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On a generalized cyclic-type system of difference equations with maximum

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Abstract. In this paper we investigate the behaviour of the solutions of the following k-dimensional cyclic system of difference equations with maximum:

$$x_i(n+1) = \max\left\{A_i, \frac{x_i^p(n)}{x_{i+1}^q(n-1)}\right\}, \qquad i = 1,2,\dots,k-1,$$
$$x_k(n+1) = \max\left\{A_k, \frac{x_k^p(n)}{x_1^q(n-1)}\right\}$$

where $n = 0,1,\dots$, $A_i > 1$, for $i = 1,2,\dots,k$, whereas the exponents p, q and the initial values $x_i(-1), x_i(0)$, $i = 1,2,\dots,k$ are positive real numbers.

Keywords: difference equations with maximum, cyclic system, equilibrium, eventually equal to equilibrium.

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1 Introduction

Undoubtedly, there is a growing interest in the study of difference equations and systems of difference equations. Among others, the study of difference equations and systems of difference equations with maximum, have attracted some attention in the last few decades (see, for instance, [1, 5–9, 11–17, 20, 22, 24, 26, 28, 29, 35–51, 54–58] and the related references therein). For some differential equations with maximum see, for example, [18, 19].

At the beginning were usually studied the difference equations and systems containing several arguments of the form $A_k(n)/x(n-k)$ where $k = 0,1,\dots,$ and $A_k(n)$ is a given sequence of real numbers (see, for example, [5, 7, 9, 15–17, 26, 28, 29, 56–58]), whereas equations and systems containing several arguments of the form $x^p(n-k)$, where p is a real number, have been usually studied recently (see, for example, [1, 6, 12–14, 35–49, 51, 52, 54, 55]).

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The motivation for the study of such difference equations and systems of difference equations stems from the study of the equations of the form

$$x(n) = a + \frac{x^p(n-k)}{x^q(n-l)}, \qquad n = 1, 2, \dots,$$

where the parameters a, p, q, and the initial values $x(j)$, $j = -\max\{k, l\}, \dots, 0$, are real or nonnegative numbers and k and l are positive integers, and their generalizations (see, for example, [2–4, 21, 23, 25, 27, 30–36] and the references cited therein).

In [10] was initiated studying cyclic systems of difference equations. The study was continued, for instance, in [11, 24, 46, 49, 52–55].

In [55] was studied the behaviour of the solutions of the following cyclic system of difference equations with maximum:

$$x_i(n+1) = \max\left\{A_i, \frac{x_i(n)}{x_{i+1}(n-1)}\right\}, \qquad i = 1, 2, \dots, k,$$

where $n = 0, 1, \dots$, the coefficients A_i, $i = 1, 2, \dots, k$ are positive constants, and the initial values $x_i(-1), x_i(0)$, $i = 1, 2, \dots, k$ are real positive numbers. Moreover, for $k = 2$ under some conditions it were found solutions which converge to periodic six solutions.

In this paper we continue the investigation of cyclic systems of difference equations by studying the behaviour of the solutions of the following generalized cyclic system of difference equations with maximum:

$$x_i(n+1) = \max\left\{A_i, \frac{x_i^p(n)}{x_{i+1}^q(n-1)}\right\}, \qquad i = 1, 2, \dots, k, \tag{1.1}$$

where $n = 0, 1, \dots$, for the coefficients A_i we assume that $A_i > 1$, $i = 1, 2, \dots, k$, the exponents p, q and the initial values $x_i(-1), x_i(0)$, $i = 1, 2, \dots, k$ are positive real numbers, and since the system is cyclic we have $A_{\lambda k+i} = A_i$, $x_{\lambda k+i}(n) = x_i(n)$, λ positive integer, $i = 1, 2, \dots, k$. To do this we use some methods and ideas in the literature mentioned above. Finally, using the results obtained for the general system (1.1), we derive some further results for system (1.1) for $k = 2$.

2 Main results

Lemma 2.1. *Consider the system of algebraic equations*

$$x_i = \max\left\{A_i, \frac{x_i^p}{x_{i+1}^q}\right\}, \qquad i = 1, 2, \dots, k, \tag{2.1}$$

where

$$A_{\lambda k+i} = A_i, \qquad x_{\lambda k+i} = x_i, \qquad i = 1, 2, \dots k, \qquad \lambda \text{ is a positive integer}, \tag{2.2}$$

and

$$A_i > 1, \qquad i = 1, 2, \dots k, \tag{2.3}$$

then

(i) *if*

$$0 < p \le 1, \qquad q > 0, \tag{2.4}$$

then system (2.1) *has a unique solution, which is*

$$(A_1, A_2, \ldots, A_k).$$

(ii) *If*

$$p > 1, \qquad 0 < q < p-1, \tag{2.5}$$

then system (2.1) *has no solutions.*

(iii) *Suppose that*

$$p > 1, \qquad q > p-1. \tag{2.6}$$

If there exist m *positive integers*

$$r_1, r_2, \ldots, r_m \in \{1,2,\ldots,k\}, \qquad r_1 < r_2 < \cdots < r_m, \qquad m \in \{1,2,\ldots,k\}, \tag{2.7}$$

