx + \frac{1}{N}\int_{\mathbb{R}^N} \left(|\nabla u|^N + V_\varepsilon |u|^N\right) dx - \int_{\mathbb{R}^N} F(u) dx \\ &\geq \frac{1}{N}\int_{\mathbb{R}^N} (|\nabla u|^N + V_\varepsilon |u|^N) dx + \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + V_\varepsilon |u|^q) dx - \varepsilon |u|_q^q - C\int_{\mathbb{R}^N} |t|^\tau \Psi_N(u) dx. \end{aligned}$$

Using Hölder's inequality, it is easy to get

$$\int_{\mathbb{R}^N} |u|^\tau \Psi_N(u)\, dx \leq \|u\|^\tau_{L^{\tau t'}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} (\Phi_N(u))^t\, dx\right)^{\frac{1}{t}},$$

where $\frac{1}{t} + \frac{1}{t'} = 1 (t' > 1,\ t > 1)$. Due to Lemma 2.3, we can find a constant $D > 0$ satisfies

$$\left(\int_{\mathbb{R}^N} (\Phi_N(u))^t\, dx\right)^{\frac{1}{t}} \leq D.$$

From (2.2), we have

$$\|u\|_{L^\nu(\mathbb{R}^N)} \leq S^{-1}_{\nu,\varepsilon} \|u\|_{X_\varepsilon}, \quad \forall u \in X_\varepsilon.$$

Hence, when $\|u\|_{X_\varepsilon}$ is small, we get

$$\begin{aligned} J_\varepsilon(u) \geq\ & \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + V_\varepsilon |u|^q) dx + \frac{1}{N}\int_{\mathbb{R}^N} (|\nabla u|^N + V_\varepsilon |u|^N) dx \\ & - \varepsilon |u|_q^q - C\int_{\mathbb{R}^N} |t|^\tau \Psi_N(u) dx \\ \geq\ & \frac{1}{q \cdot 2^{q-1}} \|u\|^q_{X_\varepsilon} - \varepsilon S^{-q}_{q,\varepsilon} \|u\|^q_{X_\varepsilon} - CDS^{-\tau}_{\tau t',\varepsilon} \|u\|^\tau_{X_\varepsilon} \\ =\ & \|u\|^q_{X_\varepsilon} \left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon} - CDS^{-\tau}_{\tau t',\varepsilon} \|u\|^{\tau - q}_{X_\varepsilon}\right). \end{aligned}$$

We see $\frac{1}{q \cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon} > 0$ for ε small enough. Let

$$h(t) = \frac{1}{q \cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon} - CDS^{-\tau}_{\tau t',\varepsilon} t^{\tau - q}, \quad t \geq 0.$$

Next, we will find $t_0 > 0$ small that satisfies $h(t_0) \geq \frac{1}{2}\left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon}\right)$. Clearly, $\lim_{t \to 0^+} h(t) = \frac{1}{q \cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon}$ and h is continuous function on $[0, +\infty)$, so there exists t_0 satisfies $h(t) \geq \frac{1}{q \cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon} - \varepsilon_1$, $\forall t \in (0, t_0)$, t_0 is small enough. Choosing $\varepsilon_1 = \frac{1}{2}\left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon}\right)$, we get that

$$h(t) \geq \frac{1}{2}\left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon}\right)$$

for all $0 \leq t \leq t_0$. In particularly,

$$h(t_0) \geq \frac{1}{2}\left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon S^{-q}_{q,\varepsilon}\right).$$

So, for $\|u\|_{X_\varepsilon} = t_0$, we have

$$J_\varepsilon(u) \geq \frac{t_0^q}{2} \cdot \left(\frac{1}{q \cdot 2^{q-1}} - \varepsilon S_{q,\varepsilon}^{-q} \right) = \rho_0 > 0.$$

Therefore, we can define c_ε as follows:

$$c_\varepsilon := \inf_{\gamma \in \Gamma_\varepsilon} \max_{s \in [0,1]} J_\varepsilon(\gamma(s)).$$

Here Γ_ε is defined by

$$\Gamma_\varepsilon := \{\gamma \in C([0,1], X_\varepsilon) \mid \gamma(1) = W_{\varepsilon,t_0}, \gamma(0) = 0\}.$$

Lemma 3.9. *There holds*

$$\overline{\lim_{\varepsilon \to 0}} c_\varepsilon \leq c_m.$$

Proof. Denote $W_{\varepsilon,0} = \lim_{t\to 0} W_{\varepsilon,t}$ in X_ε sense, then it is easy to get $W_{\varepsilon,0} = 0$. Consequently, let $\gamma(s) := W_{\varepsilon,st_0}(0 \leq s \leq 1)$, then $\gamma(s) \in \Gamma_\varepsilon$, so

$$c_\varepsilon \leq \max_{s\in[0,1]} J_\varepsilon(\gamma(s)) = \max_{t\in[0,t_0]} J_\varepsilon(W_{\varepsilon,t}).$$

Now, we only need to prove

$$\overline{\lim_{\varepsilon\to 0}} \max_{t\in[0,t_0]} J_\varepsilon(W_{\varepsilon,t}) \leq c_m.$$

In fact, similar to (3.17), we obtain that

$$\begin{aligned}\max_{t\in[0,t_0]} J_\varepsilon(W_{\varepsilon,t}) &= \max_{t\in[0,t_0]} I_m(U_t) + o(1) \\ &\leq o(1) + \max_{t\in[0,\infty)} I_m(U_t) \\ &= I_m(U) + o(1) = o(1) + c_m.\end{aligned}$$

This finishes the proof. □

Lemma 3.10. *There holds*

$$\underline{\lim_{\varepsilon\to 0}} c_\varepsilon \geq c_m.$$

Proof. Assuming $\underline{\lim}_{\varepsilon\to 0} c_\varepsilon < c_m$, we can find $\delta_0 > 0$, $\gamma_n \in \Gamma_{\varepsilon_n}$ and $\varepsilon_n \to 0$ satisfy, for $s \in [0,1]$, $J_{\varepsilon_n}(\gamma_n(s)) < c_m - \delta_0$. Now, fixed an $\varepsilon_n > 0$, we have

$$\frac{1}{N} m\varepsilon_n \left(1 + (1 + c_m)^{1/2}\right) < \min\{\delta_0, 1\}. \tag{3.18}$$

Due to $I_{\varepsilon_n}(\gamma_n(0)) = 0$ and $I_{\varepsilon_n}(\gamma_n(1)) \leq J_{\varepsilon_n}(\gamma_n(1)) = J_{\varepsilon_n}(W_{\varepsilon_n,t_0}) < -2$, we can look for an $s_n \in (0,1)$ such that $I_{\varepsilon_n}(\gamma_n(s)) \geq -1$ for $s \in [0, s_n]$ and $I_{\varepsilon_n}(\gamma_n(s_n)) = -1$. Moreover, for any $s \in [0, s_n]$, we have that

$$Q_{\varepsilon_n}(\gamma_n(s)) = J_{\varepsilon_n}(\gamma_n(s)) - I_{\varepsilon_n}(\gamma_n(s)) \leq 1 + c_m - \delta_0,$$

which implies that

$$\int_{\mathbb{R}^N \setminus (\Lambda/\varepsilon_n)} \gamma_n^N(s) \mathrm{d}x \leq \varepsilon_n \left(1 + (1 + c_m)^{1/2}\right) \quad \text{for } s \in [0, s_n].$$