such that

$$A_i < A_{r_j}^{\left(\frac{q}{p-1}\right)^{k+r_j-i}}, \quad \textit{for any } i \in \{r_j, r_j+1, \ldots, k\}, \textit{ and for any } j \in \{1,2,\ldots,m\}, \tag{2.8}$$

and

$$A_i < A_{r_j}^{\left(\frac{q}{p-1}\right)^{r_j-i}}, \quad \textit{for any } i \in \{1,2,\ldots,r_j-1\}, \textit{ and for any } j \in \{1,2,\ldots,m\}, \tag{2.9}$$

and for any $r \in \{1,2,\ldots,k\}$, $r \ne r_j$, $j \in \{1,2,\ldots,m\}$, *there exists an integer* $i \in \{1,2,\ldots,k\}$, *such that*

$$A_i > A_r^{\left(\frac{q}{p-1}\right)^{k+r-i}}, \quad \textit{for } i > r, \tag{2.10}$$

or

$$A_i > A_r^{\left(\frac{q}{p-1}\right)^{r-i}}, \quad \textit{for } i < r, \tag{2.11}$$

holds, then system (2.1) *has* $2^m - 1$ *solutions.*

(iv) *If*

$$q = p - 1 > 0, \tag{2.12}$$

then all solutions of (2.1) *are the following*

$$(x_1, x_2, \ldots, x_k) = (a, a, \ldots, a), \quad \textit{for any } a \ge A_w = \max\{A_1, A_2, \ldots, A_k\}. \tag{2.13}$$

Proof. From (2.1) and (2.3), we get

$$x_i > 1, \quad \text{for any } i \in \{1,2,\ldots,k\}. \tag{2.14}$$

(i) Suppose that (2.4) holds, then, from (2.14), we have

$$\frac{x_i^p}{x_{i+1}^q} < x_i^p \le x_i, \quad \text{for any } i \in \{1,2,\ldots,k\}. \tag{2.15}$$

Using (2.1) and (2.15), we have

$$x_i = A_i, \quad \text{for any } i \in \{1, 2, \ldots, k\}.$$

(ii) Now, suppose that (2.5) holds. We prove that system (2.1) has no solution.

On the contrary, we assume that there exists a solution of system (2.1). From (2.1), we have

$$x_i \geq \frac{x_i^p}{x_{i+1}^q}, \quad \text{for any } i \in \{1, 2, \ldots, k\}, \tag{2.16}$$

and so from (2.5), (2.14), and (2.16), we get

$$x_{i+1} \geq x_i^{\frac{p-1}{q}} > x_i, \quad \text{for any } i \in \{1, 2, \ldots, k\},$$

and obviously,

$$x_{k+1} > x_k > x_{k-1} > \cdots > x_1. \tag{2.17}$$

From (2.2) and (2.17), we get $x_1 > x_1$. So, system (2.1) has no solution.

(iii) Now, suppose that (2.6) holds.

From (2.3) and (2.6) it is obvious that (2.8) and (2.9) hold for $r_j = w$, where

$$A_w = \max\{A_1, A_2, \ldots, A_k\}.$$

So, $m \geq 1$.

First, we prove that, for every solution of (2.1), there exists a $b \in \{1, 2, \ldots, k\}$ such that

$$x_b = A_b. \tag{2.18}$$

On the contrary, suppose that

$$x_i = \frac{x_i^p}{x_{i+1}^q} = x_{i+1}^{\frac{q}{p-1}}, \quad \text{for any } i \in \{1, 2, \ldots, k\}. \tag{2.19}$$

From (2.2) and (2.19), we get

$$x_1 = x_{k+1}^{\left(\frac{q}{p-1}\right)^k} = x_1^{\left(\frac{q}{p-1}\right)^k},$$

and since k is a positive integer and (2.14) holds, we get $q = p - 1$ which contradicts with (2.6). So (2.18) is true.

To continue, we prove that

$$x_i \leq x_{i+1}^{\frac{q}{p-1}}, \quad \text{for any } i \in \{1, 2, \ldots, k\}. \tag{2.20}$$

From (2.1), we get (2.16) and so from (2.6), relation (2.20) is obvious.

In addition, from (2.1),

$$A_i \leq x_i, \quad \text{for any } i \in \{1, 2, \ldots, k\}. \tag{2.21}$$

In what follows, we prove that if there exist $i,\ r \in \{1, 2, \ldots, k\}$, such that either (2.10) or (2.11) holds, then

$$x_r = \frac{x_r^p}{x_{r+1}^q}. \tag{2.22}$$

On the contrary, suppose that

$$x_r = A_r. \tag{2.23}$$

If (2.10) holds, then, from (2.6), (2.20), and (2.21), we have

$$A_i \le x_i \le x_{i+1}^{\frac{q}{p-1}} \le \cdots \le x_k^{(\frac{q}{p-1})^{k-i}} \le x_1^{(\frac{q}{p-1})^{k-i+1}} \le \cdots \le x_r^{(\frac{q}{p-1})^{k-i+r}} = A_r^{(\frac{q}{p-1})^{k+r-i}},$$

which contradicts with (2.10). So, necessarily, if (2.10) holds, then relation (2.22) is true.

Now, suppose that (2.11) holds, then, from (2.6), (2.20), and (2.21), we have

$$A_i \le x_{i+1}^{\frac{q}{p-1}} \le x_{i+2}^{(\frac{q}{p-1})^2} \le \cdots \le x_r^{(\frac{q}{p-1})^{r-i}} = A_r^{(\frac{q}{p-1})^{r-i}},$$

which contradicts with (2.11). So, necessarily, if (2.11) holds, then relation (2.22) is true.

Finally, suppose that there exist exactly m positive integers such that (2.7), (2.8) and (2.9) hold. For any $j \in \{1, 2, \ldots, m\}$, we prove that both equations

$$x_{r_j} = A_{r_j}. \tag{2.24}$$

and

$$x_{r_j} = \frac{x_{r_j}^p}{x_{r_j+1}^q}, \tag{2.25}$$

are possible.

Since for any $i \in \{1, 2, \ldots, k\}$, $i \neq r_j$, $j \in \{1, 2, \ldots, m\}$, relation either (2.10) or (2.11) holds, from (2.22) we get

$$x_i = x_{i+1}^{\frac{q}{p-1}}, \quad \text{for any } i \in \{1, 2, \ldots, k\},\ i \neq r_j,\ j \in \{1, 2, \ldots, m\}. \tag{2.26}$$

From (2.26),

$$\begin{aligned}
&x_{r_m-1} = x_{r_m}^{\frac{q}{p-1}}, x_{r_m-2} = x_{r_m}^{(\frac{q}{p-1})^2}, \ldots, x_{r_{m-1}+1} = x_{r_m}^{(\frac{q}{p-1})^{r_m-r_{m-1}-1}},\\
&x_{r_{m-1}-1} = x_{r_{m-1}}^{\frac{q}{p-1}}, x_{r_{m-1}-2} = x_{r_{m-1}}^{(\frac{q}{p-1})^2}, \ldots, x_{r_{m-2}+1} = x_{r_{m-1}}^{(\frac{q}{p-1})^{r_{m-1}-r_{m-2}-1}},\\
&\quad\vdots\\
&x_{r_2-1} = x_{r_2}^{\frac{q}{p-1}}, x_{r_2-2} = x_{r_2}^{(\frac{q}{p-1})^2}, \ldots, x_{r_1+1} = x_{r_2}^{(\frac{q}{p-1})^{r_2-r_1-1}},\\
&x_{r_1-1} = x_{r_1}^{\frac{q}{p-1}}, \ldots, x_1 = x_{r_1}^{(\frac{q}{p-1})^{r_1-1}}, x_k = x_{r_1}^{(\frac{q}{p-1})^{r_1}}, \ldots, x_{r_m+1} = x_{r_1}^{(\frac{q}{p-1})^{k-(r_m-r_1)-1}},
\end{aligned} \tag{2.27}$$

and so from (2.1) and (2.27) for $l = 1, 2, \ldots, m-1$ we get,

$$x_{r_l} = \max\left\{A_{r_l}, \frac{x_{r_l}^p}{x_{r_l+1}^q}\right\} = \max\left\{A_{r_l}, \frac{x_{r_l}^p}{\left(x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q}\right\}.$$

Now, we prove that x_{r_l} can be equal either to A_{r_l} or to $\frac{x_{r_l}^p}{\left(x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q}$.

If $x_{r_l} = A_{r_l}$ then, from (2.1), (2.6), (2.7) we get

$$\frac{x_{r_l}^p}{\left(x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q} \le \frac{A_{r_l}^p}{\left(A_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q}. \tag{2.28}$$

Using (2.6), (2.7) and (2.9) for $i = r_l$ and $j = l+1$ we have

$$A_{r_l} < A_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l}}$$

and from (2.5)

$$A_{r_l}^{p-1} < A_{r_{l+1}}^{(p-1)(\frac{q}{p-1})^{r_{l+1}-r_l}} = A_{r_{l+1}}^{q\frac{(p-1)}{q}(\frac{q}{p-1})^{r_{l+1}-r_l}} = \left(A_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q.$$

Then,

$$\frac{A_{r_l}^p}{\left(A_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q} < A_{r_l}. \tag{2.29}$$

Therefore, from (2.28) and (2.29) we

$$\frac{x_{r_l}^p}{\left(x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q} < A_{r_l}.$$

If $x_{r_l} = \frac{x_{r_l}^p}{\left(x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l-1}}\right)^q}$ then, from (2.1), (2.6), (2.7) and (2.9) for $i = r_l$ and $j = l+1$, we get

$$x_{r_l} = x_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l}} \geq A_{r_{l+1}}^{(\frac{q}{p-1})^{r_{l+1}-r_l}} > A_{r_l},$$

and so, for any $j \in \{1, 2, \ldots, m-1\}$, both equations (2.24) and (2.25) are possible.

From (2.1) and the last equality of (2.27) we get

$$x_{r_m} = \max\left\{A_{r_m}, \frac{x_{r_m}^p}{x_{r_m+1}^q}\right\} = \max\left\{A_{r_m}, \frac{x_{r_m}^p}{\left(x_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1-1}}\right)^q}\right\}.$$

Finally, we prove that x_{r_m} can be equal either to A_{r_m} or to $\frac{x_{r_m}^p}{\left(x_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1-1}}\right)^q}$.