So for $s \in [0, s_n]$, we have

$$
\begin{aligned}
& I_{\varepsilon_n}(\gamma_n(s)) \\
& = I_m(\gamma_n(s)) + \frac{1}{N}\int_{\mathbb{R}^N} (V(\varepsilon_n x) - m)\,\gamma_n^N(s)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N} (V(\varepsilon_n x) - m)\,\gamma_n^q(s)\mathrm{d}x \\
& \geq I_m(\gamma_n(s)) + \frac{1}{N}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_n)} (V(\varepsilon_n x) - m)\,\gamma_n^N(s)\mathrm{d}x + \frac{1}{q}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_n)} (V(\varepsilon_n x) - m)\,\gamma_n^q(s)\mathrm{d}x \\
& \geq I_m(\gamma_n(s)) + \frac{1}{N}\int_{\mathbb{R}^N\setminus(\Lambda/\varepsilon_n)} (V(\varepsilon_n x) - m)\,\gamma_n^N(s)\mathrm{d}x \\
& \geq I_m(\gamma_n(s)) - \frac{1}{N} m\varepsilon_n \left(1 + (1 + c_m)^{1/2}\right).
\end{aligned}
$$

Then

$$
\begin{aligned}
I_m(\gamma_n(s_n)) &\leq I_{\varepsilon_n}(\gamma_n(s_n)) + \frac{1}{N} m\varepsilon_n \left(1 + (1 + c_m)^{1/2}\right) \\
&= -1 + \frac{1}{N} m\varepsilon_n \left(1 + (1 + c_m)^{1/2}\right) < 0,
\end{aligned}
$$

and recalling (3.3), we obtain that

$$
\max_{s \in [0, s_n]} I_m(\gamma_n(s)) \geq c_m.
$$

Therefore, we get that

$$
\begin{aligned}
c_m - \delta_0 &\geq \max_{s \in [0,1]} J_{\varepsilon_n}(\gamma_n(s)) \geq \max_{s \in [0,1]} I_{\varepsilon_n}(\gamma_n(s)) \geq \max_{s \in [0, s_n]} I_{\varepsilon_n}(\gamma_n(s)) \\
&\geq -\frac{1}{N} m\varepsilon_n \left(1 + (1 + c_m)^{1/2}\right) + \max_{s \in [0, s_n]} I_m(\gamma_n(s)),
\end{aligned}
$$

that is $0 < \delta_0 \leq \frac{1}{N} m\varepsilon_n \big(1 + (1 + c_m)^{1/2}\big)$, which contradicts (3.18). As desired. □

By using Lemmas 3.9 and 3.10, it follows

$$
0 = \lim_{\varepsilon \to 0} \left(\max_{s \in [0,1]} J_\varepsilon(\gamma_\varepsilon(s)) - c_\varepsilon \right).
$$

Here $\forall s \in [0,1]$, $\gamma_\varepsilon(s) = W_{\varepsilon, st_0}$. Denote

$$
\tilde{c}_\varepsilon := \max_{s \in [0,1]} J_\varepsilon(\gamma_\varepsilon(s)).
$$

Clearly, $c_\varepsilon \leq \tilde{c}_\varepsilon$,

$$
c_m = \lim_{\varepsilon \to 0} \tilde{c}_\varepsilon = \lim_{\varepsilon \to 0} c_\varepsilon.
$$

Now define

$$
J_\varepsilon^\alpha = \{ u \in X_\varepsilon \mid J_\varepsilon(u) \leq \alpha \}.
$$

For $\alpha > 0$ and $\forall A \subset X_\varepsilon$, set $A^\alpha = \big\{ u \in X_\varepsilon \mid \inf_{v \in A} \|u - v\|_{X_\varepsilon} \leq \alpha \big\}$.

Lemma 3.11. *Assume $\{\varepsilon_i\}_{i=1}^\infty$ satisfies $\lim_{i\to\infty} \varepsilon_i = 0$, $\{u_{\varepsilon_i}(\cdot)\} \subset Y_{\varepsilon_i}^d$ and*

$$
\lim_{i \to \infty} J'_{\varepsilon_i}(u_{\varepsilon_i}(\cdot)) = 0, \quad \lim_{i \to \infty} J_{\varepsilon_i}(u_{\varepsilon_i}(\cdot)) \leq c_m.
$$

Then, $\forall d > 0$ small enough, up to a subsequence, there exist $x \in \mathcal{M}$, $\{y_i\}_{i=1}^\infty \subset \mathbb{R}^N$, $U \in \mathcal{S}_m$ satisfy

$$
\lim_{i \to \infty} \|\varphi_{\varepsilon_i}(\cdot - y_i)\, U(\cdot - y_i) - u_{\varepsilon_i}(\cdot)\|_{X_{\varepsilon_i}} = 0 \quad \text{and} \quad \lim_{i \to \infty} |x - \varepsilon_i y_i| = 0.
$$

Proof. Now, write ε_i as ε. According to

$$Y_\varepsilon := \left\{ \varphi\left(\varepsilon y - x\right) U\left(y - \frac{x}{\varepsilon}\right) : x \in \mathcal{M}^\beta, U \in \mathcal{S}_m \right\},$$

we can find $\{U_\varepsilon\} \subset \mathcal{S}_m$ and $\{x_\varepsilon\} \subset \mathcal{M}^\beta$ satisfy

$$\left\| \varphi_\varepsilon\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) U_\varepsilon\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) - u_\varepsilon(\cdot) \right\|_{X_\varepsilon} \leq d.$$

Due to $\mathcal{S}_m$, $\mathcal{M}^\beta$ are compact, there exist $Z \in \mathcal{S}_m, x \in \mathcal{M}^\beta$ satisfy $U_\varepsilon \to Z$ in X_ε and $x_\varepsilon \to x$. Hence, for $\varepsilon > 0$ small enough,

$$\left\| \varphi_\varepsilon\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) Z\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) - u_\varepsilon(\cdot) \right\|_{X_\varepsilon} \leq 2d. \tag{3.19}$$

In addition, according to (f_2), we can suppose that $\sup \|u_\varepsilon\|_{X_\varepsilon} \leq 1$.

Step 1. First we will prove

$$0 = \liminf_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B(y,1)} |u_\varepsilon|^N \,\mathrm{d}x, \tag{3.20}$$

here $A_\varepsilon = B\left(\frac{x_\varepsilon}{\varepsilon}, \frac{3\beta}{\varepsilon}\right) \backslash B\left(\frac{x_\varepsilon}{\varepsilon}, \frac{\beta}{2\varepsilon}\right)$.

Assume the formula (3.20) is true, according to Lions' lemma, for any $\xi > N$, we have that $u_\varepsilon \to 0$ in $L^\xi(B_\varepsilon)$, where $B_\varepsilon = B\left(\frac{x_\varepsilon}{\varepsilon}, \frac{2\beta}{\varepsilon}\right) \backslash B\left(\frac{x_\varepsilon}{\varepsilon}, \frac{\beta}{\varepsilon}\right)$.

Now, we assume the formula (3.20) is not true, then we can find $r > 0$ that satisfies

$$\liminf_{\varepsilon\to 0} \sup_{y\in A_\varepsilon} \int_{B(y,1)} |u_\varepsilon|^N \,\mathrm{d}x = 2r > 0.$$

So, for $\varepsilon > 0$ small enough, we also can find that $y_\varepsilon \in A_\varepsilon$ satisfies $\int_{B(y_\varepsilon,1)} |u_\varepsilon|^N \,\mathrm{d}x \geq r$. It is necessary to mention that, there is $x_0 \in \mathcal{M}^{4\beta} \subset \Lambda$ satisfying $\varepsilon y_\varepsilon \to x_0$. Assume $v_\varepsilon(y) = u_\varepsilon\left(y + y_\varepsilon\right)$, it is easy to obtain that

$$\begin{aligned} &-\Delta_N v_\varepsilon - \Delta_q v_\varepsilon + V_\varepsilon\left(y + y_\varepsilon\right) |v_\varepsilon|^{N-2} v_\varepsilon - g\left(\varepsilon y + \varepsilon y_\varepsilon, v_\varepsilon\right) + V_\varepsilon\left(y + y_\varepsilon\right) |v_\varepsilon|^{q-2} v_\varepsilon \\ &\quad = h_\varepsilon - 2NQ_\varepsilon^{1/2}\left(u_\varepsilon\right) \chi_\varepsilon\left(y + y_\varepsilon\right) |v_\varepsilon|^{N-2} v_\varepsilon. \end{aligned} \tag{3.21}$$

Taking ε adequately small, we have

$$\int_{B(0,1)} |v_\varepsilon|^N \,\mathrm{d}y \geq r. \tag{3.22}$$

Going if necessary to a subsequence, then we get $v_\varepsilon \rightharpoonup v$ in X_ε, and almost everywhere in $\mathbb{R}^N$. Note that the embedding $X_\varepsilon \hookrightarrow L^N(B(0,1))$ is compact, by using (3.22), we get $v \not\equiv 0$. Next, we will prove v satisfies

$$-\Delta_q v - \Delta_N v + V\left(x_0\right) |v|^{q-2} v + V\left(x_0\right) |v|^{N-2} v = f(v) \quad \text{ in } \mathbb{R}^N. \tag{3.23}$$

Indeed, for any $\varphi \in C_0^\infty\left(\mathbb{R}^N\right)$, in (3.21), we use $\left(v_\varepsilon - v\right)\varphi$ as a test function. For ε small enough, according to χ and g, we have that

$$\chi_\varepsilon\left(y + y_\varepsilon\right) |v_\varepsilon|^{N-2} v_\varepsilon\left(v_\varepsilon - v\right)\varphi = 0, \quad \forall y \in \mathbb{R}^N,$$

$$g\left(\varepsilon y+\varepsilon y_{\varepsilon},v_{\varepsilon}\right)\left(v_{\varepsilon}-v\right)\varphi=f\left(v_{\varepsilon}\right)\left(v_{\varepsilon}-v\right)\varphi,\quad \forall y\in\mathbb{R}^{N},$$
$$\chi_{\varepsilon}\left(y+y_{\varepsilon}\right)\left|v_{\varepsilon}\right|^{q-2}v_{\varepsilon}\left(v_{\varepsilon}-v\right)\varphi=0,\quad \forall y\in\mathbb{R}^{N}.$$

$\forall\xi\geq N$, we know that the embedding $X_{\varepsilon}\hookrightarrow L^{\xi}\left(\mathbb{R}^{N}\right)$ is local compact. Hence,

$$\int_{\mathbb{R}^{N}}V_{\varepsilon}\left(y+y_{\varepsilon}\right)\left|v_{\varepsilon}\right|^{N-2}v_{\varepsilon}\varphi\mathrm{d}y\to\int_{\mathbb{R}^{N}}V\left(x_{0}\right)|v|^{N-2}v\varphi\mathrm{d}y$$

and

$$\int_{\mathbb{R}^{N}}V_{\varepsilon}\left(y+y_{\varepsilon}\right)\left|v_{\varepsilon}\right|^{q-2}v_{\varepsilon}\varphi\mathrm{d}y\to\int_{\mathbb{R}^{N}}V\left(x_{0}\right)|v|^{q-2}v\varphi\mathrm{d}y.$$

By Lemma 2.2, (f_1), and $\|f\left(v_{\varepsilon}\right)\|_{N}<\infty$, we obtain that

$$\int_{\mathbb{R}^{N}}f\left(v_{\varepsilon}\right)\left(v_{\varepsilon}-v\right)\varphi\mathrm{d}y=\int_{\mathbb{R}^{N}}g\left(\varepsilon y+\varepsilon y_{\varepsilon},v_{\varepsilon}\right)\left(v_{\varepsilon}-v\right)\varphi\mathrm{d}y\to 0.$$

Therefore, similar to the proof of Lemma 3 in [6], we have that

$$\int_{\mathbb{R}^{N}}\left|\nabla v_{\varepsilon}\right|^{N-2}\nabla v_{\varepsilon}\nabla\varphi\mathrm{d}y\to\int_{\mathbb{R}^{N}}|\nabla v|^{N-2}\nabla v\nabla\varphi\mathrm{d}y$$

and

$$\int_{\mathbb{R}^{N}}\left|\nabla v_{\varepsilon}\right|^{q-2}\nabla v_{\varepsilon}\nabla\varphi\mathrm{d}y\to\int_{\mathbb{R}^{N}}|\nabla v|^{q-2}\nabla v\nabla\varphi\mathrm{d}y.$$

According to (f_1), (f_2), the compactness lemma of Strauss [32] and Lemma 2.2, we get that

$$\int_{\mathbb{R}^{N}}g\left(\varepsilon y+\varepsilon y_{\varepsilon},v_{\varepsilon}\right)\varphi\mathrm{d}y\to\int_{\mathbb{R}^{N}}f(v)\varphi\mathrm{d}y.$$

Therefore, (3.23) has a nontrivial solution v. According to definition, $I_{V(x_0)}(v)\geq c_{V(x_0)}$. For $R>0$ large enough, because of Fatou's lemma, it is easy to get

$$\liminf_{\varepsilon\to0}\int_{B(x_{\varepsilon},R)}\left|\nabla u_{\varepsilon}\right|^{N}\mathrm{d}y\geq\frac{1}{2}\int_{\mathbb{R}^{N}}|\nabla v|^{N}\mathrm{d}y,\tag{3.24}$$

and

$$\liminf_{\varepsilon\to0}\int_{B(x_{\varepsilon},R)}\left|\nabla u_{\varepsilon}\right|^{q}\mathrm{d}y\geq\frac{1}{2}\int_{\mathbb{R}^{N}}|\nabla v|^{q}\mathrm{d}y.\tag{3.25}$$

Now, recalling from Remark 3.7 that $c_a>c_b$ when $a>b$, it is easy to see that $c_{V(x_0)}\geq c_m$ because of $V\left(x_0\right)\geq m$. According to Pohožaev identity, for any $U\in\mathcal{S}_m$,

$$\frac{1}{N}\left(\int_{\mathbb{R}^{N}}|\nabla U|^{N}\mathrm{d}x+\int_{\mathbb{R}^{N}}|\nabla U|^{q}\mathrm{d}x\right)=I_m(U).\tag{3.26}$$

Thus,it follows from (3.24), (3.25) and (3.26) that

$$\liminf_{\varepsilon\to0}\int_{B(y_{\varepsilon},R)}\left|\nabla u_{\varepsilon}\right|^{N}\mathrm{d}y+\liminf_{\varepsilon\to0}\int_{B(y_{\varepsilon},R)}\left|\nabla u_{\varepsilon}\right|^{q}\mathrm{d}y\geq\frac{N}{2}I_{V(x_0)}(v)\geq\frac{N}{2}c_m>0.$$

When d is small enough, this is a contradiction with (3.19) .

Step 2. Define $u_{\varepsilon}^{2}=u_{\varepsilon}-u_{\varepsilon}^{1}$, where $u_{\varepsilon}^{1}(y)=\varphi_{\varepsilon}\left(y-x_{\varepsilon}/\varepsilon\right)u_{\varepsilon}(y)$. For $d>0$ small enough, we will prove, $J_{\varepsilon}\left(u_{\varepsilon}^{2}\right)\geq0$ and

$$J_{\varepsilon}\left(u_{\varepsilon}\right)\geq o(1)+J_{\varepsilon}\left(u_{\varepsilon}^{1}\right)+J_{\varepsilon}\left(u_{\varepsilon}^{2}\right)\qquad\text{as }\varepsilon\to0.\tag{3.27}$$