If $x_{r_m} = A_{r_m}$, then, from (2.1), (2.6), (2.7), we get

$$\frac{x_{r_m}^p}{\left(x_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_l-1}}\right)^q} \leq \frac{A_{r_m}^p}{\left(A_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1-1}}\right)^q}. \tag{2.30}$$

Using (2.6), (2.7) and (2.8) for $i = r_m$ and $j = 1$, we have

$$A_{r_m} < A_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1}}$$

and so, arguing as to prove (2.29)

$$\frac{A_{r_m}^p}{\left(A_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1-1}}\right)^q} < A_{r_m}. \tag{2.31}$$

Therefore, from (2.30) and (2.31), we take

$$\frac{x_{r_m}^p}{\left(x_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1-1}}\right)^q} < A_{r_m}.$$

If $x_{r_m} = \frac{x_{r_m}^p}{\left(x_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1-1}}\right)^q}$ then, from (2.1), (2.6), (2.7) and (2.8) for $i = r_m$ and $j = 1$, we get

$$x_{r_m} = x_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1}} \geq A_{r_1}^{(\frac{q}{p-1})^{k-r_m+r_1}} > A_{r_m}$$

and so for any $j \in \{1, 2, \ldots, m\}$ both equations (2.24) and (2.25) are possible.

From (2.7), (2.8), (2.9), (2.24), (2.25), and (2.26), and since, for every solution of (2.1) there exists at least one r such that (2.18) holds, we have that system (2.1) has $2^m - 1$ solutions.

(iv) Finally, suppose that (2.12) holds. From (2.1) and (2.12), we get

$$x_i \geq \frac{x_i^p}{x_{i+1}^{p-1}}, \quad \text{for any } i \in \{1, 2, \ldots, k\},$$

and so

$$x_{i+1} \geq x_i, \quad \text{for any } i \in \{1, 2, \ldots, k\}. \tag{2.32}$$

From (2.2) and (2.32), we have

$$x_{k+1} = x_1 \geq x_k \geq x_{k-1} \geq \cdots \geq x_2 \geq x_1,$$

which means that

$$x_1 = x_2 = \cdots = x_k.$$

Then, from (2.1) and (2.12), if we set $x_i = a$, $i = 1, 2, \ldots, k$, we get

$$a = \max\{A_i, a\}, \qquad i = 1, 2, \ldots, k.$$

Therefore, if $a \geq A_w$, we get that all the solutions of (2.1), if (2.12), holds are given by (2.13). This completes the proof of the Lemma 2.1. □

In the following proposition we give a result concerning the global behavior of the solutions of (1.1). Since the proof is similar to the proof of Proposition 2.2 of [55], we omit it.

Proposition 2.2. *Consider the system of difference equations* (1.1). *If* (2.4) *holds, then every solution of* (1.1) *is eventually equal to the unique equilibrium* $(A_1, A_2, \ldots, A_k)$.

In the following lemma we prove some results concerning the solutions of (1.1), which can be used in order to study the behavior of these solutions.

Lemma 2.3. *Consider the system of difference equations* (1.1) *where*

$$p > 1 \quad \textit{and} \quad q > 0. \tag{2.33}$$

For a solution of (1.1), *suppose that there exist a* $j \in \{1, 2, \ldots, k\}$, *a positive integer* $S_j \geq 2$, *and a constant* $a > 0$, *such that*

$$x_j(n) = a, \quad \textit{for any } n \geq S_j, \tag{2.34}$$

then

(i) If

$$x_{j-1}(S_j+1) > a^{\frac{q}{p-1}}, \tag{2.35}$$

then the solution of (1.1) *is unbounded.*

(ii) If

$$x_{j-1}(S_j+1) < a^{\frac{q}{p-1}}, \tag{2.36}$$

then there exists an integer $S_{j-1} \geq S_j+1$*, such that*

$$x_{j-1}(n) = A_{j-1}, \quad \text{for any } n \geq S_{j-1}. \tag{2.37}$$

(iii) If

$$x_{j-1}(S_j+1) = a^{\frac{q}{p-1}}, \tag{2.38}$$

then

$$x_{j-1}(n) = a^{\frac{q}{p-1}}, \quad \text{for any } n \geq S_j+1. \tag{2.39}$$

Proof. (i) From (1.1) and (2.34), we get

$$x_{j-1}(S_j+2) \geq \frac{x_{j-1}^p(S_j+1)}{x_j^q(S_j)} = \frac{x_{j-1}^p(S_j+1)}{a^q},$$
$$x_{j-1}(S_j+3) \geq \frac{x_{j-1}^p(S_j+2)}{x_j^q(S_j+1)} \geq \frac{x_{j-1}^{p^2}(S_j+1)}{a^{q(1+p)}},$$

and working inductively we have

$$x_{j-1}(S_j+m) \geq \frac{x_{j-1}^{p^{m-1}}(S_j+1)}{a^{q(1+p+p^2+\cdots+p^{m-2})}} = \frac{x_{j-1}^{p^{m-1}}(S_j+1)}{a^{q\frac{p^{m-1}-1}{p-1}}} = a^{\frac{q}{p-1}}\left(\frac{x_{j-1}(S_j+1)}{a^{\frac{q}{p-1}}}\right)^{p^{m-1}}, \quad m \geq 2. \tag{2.40}$$

From (2.33), (2.35), and (2.40), we get

$$\lim_{n\to\infty} x_{j-1}(n) = \infty,$$

and so, the solution of (1.1) is unbounded.