Clearly, for small enough $\varepsilon > 0$, we have $Q_\varepsilon\left(u_\varepsilon^1\right) = 0$ and $Q_\varepsilon\left(u_\varepsilon\right) = Q_\varepsilon\left(u_\varepsilon^2\right)$. Moreover, $\forall y \in \mathbb{R}^N$, $u_\varepsilon^1(y)u_\varepsilon^2(y) \geq 0$, we get

$$\begin{aligned}|u_\varepsilon(y)|^q &= \left(\left|u_\varepsilon^1(y)\right|^2 + \left|u_\varepsilon^2(y)\right|^2 + 2u_\varepsilon^1(y)u_\varepsilon^2(y)\right)^{q/2}\\ &\geq \left(\left|u_\varepsilon^1(y)\right|^2 + \left|u_\varepsilon^2(y)\right|^2\right)^{q/2}\\ &\geq \left|u_\varepsilon^1(y)\right|^q + \left|u_\varepsilon^2(y)\right|^q\end{aligned}$$

and

$$\begin{aligned}|u_\varepsilon(y)|^N &= \left(\left|u_\varepsilon^1(y)\right|^2 + \left|u_\varepsilon^2(y)\right|^2 + 2u_\varepsilon^1(y)u_\varepsilon^2(y)\right)^{N/2}\\ &\geq \left(\left|u_\varepsilon^1(y)\right|^2 + \left|u_\varepsilon^2(y)\right|^2\right)^{N/2}\\ &\geq \left|u_\varepsilon^1(y)\right|^N + \left|u_\varepsilon^2(y)\right|^N.\end{aligned}$$

So

$$\int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon^1\right|^N \mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon^2\right|^N \mathrm{d}y \leq \int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon\right|^N \mathrm{d}y$$

and

$$\int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon\right|^q \mathrm{d}y \geq \int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon^1\right|^q \mathrm{d}y + \int_{\mathbb{R}^N} V_\varepsilon \left|u_\varepsilon^2\right|^q \mathrm{d}y.$$

Moreover, it is easy to verify that

$$\begin{aligned}\int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^1\right|^N \mathrm{d}y &= \int_{\mathbb{R}^N} \varphi_\varepsilon^N\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) \left|\nabla u_\varepsilon\right|^N \mathrm{d}y + o(1),\\ \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^2\right|^N \mathrm{d}y &= \int_{\mathbb{R}^N} \left(1 - \varphi_\varepsilon\left(-\frac{x_\varepsilon}{\varepsilon}\right)\right)^N \left|\nabla u_\varepsilon\right|^N \mathrm{d}y + o(1),\\ \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^2\right|^q \mathrm{d}y &= \int_{\mathbb{R}^N} \left(1 - \varphi_\varepsilon\left(-\frac{x_\varepsilon}{\varepsilon}\right)\right)^q \left|\nabla u_\varepsilon\right|^N \mathrm{d}y + o(1),\\ \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^1\right|^q \mathrm{d}y &= \int_{\mathbb{R}^N} \varphi_\varepsilon^N\left(\cdot - \frac{x_\varepsilon}{\varepsilon}\right) \left|\nabla u_\varepsilon\right|^q \mathrm{d}y + o(1).\end{aligned}$$

Obviously, for any $y \in \mathbb{R}^N$, we have

$$\varphi_\varepsilon^2\left(y - x_\varepsilon/\varepsilon\right) \left|\nabla u_\varepsilon(y)\right|^2 + \left(1 - \varphi_\varepsilon\left(y - x_\varepsilon/\varepsilon\right)\right)^2 \left|\nabla u_\varepsilon(y)\right|^2 \leq \left|\nabla u_\varepsilon(y)\right|^2.$$

Therefore, we have

$$\int_{\mathbb{R}^N} \left|\nabla u_\varepsilon\right|^N \mathrm{d}y \geq \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^1\right|^N \mathrm{d}y + \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^2\right|^N \mathrm{d}y + o(1)$$

and

$$\int_{\mathbb{R}^N} \left|\nabla u_\varepsilon\right|^q \mathrm{d}y \geq \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^1\right|^q \mathrm{d}y + \int_{\mathbb{R}^N} \left|\nabla u_\varepsilon^2\right|^q \mathrm{d}y + o(1).$$

Hence, we have that

$$J_\varepsilon\left(u_\varepsilon\right) \geq o(1) - \int_{B_\varepsilon} \left(G\left(\varepsilon y, u_\varepsilon\right) - G(\varepsilon y, u_\varepsilon^1) - G\left(\varepsilon y, u_\varepsilon^2\right)\right) \mathrm{d}y + J_\varepsilon(u_\varepsilon^1) + J_\varepsilon(u_\varepsilon^2).$$

According to (f_1) and (f_2), then we obtain

$$\varepsilon|t|^q + C|t|^\tau \Psi_N(t) \geq |F(t)|. \tag{3.28}$$

Using the same proof as that in Lemma 3.1, we get

$$\int_{\mathbb{R}^N} |u|^\tau \Psi_N(u)\,\mathrm{d}x \leq \|u\|^\tau_{L^{\tau t'}(\mathbb{R}^N)} \left(\int_{\mathbb{R}^N} (\Phi_N(u))^t\,\mathrm{d}x \right)^{\frac{1}{t}}.$$

By using Step 1, we know that $u_\varepsilon \to 0$ in $L^q(B_\varepsilon)$, so

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} \int_{B_\varepsilon} \left(G(\varepsilon y, u_\varepsilon) - G(\varepsilon y, u_\varepsilon^2) - G\left(\varepsilon y, u_\varepsilon^1\right) \right) \mathrm{d}y \\
&= \limsup_{\varepsilon\to 0} \left| \int_{B_\varepsilon} \left(F(u_\varepsilon) - F\left(u_\varepsilon^1\right) - F(u_\varepsilon^2) \right) \mathrm{d}y \right| \\
&\leq \limsup_{\varepsilon\to 0} \int_{B_\varepsilon} \left(C|u_\varepsilon|^\tau \Psi_N(u_\varepsilon) + \varepsilon |u_\varepsilon|^q \right) \mathrm{d}y \\
&\leq c\varepsilon.
\end{aligned}$$

Due to ε being arbitrary, as $\varepsilon \to 0$ we get $\int_{B_\varepsilon} \left(F(u_\varepsilon) - F\left(u_\varepsilon^1\right) - F\left(u_\varepsilon^2\right) \right) \mathrm{d}y = o(1)$. So there is $C > 0$ satisfies

$$\begin{aligned}
J_\varepsilon\left(u_\varepsilon^2\right) \geq I\left(u_\varepsilon^2\right) &\geq \frac{1}{N} \|u_\varepsilon^2\|^N_{X_\varepsilon} + \frac{1}{q} \|u_\varepsilon^2\|^q_{X_\varepsilon} - C \int_{\mathbb{R}^N} |u_\varepsilon|^\tau \Psi_N\left(u_\varepsilon^2\right) \mathrm{d}y - \varepsilon \|u_\varepsilon^2\|^q_{X_\varepsilon} \\
&\geq \frac{1}{q \cdot 2^{q-1}} \|u_\varepsilon^2\|^q_{X_\varepsilon} - C \|u_\varepsilon^2\|^\tau_{X_\varepsilon}.
\end{aligned}$$

Hence, by using $\tau > q$, we get that $J_\varepsilon\left(u_\varepsilon^2\right) \geq 0$ for $d > 0$ small.

Step 3. Now, assume $w_\varepsilon(y) := u_\varepsilon^1\left(y + \frac{x_\varepsilon}{\varepsilon}\right) = \varphi_\varepsilon(y) u_\varepsilon\left(y + \frac{x_\varepsilon}{\varepsilon}\right)$. Up to a subsequence, we have $w_\varepsilon \rightharpoonup w$ in $X_\varepsilon, w_\varepsilon \to w$ almost everywhere in $\mathbb{R}^N$. Next, we will prove that

$$w_\varepsilon \to w \quad \text{in } L^\tau(\mathbb{R}^N),$$

where τ is given in (3.28). By contradiction, if there is $r > 0$ that satisfies

$$0 < 2r = \liminf_{\varepsilon\to 0} \sup_{z\in\mathbb{R}^N} \int_{B(z,1)} |w_\varepsilon - w|^\tau \,\mathrm{d}y.$$

So there is $z_\varepsilon \in \mathbb{R}^N$ that satisfies

$$\liminf_{\varepsilon\to 0} \int_{B(z_\varepsilon,1)} |w_\varepsilon - w|^\tau > r.$$

It is easy to see that (z_ε) is unbounded. We may assume that $|z_\varepsilon| = \infty$ as $\varepsilon \to 0$, then,

$$r \leq \liminf_{\varepsilon\to 0} \int_{B(z_\varepsilon,1)} |w_\varepsilon|^\tau \,\mathrm{d}y,$$

i.e.

$$\liminf_{\varepsilon\to 0} \int_{B(z_\varepsilon,1)} \left| \varphi_\varepsilon(y) u_\varepsilon\left(y + \frac{x_\varepsilon}{\varepsilon}\right) \right|^\tau \mathrm{d}y \geq r.$$

Using the same proof method as [9], for ε small enough, we have that $|z_\varepsilon| \leq \frac{\beta}{2\varepsilon}$. Assume that

$$\varepsilon z_\varepsilon \to z_0 \in \overline{B(0,\beta/2)},$$

$$\widetilde{w}_{\varepsilon} = w_{\varepsilon}(y + z_{\varepsilon}) \rightharpoonup \widetilde{w} \quad \text{in } X_{\varepsilon},$$
$$\widetilde{w}_{\varepsilon} \to \widetilde{w} \quad \text{a.e. in } \mathbb{R}^N.$$

So $\widetilde{w} \not\equiv 0$ and according to Step 1, $\widetilde{w}$ satisfies

$$\begin{aligned} -\Delta_q \widetilde{w}(y) - \Delta_N \widetilde{w}(y) + V(x + z_0) |\widetilde{w}(y)|^{q-2}\widetilde{w}(y) + V(x + z_0) |\widetilde{w}(y)|^{N-2}\widetilde{w}(y) \\ = f(\widetilde{w}(y)), \quad y \in \mathbb{R}^N. \end{aligned}$$

Using the same approach as Step 1, we obtain a contradiction for $d > 0$ small enough. Therefore, $w_{\varepsilon} \to w$ in $L^{\tau}(\mathbb{R}^N)$.

Step 4. According to Step 3, it follows that

$$\begin{aligned} \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} G\left(\varepsilon x, u_{\varepsilon}^1\right) \mathrm{d}x &= \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} G(\varepsilon x + x_{\varepsilon}, w_{\varepsilon}) \mathrm{d}x \\ &= \lim_{\varepsilon \to 0} \int_{\Lambda_{\varepsilon} - x_{\varepsilon}/\varepsilon} F(w_{\varepsilon}) \mathrm{d}x = \int_{\mathbb{R}^N} F(w) \mathrm{d}x. \end{aligned} \tag{3.29}$$

By using $w_{\varepsilon} \rightharpoonup w$ in X_{ε}, we have

$$\begin{aligned} &\lim_{\varepsilon \to 0} J_{\varepsilon}\left(u_{\varepsilon}^1\right) \\ &\geq \liminf_{\varepsilon \to 0} I_{\varepsilon}\left(u_{\varepsilon}^1\right) \\ &= \liminf_{\varepsilon \to 0} \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla w_{\varepsilon}(y)|^N + V_{\varepsilon}|w_{\varepsilon}(y)|^N) \mathrm{d}y + \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla w_{\varepsilon}(y)|^q + V_{\varepsilon}|w_{\varepsilon}(y)|^q) \mathrm{d}y \\ &\quad - \int_{\mathbb{R}^N} F(w_{\varepsilon}(y)) \mathrm{d}y \\ &\geq \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla w|^N + m|w|^N) \mathrm{d}y - \int_{\mathbb{R}^N} F(w) \mathrm{d}y + \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla w|^q + m|w|^q) \mathrm{d}y \\ &\geq c_m. \end{aligned} \tag{3.30}$$

On the other hand, since $\lim_{\varepsilon \to 0} J_{\varepsilon}(u_{\varepsilon}) \leq c_m$, $J_{\varepsilon}\left(u_{\varepsilon}^2\right) \geq 0$ and (3.27), we have

$$\limsup_{\varepsilon \to 0} J_{\varepsilon}\left(u_{\varepsilon}^1\right) \leq c_m. \tag{3.31}$$

Combining (3.30) and (3.31), we obtain that $J_{\varepsilon}(w) = c_m$. Similar to [25], we can obtain that $x \in \mathcal{M}$. So it is easy to see that $w(y) = U(y - z)$, $U \in \mathcal{S}_m$, $z \in \mathbb{R}^N$.

Lastly, due to (3.29) and (3.31) and $V(y) \geq m$ on Λ, by using (3.30), we have

$$\begin{aligned} \int_{\mathbb{R}^N} \left(|\nabla w|^N + m|w|^N\right) \mathrm{d}y &\geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} \left(\left|\nabla u_{\varepsilon}^1(y)\right|^N + V(\varepsilon y)|u_{\varepsilon}^1(y)|^N\right) \mathrm{d}y \\ &\geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} \left(\left|\nabla u_{\varepsilon}^1(y)\right|^N + m|u_{\varepsilon}^1(y)|^N\right) \mathrm{d}y \\ &\geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} \left(|\nabla w_{\varepsilon}(y)|^N + m|w_{\varepsilon}(y)|^N\right) \mathrm{d}y \end{aligned}$$

and

$$\begin{aligned} \int_{\mathbb{R}^N} (|\nabla w|^q + m|w|^q) \mathrm{d}y &\geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} \left(\left|\nabla u_{\varepsilon}^1(y)\right|^q + V(\varepsilon y)|u_{\varepsilon}^1(y)|^q\right) \mathrm{d}y \\ &\geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} \left(\left|\nabla u_{\varepsilon}^1(y)\right|^q + m|u_{\varepsilon}^1(y)|^q\right) \mathrm{d}y \\ &\geq \limsup_{\varepsilon \to 0} \int_{\mathbb{R}^N} (|\nabla w_{\varepsilon}(y)|^q + m|w_{\varepsilon}(y)|^q) \mathrm{d}y. \end{aligned}$$

Moreover, by using weak lower semi-continuity, we prove $u_\varepsilon^1 \to w$ in X_ε. Especially, let $y_\varepsilon = z + \frac{x}{\varepsilon}$, then $u_\varepsilon^1 \to U(\cdot - y_\varepsilon)\, \varphi_\varepsilon(\cdot - y_\varepsilon)$ in X_ε. So we get $u_\varepsilon^1 \to U(\cdot - y_\varepsilon)\, \varphi_\varepsilon(\cdot - y_\varepsilon)$ in X_ε.

In order to prove the desired conclusion, we only prove that $u_\varepsilon^2 \to 0$ in X_ε. Since $\{u_\varepsilon\}_\varepsilon$ is bounded, for small $\varepsilon > 0$, it is easy to see from (3.19) that $\|u_\varepsilon^2\|_\varepsilon \le 4d$. Now, using (3.27), $\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon^1) = c_m$ and the estimation of $J_\varepsilon(u_\varepsilon^2)$, we have that for some $C > 0$,

$$c_m \ge \lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) \ge c_m + \|u_\varepsilon^2\|_{X_\varepsilon}^q \left(\frac{1}{q \cdot 2^{q-1}} - C(4d)^{\tau - q} \right) + o(\varepsilon).$$

This proves that $u_\varepsilon^2 \to 0$ in X_ε, which completes the proof. □

Lemma 3.12. *For $0 < d_2 < d_1$ small enough, there exist $\omega > 0$ and $\varepsilon_0 > 0$ that satisfy $|J_\varepsilon'(u)| \ge \omega$, where $\varepsilon \in (0, \varepsilon_0)$, $u \in J_\varepsilon^{\tilde{c}_\varepsilon} \cap (Y_\varepsilon^{d_1} \backslash Y_\varepsilon^{d_2})$.*