(ii) Now, suppose that (2.36) holds.

First, we prove that there exists a positive integer $S_{j-1} \geq S_j+1$, such that

$$x_{j-1}(S_{j-1}) = A_{j-1}. \tag{2.41}$$

If

$$x_{j-1}(S_j+1) = A_{j-1},$$

then (2.41) holds for $S_{j-1} = S_j+1$.

Now, suppose that

$$x_{j-1}(n) > A_{j-1}, \quad \text{for any } n \geq S_j+1, \tag{2.42}$$

then, from (1.1) and (2.34), and working as to prove (2.40), we have

$$x_{j-1}(S_j+m) = a^{\frac{q}{p-1}}\left(\frac{x_{j-1}(S_j+1)}{a^{\frac{q}{p-1}}}\right)^{p^{m-1}}, \qquad m \geq 2. \tag{2.43}$$

From (2.33), (2.36) and (2.43), we have that there exists a positive integer $n_0 \geq S_j + 2$, such that

$$x_{j-1}(n) < A_{j-1}, \quad \text{for any } n \geq n_0,$$

which contradicts with (2.42). So, in any case, there exists a positive integer $S_{j-1} \geq S_j + 1$, such that (2.41) holds.

Now, we prove that (2.37) holds for any $n \geq S_{j-1}$.

From (1.1) and (2.36), we get

$$A_{j-1} < a^{\frac{q}{p-1}}. \tag{2.44}$$

From (2.34), (2.41) and (2.44), we have

$$\frac{x_{j-1}^p(S_{j-1})}{x_j^q(S_{j-1}-1)} = \frac{A_{j-1}^p}{a^q} < \frac{A_{j-1}^p}{A_{j-1}^{p-1}} = A_{j-1},$$

and so, from (1.1), we have

$$x_{j-1}(S_{j-1}+1) = A_{j-1},$$

and working inductively we get (2.37).

(iii) Finally, suppose that (2.38) holds.

From (1.1) and (2.38), we get

$$A_{j-1} \leq a^{\frac{q}{p-1}}. \tag{2.45}$$

Using (2.34), (2.38) and (2.45), we get

$$\frac{x_{j-1}^p(S_j+1)}{x_j^q(S_j)} = \frac{a^{\frac{pq}{p-1}}}{a^q} = a^{\frac{q}{p-1}} \geq A_{j-1},$$

and so, from (1.1), we get

$$x_{j-1}(S_j+2) = a^{\frac{q}{p-1}},$$

and working inductively (2.39) is true.

So, the proof of Lemma 2.3 is completed. □

In the following propositions, we give furthermore results for system (1.1), where $k = 2$ and relation (2.6) or (2.12) holds. Our aim is to present how the results of Lemma 2.3 can be used, in order to find out how a solution of (1.1) behaves.

In what follows, without loss of generality, we assume that $A_2 = \max\{A_1, A_2\}$. If, in addition, (2.6) holds, and since $A_2 > 1$, we have that

$$A_1 < A_2^{\frac{q}{p-1}}. \tag{2.46}$$

Proposition 2.4. *Consider the system of difference equations*

$$\begin{aligned} x_1(n+1) &= \max\left\{A_1, \frac{x_1^p(n)}{x_2^q(n-1)}\right\}, \\ x_2(n+1) &= \max\left\{A_2, \frac{x_2^p(n)}{x_1^q(n-1)}\right\}, \end{aligned} \tag{2.47}$$

where $n = 0, 1, \ldots$, A_1, $A_2 > 1$, and the initial values $x_i(-1), x_i(0)$, $i = 1, 2$, are positive real numbers. Suppose that (2.6) holds.

The following statements are true:

I. *Suppose that*

$$A_2 > A_1^{\frac{q}{p-1}}. \tag{2.48}$$

Then system (2.47) has a unique equilibrium which is

$$(A_2^{\frac{q}{p-1}}, A_2). \tag{2.49}$$

Furthermore, we have:

(a) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), for which, there exists an integer $r \geq 2$, such that*

$$x_1(r) < A_2^{\frac{q}{p-1}}. \tag{2.50}$$

These solutions are unbounded.

(b) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), such that*

$$x_1(n) \geq A_2^{\frac{q}{p-1}}, \quad \text{for any } n \geq 2, \tag{2.51}$$

and

$$x_1(z) = A_2^{\frac{q}{p-1}}, \quad \text{for an integer } z \geq 2. \tag{2.52}$$

These solutions are eventually equal to the unique equilibrium (2.49).

(c) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), such that*

$$x_1(n) > A_2^{\frac{q}{p-1}}, \quad \text{for any } n \geq 2, \tag{2.53}$$

and

$$x_2(d) = A_2, \quad \text{for an integer } d \geq 2. \tag{2.54}$$

These solutions are unbounded.

II. *Suppose that*

$$A_2 < A_1^{\frac{q}{p-1}}. \tag{2.55}$$

Then system (2.47) has three equilibria, the one given by (2.49), and the following two,

$$(A_1, A_1^{\frac{q}{p-1}}), \tag{2.56}$$

and

$$(A_1, A_2). \tag{2.57}$$

Furthermore, we have:

(a) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), for which, there exists an integer $r \geq 2$, such that (2.50) holds. These solutions are unbounded or eventually equal to the equilibrium (2.56) or eventually equal to the equilibrium (2.57).*

(b) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), such that (2.51) and (2.52) hold. These solutions are eventually equal to the equilibrium (2.49).*

(c) *There exist solutions $(x_1(n), x_2(n))$ of (2.47), such that (2.53) and (2.54) hold. These solutions are unbounded.*

Proof. **(I.)** From (2.46), (2.48) and (iii) of Lemma 2.1, we have that system (2.47) has a unique equilibrium given by (2.49).

I(a). First, we prove that there exist solutions $(x_1(n), x_2(n))$ of (2.47), for which there exists an integer $r \geq 2$, such that (2.50) holds.
Indeed, if, for instance,

$$x_1(-1) > 0,\ x_1(0) > 0 \quad \text{and} \quad x_2(-1) \geq \frac{x_1^{\frac{p}{q}}(0)}{A_1^{\frac{1}{q}}}, \qquad x_2(0) > \frac{A_1^{\frac{p}{q}}}{A_2^{\frac{1}{p-1}}},$$

then, it is easy to prove that

$$x_1(2) < A_2^{\frac{q}{p-1}},$$

and so (2.50) is true for $r = 2$.

Now, we prove that, if for a solution of (2.47), relation (2.50) is satisfied, then the solution is unbounded.

At the beginning, we prove that there exists a positive integer $s \geq r$, such that

$$x_1(s) = A_1. \tag{2.58}$$

On the contrary, suppose that

$$x_1(n) > A_1, \quad \text{for any } n \geq r, \tag{2.59}$$

then, from (2.47), we have

$$\begin{aligned} x_1(r+1) &= \frac{x_1^p(r)}{x_2^q(r-1)} \leq \frac{x_1^p(r)}{A_2^q}, \\ x_1(r+2) &= \frac{x_1^p(r+1)}{x_2^q(r)} \leq \frac{x_1^{p^2}(r)}{A_2^{q(1+p)}}, \end{aligned}$$

and working inductively and as in (2.40), we get

$$x_1(r+m) \leq A_2^{\frac{q}{p-1}} \left(\frac{x_1(r)}{A_2^{\frac{q}{p-1}}} \right)^{p^m}, \qquad m \geq 1. \tag{2.60}$$

From (2.6), (2.50) and (2.60), we have that there exists a positive integer $n_0 \geq r$, such that

$$x_1(n) < A_1, \quad \text{for any } n \geq n_0,$$

which contradicts with (2.59). So, if (2.50) holds, then there exists a positive integer $s \geq r$, such that (2.58) holds.

Now, we prove that

$$x_1(n) = A_1, \quad \text{for any } n \geq s. \tag{2.61}$$

From (2.46), (2.47) and (2.58), we get

$$\frac{x_1^p(s)}{x_2^q(s-1)} \leq \frac{A_1^p}{A_2^q} \leq \frac{A_1^p}{A_1^{p-1}} = A_1. \tag{2.62}$$

From (2.47) and (2.62), obviously,

$$x_1(s+1) = A_1,$$

and working inductively we get (2.61).

From (2.47) and (2.48), we have

$$x_2(s+1) \geq A_2 > A_1^{\frac{q}{p-1}},$$

and so, from (2.61) and (i) of Lemma 2.3 for $a = A_1$, we have that the solution is unbounded.

I(b). We show that there exist solutions $(x_1(n), x_2(n))$ of (2.47) and an integer $z \geq 2$, such that (2.51) and (2.52) hold.

Indeed, if, for instance,

$$x_1(0) > A_2^{\frac{p-1}{q}}, \qquad x_1(-1) > A_2^{\frac{p-1}{q}}, \qquad x_2(0) = A_2, \qquad x_2(-1) = \frac{x_1^{\frac{p}{q}}(0)}{A_2^{\frac{1}{p-1}}},$$

it is easy to prove that

$$x_1(n) \geq A_2^{\frac{q}{p-1}}, \qquad n \geq -1 \quad \text{and} \quad x_1(2) = A_2^{\frac{q}{p-1}}.$$

Now, we prove that, if for a solution of (2.47), relations (2.51) and (2.52) hold, then the solution is eventually equal to the unique equilibrium (2.49).

From (2.47) and (2.52), we have

$$\frac{x_1^p(z)}{x_2^q(z-1)} \leq \frac{\left(A_2^{\frac{q}{p-1}}\right)^p}{A_2^q} = A_2^{\frac{q}{p-1}},$$

and so, from (2.46) and (2.47), we get

$$x_1(z+1) \leq A_2^{\frac{q}{p-1}},$$

and from (2.51) we have

$$x_1(z+1) = A_2^{\frac{q}{p-1}}.$$

Working inductively, we get

$$x_1(n) = A_2^{\frac{q}{p-1}} > A_1, \quad \text{for any } n \geq z. \tag{2.63}$$

From (2.47) and (2.63), we get

$$A_2^{\frac{q}{p-1}} = \max\left\{A_1, \frac{A_2^{\frac{pq}{p-1}}}{x_2^q(n)}\right\}, \qquad n \geq z-1,$$

and so, from (2.46), we have

$$x_2(n) = A_2, \quad \text{for any } n \geq z-1. \tag{2.64}$$

From (2.63) and (2.64), we have that the solution is eventually equal to the unique equilibrium (2.49).