Proof. By contradiction, we can suppose $0 < d_2 < d_1$ small enough, there are $\{\varepsilon_i\}_{i=1}^\infty$ with $\lim_{i\to\infty} \varepsilon_i = 0$ and $u_{\varepsilon_i} \in Y_{\varepsilon_i}^{d_1} \backslash Y_{\varepsilon_i}^{d_2}$ satisfying $\lim_{i\to\infty} J_{\varepsilon_i}(u_{\varepsilon_i}) \le c_m$ and $\lim_{i\to\infty} \left|J_{\varepsilon_i}'(u_{\varepsilon_i})\right| = 0$. For the convenience of description, we write ε for ε_i. Due to Lemma 3.11, for some $U \in \mathcal{S}_m$ and $x \in \mathcal{M}$, there is $\{y_\varepsilon\}_\varepsilon \subset \mathbb{R}^N$ such that

$$\lim_{\varepsilon\to 0} \|\varphi_\varepsilon(\cdot - y_\varepsilon)\, U(\cdot - y_\varepsilon) - u_\varepsilon\|_{X_\varepsilon} = 0 \quad \text{and} \quad \lim_{\varepsilon\to 0} |x - \varepsilon y_\varepsilon| = 0.$$

It follows from Y_ε that $\lim_{\varepsilon\to 0} \operatorname{dist}(Y_\varepsilon, u_\varepsilon) = 0$. Obviously contradictory because of $u_\varepsilon \notin Y_\varepsilon^{d_2}$. □

According to Lemma 3.12, fix a $d > 0$ small enough, there exist $\omega > 0$ and $\varepsilon_0 > 0$ that satisfy $|J_\varepsilon'(u)| \ge \omega$, where $\varepsilon \in (0, \varepsilon_0)$, $u \in J_\varepsilon^{\tilde{c}_\varepsilon} \cap (Y_\varepsilon^{d_1} \setminus Y_\varepsilon^{d_2})$. So we have

Lemma 3.13. *For $\varepsilon > 0$ small enough, we can find $\alpha > 0$ satisfies $J_\varepsilon(\gamma_\varepsilon(s)) \ge c_\varepsilon - \alpha$, then $\gamma_\varepsilon(s) \in Y_\varepsilon^{d/2}$ where $\gamma_\varepsilon(s) = W_{\varepsilon, st_0}(s)$.*

Proof. For each $s \in [0, 1]$, due to $\mathcal{M}_\varepsilon^{2\beta} \supset \operatorname{supp}(\gamma_\varepsilon(s))$, we have $I_\varepsilon(\gamma_\varepsilon(s)) = J_\varepsilon(\gamma_\varepsilon(s))$. In addition, it is easy to see that

$$\begin{aligned}
I_\varepsilon(\gamma_\varepsilon(s)) &= \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla \gamma_\varepsilon(s)|^q + V_\varepsilon |\gamma_\varepsilon(s)|^q) \mathrm{d}x + \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla \gamma_\varepsilon(s)|^N + V_\varepsilon |\gamma_\varepsilon(s)|^N) \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} F(\gamma_\varepsilon(s)) \mathrm{d}x \\
&= \frac{1}{q} \int_{\mathbb{R}^N} (|\nabla \gamma_\varepsilon(s)|^q + m |\gamma_\varepsilon(s)|^q) \mathrm{d}x + \frac{1}{N} \int_{\mathbb{R}^N} (|\nabla \gamma_\varepsilon(s)|^N + m |\gamma_\varepsilon(s)|^N) \mathrm{d}x \\
&\quad + \frac{1}{q} \int_{\mathbb{R}^N} (V_\varepsilon(x) - m) |\gamma_\varepsilon(s)|^q) \mathrm{d}x + \frac{1}{N} \int_{\mathbb{R}^N} (V_\varepsilon(x) - m) |\gamma_\varepsilon(s)|^N) \mathrm{d}x \\
&\quad - \int_{\mathbb{R}^N} F(\gamma_\varepsilon(s)) \mathrm{d}x \\
&= \frac{1}{N} \int_{\mathbb{R}^N} |\nabla U|^N \mathrm{d}x + \frac{(st_0)^{N-q}}{q} \int_{\mathbb{R}^N} |\nabla U|^q \mathrm{d}x + \frac{(st_0)^N}{N} \int_{\mathbb{R}^N} m |U|^N \mathrm{d}x \\
&\quad + \frac{(st_0)^N}{q} \int_{\mathbb{R}^N} m |U|^q \mathrm{d}x - (st_0)^N \int_{\mathbb{R}^N} F(U) \mathrm{d}x + O(\varepsilon).
\end{aligned}$$

Using the Pohozǎev identity, we have

$$
\begin{aligned}
J_\varepsilon\left(\gamma_\varepsilon(s)\right) &= I_\varepsilon\left(\gamma_\varepsilon(s)\right) \\
&= \frac{1}{N}\int_{\mathbb{R}^N}|\nabla U|^N \mathrm{d}x + \frac{(st_0)^{N-q}}{q}\int_{\mathbb{R}^N}|\nabla U|^q \mathrm{d}x - \frac{N-q}{Nq}(st_0)^N\int_{\mathbb{R}^N}|\nabla U|^q \mathrm{d}x + O(\varepsilon) \\
&= \frac{1}{N}\int_{\mathbb{R}^N}|\nabla U|^N \mathrm{d}x + \left(\frac{(st_0)^{N-q}}{q} - \frac{N-q}{Nq}(st_0)^N\right)\int_{\mathbb{R}^N}|\nabla U|^q \mathrm{d}x + O(\varepsilon).
\end{aligned}
$$

Note that

$$
c_m = \left(\frac{t^{N-q}}{q} - \frac{N-q}{Nq}t^N\right)\int_{\mathbb{R}^N}|\nabla U|^q \mathrm{d}x + \frac{1}{N}\int_{\mathbb{R}^N}|\nabla U|^N \mathrm{d}x
$$

and $\lim_{\varepsilon\to 0} c_\varepsilon = c_m$. Denote $g_1(t) = -\frac{N-q}{Nq}t^N + \frac{t^{N-q}}{q}$, then

$$
g_1'(t)\begin{cases} <0, & t>1, \\ =0, & t=1, \\ >0, & t\in(0,1). \end{cases}
$$

So we have $g_1''(1) = q - N < 0$, the conclusion follows. □

Lemma 3.14. *For $\varepsilon > 0$ small enough, we can find $\{u_n\}_{n=1}^\infty \subset Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$ satisfies as $n \to \infty$, $J_\varepsilon'(u_n) \to 0$.*

Proof. According to Lemma 3.13, for $\varepsilon > 0$ small enough, due to $\exists \alpha > 0$ satisfies $J_\varepsilon(\gamma_\varepsilon(s)) \geq c_\varepsilon - \alpha$. So $\gamma_\varepsilon(s) \in Y_\varepsilon^{d/2}$. Now, we assume that Lemma 3.14 is not true, then for $\varepsilon > 0$ small enough, we can find $a(\varepsilon) > 0$ satisfies $|J_\varepsilon'(u)| \geq a(\varepsilon)$ on $Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$. Moreover, by using Lemma 3.12, we also can find $\omega > 0$, independent of $\varepsilon > 0$, satisfies for $u \in J_\varepsilon^{\tilde{c}_\varepsilon} \cap \left(Y_\varepsilon^d \backslash Y_\varepsilon^{d/2}\right)$, $|J_\varepsilon'(u)| \geq \omega$. Therefore, recalling that $\lim_{\varepsilon\to 0}(c_\varepsilon - \tilde{c}_\varepsilon) = 0$, according to a deformation lemma, for $\varepsilon > 0$ small enough, we can construct a path $\gamma \in \Gamma_\varepsilon$ satisfying $J_\varepsilon(\gamma(s)) < c_\varepsilon, s \in [0,1]$. Obviously contradictory. □

Lemma 3.15. *For $\varepsilon > 0$ sufficiently small, $u_\varepsilon \in Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$ is a critical point of J_ε.*

Proof. For $\varepsilon > 0$ sufficiently small. According to Lemma 3.14, there exists a sequence $\{u_{n,\varepsilon}\}_{n=1}^\infty \subset Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$ that satisfies, as $n \to \infty$, $|J_\varepsilon'(u_{n,\varepsilon})| \to 0$. Due to Y_ε^d is bounded, so as $n \to \infty$, $u_{n,\varepsilon} \rightharpoonup u_\varepsilon$ in X_ε. Using the same proof as [10, Proposition 3], we obtain that

$$
0 = \lim_{R\to\infty}\sup_{n\geq 1}\int_{|x|\geq R}\left(V_\varepsilon|u_{n,\varepsilon}|^N + |\nabla u_{n,\varepsilon}|^N\right)\mathrm{d}x \tag{3.32}
$$

and

$$
0 = \lim_{R\to\infty}\sup_{n\geq 1}\int_{|x|\geq R}\left(V_\varepsilon|u_{n,\varepsilon}|^q + |\nabla u_{n,\varepsilon}|^q\right)\mathrm{d}x, \tag{3.33}
$$

so as $n \to \infty$, $u_{n,\varepsilon} \to u_\varepsilon$ in $L^r\left(\mathbb{R}^N\right)$ $(N \leq r < +\infty)$. In addition, using (f_1)–(f_2), we have $\sup\|f(u_{n,\varepsilon})\| < \infty$. Now, $\forall \varphi \in C_0^\infty(\mathbb{R}^N)$,

$$
\int_{\mathbb{R}^N} f(u_{n,\varepsilon})(u_{n,\varepsilon} - u_\varepsilon)\varphi \mathrm{d}x \to 0, \quad n \to \infty.
$$

Using the same argument as in [21, Proposition 5.3], we have $u_{n,\varepsilon} \to u_\varepsilon$ in X_ε as $n \to \infty$. Hence, $u_\varepsilon \in Y_\varepsilon^d \cap J_\varepsilon^{\tilde{c}_\varepsilon}$ and $J_\varepsilon'(u_\varepsilon) = 0$ in X_ε. This completes the proof. □

Next, we will use Moser iteration in [27] to obtain L^∞-estimate.

Lemma 3.16. *Let (u_n) is the sequence in Lemma 3.11. Then, $J_{\varepsilon_n}(u_n) \to c_m$ in $\mathbb{R}$ as $n \to \infty$, and there is some sequence $(\hat{y}_n) \subset \mathbb{R}^N$ that satisfies $v_n(\cdot) := u_n(\cdot + \hat{y}_n) \in L^\infty(\mathbb{R}^N)$ and $|v_n|_{L^\infty(\mathbb{R}^N)} \leqslant C$ for all $n \in \mathbb{N}$.*

Proof. Proceeding as in the proof of Lemmas 3.9 and 3.10, as $n \to \infty$, we know that $J_{\varepsilon_n}(u_n) \to c_m$ in $\mathbb{R}$. According to Lemma 3.11, as $n \to \infty$, we can find $(\hat{y}_n) \subset \mathbb{R}^N$ satisfies $v_n(\cdot) := u_n(\cdot + \hat{y}_n) \to v(\cdot) \in X_\varepsilon$ and $y_n := \varepsilon_n \hat{y}_n \to y_0 \in \mathcal{M}$.

For all $L > 0$ and $\beta > 1$, consider

$$\phi(v_n) = \phi_{L,\beta}(v_n) = v_n v_{L,n}^{N(\beta-1)} \in X_\varepsilon, v_{L,n} = \min\{v_n, L\}.$$

Set

$$\Phi(t) = \int_0^t (\phi'(t))^{\frac{1}{N}} \mathrm{d}\tau, \quad \Upsilon(t) = \frac{|t|^N}{N}.$$

According to [5], we have

$$|\Phi(a) - \Phi(b)|^N \le \Upsilon'(a-b)(\phi(a) - \phi(b)), \quad \forall a \in \mathbb{R},\ b \in \mathbb{R}. \tag{3.34}$$

According to (3.34), we have

$$\begin{aligned} &|\Phi(v_n(x)) - \Phi(v_n(y))|^N \\ &\quad \le (v_n(x) - v_n(y)) \left(\left(v_n v_{L,n}^{N(\beta-1)}\right)(x) - \left(v_n v_{L,n}^{N(\beta-1)}\right)(y) \right) |v_n(x) - v_n(y)|^{N-2}. \end{aligned} \tag{3.35}$$

Therefore, taking $\phi(v_n) = v_n v_{L,n}^{N(\beta-1)}$ as a test function, we obtain that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla v_n|^{N-1} \phi(v_n)\, \mathrm{d}x + \int_{\mathbb{R}^N} |\nabla v_n|^{q-1} \phi(v_n)\, \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} V(y_n + \varepsilon_n x) |v_n|^{N-2} v_n \phi(v_n)\, \mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |v_n|^{q-2} v_n \phi(v_n)\, \mathrm{d}x \\ &= \int_{\mathbb{R}^N} f(\varepsilon_n x + y_n, v_n)\, \phi(v_n)\, \mathrm{d}x. \end{aligned}$$

Due to (f_1) and (f_2), $\forall \varepsilon > 0$, we can find $C(\varepsilon) > 0$ satisfies

$$|f(t)| \le \varepsilon |t|^{q-1} + C(\varepsilon)|t|^{N-1} \Psi_N(t), \quad \forall t \in \mathbb{R}.$$

According to method of [5], it is easy to get

$$\int_{\mathbb{R}^N} |\nabla v_n|^N v_{L,n}^{p(\beta-1)} \mathrm{d}x + \int_{\mathbb{R}^N} V(\varepsilon_n x + y_n) |v_n|^N v_{L,n}^{p(\beta-1)} \mathrm{d}x \le \int_{\mathbb{R}^N} f(v_n)\, v_n v_{L,n}^{N(\beta-1)} \mathrm{d}x.$$

Since $\Phi(v_n) \ge \frac{1}{\beta} v_n v_{L,n}^{\beta-1}, v_n v_{L,n}^{\beta-1} \ge \Phi(v_n)$ and the embedding from $X_\varepsilon \to L^{N^*}(\mathbb{R}^N)$ $(N^* > N)$ is continuous, so we can find $S_* > 0$ that satifies

$$\frac{1}{\beta^N} S_* \left\| v_n v_{L,n}^{\beta-1} \right\|_{L^{N^*}(\mathbb{R}^N)}^N \le S_* \|\Phi(v_n)\|_{L^{N^*}(\mathbb{R}^N)}^N \le \|\Phi(v_n)\|_{X_\varepsilon}^N. \tag{3.36}$$

Since $X_\varepsilon \to L^\nu(\mathbb{R}^N)$ $(\nu \ge N)$ is continuous, there exists $\mathcal{S}_\nu$ satisfying

$$\mathcal{S}_\nu = \inf_{u \neq 0, u \in X_\varepsilon} \frac{\|u\|_{X_\varepsilon}}{\|u\|_{L^\nu(\mathbb{R}^N)}}, \quad \nu \ge N.$$

This implies

$$\|u\|_{L^N(\mathbb{R}^N)} \le \mathcal{S}_N^{-1}\|u\|_{X_\varepsilon}, \quad \forall u \in X_\varepsilon. \tag{3.37}$$