I(c). We show that there exist solutions $(x_1(n), x_2(n))$ of (2.47) and an integer $d \geq 3$, such that (2.53) and (2.54) hold.

Indeed, if, for instance,

$$x_1(-1) > A_2^{\frac{p-1}{q}}, \qquad x_1(0) > A_2^{\frac{q}{p-1}} \quad \text{and} \quad x_2(-1) \leq A_2, \qquad x_2(0) \leq A_2,$$

it is easy to prove that

$$x_1(n) > A_2^{\frac{q}{p-1}}, \quad \text{for any } n \geq 2 \quad \text{and} \quad x_2(3) = A_2.$$

Now, we prove that, if for a solution of (2.47), relations (2.53) and (2.54) hold, then the solution is unbounded.

From (2.53) and (2.54), we have

$$\frac{x_2^p(d)}{x_1^q(d-1)} < \frac{A_2^p}{\left(A_2^{\frac{q}{p-1}}\right)^q} < A_2, \tag{2.65}$$

and so, from (2.47),

$$x_2(d+1) = A_2, \tag{2.66}$$

and working inductively, obviously,

$$x_2(n) = A_2, \quad \text{for any } n \geq d. \tag{2.67}$$

Since (2.53) hold, then from (2.67) and (i) of Lemma 2.3 for $a = A_2$, we have that the solution is unbounded.

II. From (2.46), (2.55) and (iii) of Lemma 2.1 we have that system (2.47) has three equilibria, which are given by (2.49), (2.56) and (2.57).

II(a). For a solution $(x_1(n), x_2(n))$ of (2.47) suppose that there exists an integer $r \geq 2$, such that (2.50) holds. Then, arguing as in **I(a)**, we get that there exists a positive integer $s \geq r$, such that (2.61) holds.

If

$$x_2(s+1) > A_1^{\frac{q}{p-1}}, \tag{2.68}$$

then from (2.61), (2.68) and (i) of Lemma 2.3 for $a = A_1$, we have that the solution is unbounded.

If

$$x_2(s+1) < A_1^{\frac{q}{p-1}}, \tag{2.69}$$

then from (2.61), (2.69) and (ii) of Lemma 2.3 for $a = A_1$, we have that there exists an integer $s_2 \geq s+1$, such that

$$x_2(n) = A_2, \quad \text{for any } n \geq s_2. \tag{2.70}$$

From (2.61) and (2.70), we have that the solution is eventually equal to the equilibrium (2.57).

If

$$x_2(s+1) = A_1^{\frac{q}{p-1}}, \tag{2.71}$$

then from (2.61), (2.71) and (iii) of Lemma 2.3 for $a = A_1$, we have that

$$x_2(n) = A_1^{\frac{q}{p-1}}, \quad \text{for any } n \geq s+1. \tag{2.72}$$

From (2.61) and (2.72) we have that the solution is eventually equal to the equilibrium (2.56).

Now, we show that there exist solutions $(x_1(n), x_2(n))$ of (2.47) and integers r, s, $r \geq 2$, $s \geq r$, such that (2.50) and (2.68) hold.

Indeed, if, for instance,

$$x_1(0) > 0, \qquad x_2(0) > A_1^{\frac{p-1}{q}} \quad \text{and} \quad x_1(-1) < \frac{x_2^{\frac{p}{q}}(0)}{A_1^{\frac{1}{p(p-1)}} x_1^{\frac{1}{p}}(0)}, \qquad x_2(-1) \geq \frac{x_1^{\frac{p}{q}}(0)}{A_1^{\frac{1}{q}}}, \tag{2.73}$$

it is easy to prove that

$$x_1(2) = A_1 < A_2^{\frac{q}{p-1}} \quad \text{and} \quad x_2(3) > A_1^{\frac{q}{p-1}},$$

and so these solutions are unbounded.

In addition, we show that there exist solutions $(x_1(n), x_2(n))$ of (2.47) and integers r, s, $r \geq 2$, $s \geq r$, such that (2.50) and (2.69) hold.

Indeed, if, for instance,

$$x_1(0) > \frac{A_2^{\frac{p}{q}}}{A_1^{\frac{1}{p-1}}}, \qquad x_2(0) > A_1^{\frac{p-1}{q}}, \qquad x_2(-1) \geq \frac{x_1^{\frac{p}{q}}(0)}{A_1^{\frac{1}{q}}}, \qquad x_1(-1) > \frac{x_2^{\frac{p}{q}}(0)}{A_1^{\frac{1}{p(p-1)}} x_1^{\frac{1}{p}}(0)},$$

it is easy to prove that

$$x_1(2) = A_1 < A_2^{\frac{q}{p-1}} \quad \text{and} \quad x_2(3) < A_1^{\frac{q}{p-1}},$$

and so these solutions are eventually equal to the equilibrium (2.57).

Finally, we show that there exist solutions $(x_1(n), x_2(n))$ of (2.47) and integers r, s, $r \geq 2$, $s \geq r$, such that (2.50) and (2.71) hold.