Then we obtain

$$\begin{aligned}\|\Phi(v_n)\|_{m,X(\mathbb{R}^N)}^N &\le \varepsilon \int_{\mathbb{R}^N} \left|v_n v_{L,n}^{\beta-1}\right|^N \mathrm{d}x + C(\varepsilon)\int_{\mathbb{R}^N} \Psi_N(v_n)\left|v_n v_{L,n}^{\beta-1}\right|^p \mathrm{d}x \\ &\le \varepsilon\beta^N \int_{\mathbb{R}^N} |\Phi(v_n)|^N \mathrm{d}x + C(\varepsilon)\int_{\mathbb{R}^N} \Psi_N(v_n)\left|v_n v_{L,n}^{\beta-1}\right|^N \mathrm{d}x \\ &\le \varepsilon\beta^N \mathcal{S}_N^{-N}\|\Phi(v_n)\|_{m,X(\mathbb{R}^N)}^N + C(\varepsilon)\int_{\mathbb{R}^N} \Psi_N(v_n)\left|v_n v_{L,n}^{\beta-1}\right|^N \mathrm{d}x.\end{aligned} \tag{3.38}$$

Choose $0 < \varepsilon < \beta^{-N}\mathcal{S}_N^N$, then (3.38) implies

$$\begin{aligned}&\frac{1}{\beta^N} S_*\left(1-\varepsilon\beta^N\mathcal{S}_N^{-N}\right)\left\|v_n v_{L,n}^{\beta-1}\right\|_{L^{N^*}(\mathbb{R}^N)}^N \\ &\le C(\varepsilon)\left(\int_{\mathbb{R}^N}(\Psi_N(v_n))^{q'}\mathrm{d}x\right)^{\frac{1}{q'}}\left(\int_{\mathbb{R}^N}\left|v_n v_{L,n}^{\beta-1}\right|^{qN}\mathrm{d}x\right)^{\frac{1}{q}}.\end{aligned}$$

Now, by the Trudinger–Moser inequality with $N << q$ such that $N^* > qN = N^{**}$. Note that, q' near 1 but $q' > 1$. So we can find $D > 0$ satisfies

$$\left\|v_n v_{L,n}^{\beta-1}\right\|_{L^{N^*}(\mathbb{R}^N)}^N \le D\beta^N\left\|v_n v_{L,n}^{\beta-1}\right\|_{L^{qN}(\mathbb{R}^N)}^N.$$

Let $L \to +\infty$, we obtain

$$\|v_n\|_{L^{N^*\beta}} \le D^{\frac{1}{N\beta}}\beta^{\frac{1}{\beta}}\|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)}. \tag{3.39}$$

Let $\beta = \frac{N^*}{N^{**}} > 1$. Then $\beta^2 N^{**} = \beta N^*$. Replace β with β^2, (3.39) holds. Hence,

$$\begin{aligned}\|v_n\|_{L^{N^*\beta^2}} &\le D^{\frac{1}{N\beta^2}}\beta^{\frac{2}{\beta^2}}\|v_n\|_{L^{N^{**}\beta^2}(\mathbb{R}^N)} \\ &= D^{\frac{1}{N\beta^2}}\beta^{\frac{2}{\beta^2}}\|v_n\|_{L^{N^*\beta}(\mathbb{R}^N)} \\ &\le D^{\frac{1}{N}\left(\frac{1}{\beta}+\frac{1}{\beta^2}\right)}\beta^{\frac{1}{\beta}+\frac{2}{\beta^2}}\|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)}.\end{aligned} \tag{3.40}$$

Now iterating the process, as shown in (3.40), for any positive integer m, we get that

$$\|v_n\|_{L^{N^*\beta^\sigma}} \le D^{\sum\limits_{j=1}^{\sigma}\frac{1}{N\beta^j}}\beta^{\sum\limits_{j=1}^{\sigma} j\beta^{-j}}\|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)}. \tag{3.41}$$

Taking the limit in (3.41) as $\sigma \to \infty$, we have

$$\|v_n\|_{L^\infty(\mathbb{R}^N)} \le C$$

for all n, where $C = D^{\sum\limits_{j=1}^{\infty}\frac{1}{N\beta^j}}\beta^{\sum_{j=1}^{\infty} j\beta^{-j}}\sup\limits_n \|v_n\|_{L^{N^{**}\beta}(\mathbb{R}^N)} < +\infty.$ □

Proof of Theorem 1.1. For $\varepsilon \in (0, \varepsilon_0)$, according to Lemma 3.15, there are $d,\ \varepsilon_0 > 0$ that satisfy J_ε has a critical point $u_\varepsilon \in Y_\varepsilon^d \cap \Gamma_\varepsilon^{\tilde{c}_\varepsilon}$. Since u_ε satisfies

$$-\Delta_N u_\varepsilon - \Delta_q u_\varepsilon + V(\varepsilon x)(|u_\varepsilon|^{N-2}u_\varepsilon + |u_\varepsilon|^{q-2}u_\varepsilon) = f(u_\varepsilon) + 4\left(\int_{\mathbb{R}^N}\chi_\varepsilon u_\varepsilon^p \mathrm{d}x - 1\right)_+ \chi_\varepsilon u_\varepsilon \quad \text{in } \mathbb{R}^N.$$

When $t \leq 0$, we know $f(t) = 0$. So $u_\varepsilon > 0$ in $\mathbb{R}^N$. In addition, by using Lemma 3.16, it is easy to get $\{\|u_\varepsilon\|_{L^\infty}\}_\varepsilon$ is bounded. Now by using Lemma 3.11, we have

$$\lim_{\varepsilon \to 0} \left[\frac{1}{N} \left(\int_{\mathbb{R}^N \setminus \mathcal{M}_\varepsilon^{2\delta}} |\nabla u_\varepsilon|^N + V_\varepsilon (u_\varepsilon)^N \, dx \right) + \frac{1}{q} \left(\int_{\mathbb{R}^N \setminus \mathcal{M}_\varepsilon^{2\delta}} |\nabla u_\varepsilon|^q + V_\varepsilon (u_\varepsilon)^q \, dx \right) \right] = 0.$$

According to elliptic estimates in [20], we know

$$\lim_{\varepsilon \to 0} \|u_\varepsilon\|_{L^\infty(\mathbb{R}^N \setminus \mathcal{M}_\varepsilon^{2\delta})} = 0.$$

Similar to [35], there are $C > 0$, $c > 0$ that satisfy

$$u(x) \leq Ce^{-c|x|}.$$

In fact, by using the Radial Lemma in [7], one has

$$u(x) \leq C \frac{\|u\|_{L^N}}{|x|}, \quad \forall x \neq 0,$$

here C is related to N, p. Therefore, for $u \in \mathcal{S}_m$, we have $\lim_{|x| \to \infty} u(x) = 0$ uniformly. According to the comparison principle, we have that $C > 0$, $c > 0$ satisfy

$$u(x) \leq Ce^{-c|x|}, \quad \forall x \in \mathbb{R}^N.$$

According to a comparison principle, for some $C, c > 0$, we obtain that

$$u_\varepsilon(x) \leq C \exp\left(-c \operatorname{dist}\left(x, \mathcal{M}_\varepsilon^{2\delta}\right)\right).$$

So $Q_\varepsilon(u_\varepsilon) = 0$, then u_ε satisfies (1.1). Lastly, assume u_ε has a maximum point x_ε. According to Lemma 3.8 and Lemma 3.11, for some $x \in \mathcal{M}$, we get that $\varepsilon x_\varepsilon \to x$ as $\varepsilon \to 0$. Moreover, as to $C > 0$, $c > 0$,

$$u_\varepsilon(x) \leq Ce^{-c|x - x_\varepsilon|}.$$

This completes the final proof. □

Acknowledgements

The authors express their sincere gratitude to the referee for his/her careful reading and helpful suggestions. L. Wang was supported by National Natural Science Foundation of China (No. 12161038), Science and Technology project of Jiangxi provincial Department of Education (No. GJJ212204 and GJJ2200635), Jiangxi Provincial Natural Science Foundation (Grant No. 20202BABL211004). B. Zhang was supported by National Natural Science Foundation of China (No. 11871199 and No. 12171152) and Cultivation Project of Young and Innovative Talents in Universities of Shandong Province.

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