Indeed, if, for instance,

$$x_1(0) > \frac{A_2^{\frac{p}{q}}}{A_1^{\frac{1}{p-1}}}, \qquad x_2(0) > A_1^{\frac{p-1}{q}}, \qquad x_2(-1) \geq \frac{x_1^{\frac{p}{q}}(0)}{A_1^{\frac{1}{q}}}, \qquad x_1(-1) = \frac{x_2^{\frac{p}{q}}(0)}{A_1^{\frac{1}{p(p-1)}} x_1^{\frac{1}{p}}(0)}, \tag{2.74}$$

it is easy to prove that

$$x_1(2) = A_1 < A_2^{\frac{q}{p-1}} \quad \text{and} \quad x_2(3) = A_1^{\frac{q}{p-1}},$$

and so these solutions are eventually equal to the equilibrium (2.56).

II(b). The proof is the same as in **I**(b).

II(c). The proof is the same as in **I**(c). □

Proposition 2.5. *Consider the system of difference equations*

$$\begin{aligned} x_1(n+1) &= \max\left\{A_1, \frac{x_1^p(n)}{x_2^{p-1}(n-1)}\right\}, \\ x_2(n+1) &= \max\left\{A_2, \frac{x_2^p(n)}{x_1^{p-1}(n-1)}\right\}, \end{aligned} \tag{2.75}$$

where $n = 0, 1, \ldots,$ $A_1, A_2 > 1$, and the initial values $x_i(-1), x_i(0)$, $i = 1, 2$, are positive real numbers.

The following statements are true.

(a) *There exist solutions* $(x_1(n), x_2(n))$ *of* (2.75)*, for which, there exists an integer* $r \geq 2$*, such that*

$$x_1(r) < A_2. \tag{2.76}$$

These solutions are unbounded.

(b) *There exist solutions* $(x_1(n), x_2(n))$ *of* (2.75)*, such that*

$$x_1(n) \geq A_2, \quad \text{for any } n \geq 2, \tag{2.77}$$

and

$$x_1(z) = A_2, \quad \text{for an integer } z \geq 2. \tag{2.78}$$

These solutions are unbounded or eventually equal to the equilibrium (A_2, A_2)*.*

(c) *There exist solutions* $(x_1(n), x_2(n))$ *of* (2.75)*, such that*

$$x_1(n) > A_2, \quad \text{for any } n \geq 2, \tag{2.79}$$

and

$$x_2(d) = A_2, \quad \text{for an integer } d \geq 2. \tag{2.80}$$

These solutions are unbounded.

(d) *The solution* $(x_1(n), x_2(n)) = (a, a)$, $n \geq -1$, $a > A_2$*, is the only solution of* (2.75)*, which is eventually equal to the equilibrium* (a, a)*.*

Proof. **(a)** Since Lemma 2.3 holds for $q = p - 1 > 0$ and, from (2.75) and (2.76), we get that $A_1 < A_2$, the proof of **(a)** is exactly the same with the proof of **I(a)** of Proposition 2.4, and we omit it.

(b) If $A_1 < A_2$, then arguing as in the proof of **I(b)** of Proposition 2.4, we can prove that, there exist solutions $(x_1(n), x_2(n))$ of (2.75), such that relations (2.77) and (2.78) hold, and these solutions are eventually equal to the equilibrium (A_2, A_2).

If $A_1 = A_2$, then for a solution $(x_1(n), x_2(n))$ of (2.75), such that (2.77) and (2.78) hold, we have

$$x_1(z) = A_1, \quad \text{for an integer } z \geq 2, \tag{2.81}$$

and so, arguing as to prove (2.61), we get

$$x_1(n) = A_1, \quad \text{for any } n \geq z. \tag{2.82}$$

If

$$x_2(z+1) = A_2 = A_1, \tag{2.83}$$

then, from (2.82), (2.83), and (iii) of Lemma 2.3 for $a = A_1$ and $q = p - 1 > 0$, we have that

$$x_2(n) = A_2 = A_1, \quad \text{for any } n \geq z + 1. \tag{2.84}$$

From (2.82) and (2.84), we have that the solution is eventually equal to the equilibrium (A_2, A_2).

If

$$x_2(z+1) > A_2 = A_1, \tag{2.85}$$

then, from (2.82), (2.85), and (i) of Lemma 2.3 for $a = A_1$ and $q = p - 1 > 0$, we have that the solution is unbounded.

Now, we show that there exist solutions $(x_1(n), x_2(n))$ of (2.75), such that (2.81) and (2.85) hold for an integer z, $z \geq 2$. Indeed, if, for instance, relations (2.74) hold for $q = p - 1 > 0$, then it is easy to prove that

$$x_1(2) = A_1 = A_2 \quad \text{and} \quad x_2(3) = A_1 = A_2,$$

and so these solutions are eventually equal to the equilibrium (A_2, A_2).

In addition, we show that there exist solutions $(x_1(n), x_2(n))$ of (2.75), such that (2.81) and (2.83) hold for an integer z, $z \geq 2$,.

Indeed, if, for instance, relations (2.73) hold for $q = p - 1 > 0$, then it is easy to prove that

$$x_1(2) = A_1 = A_2 \quad \text{and} \quad x_2(3) > A_1 = A_2,$$

and so these solutions are unbounded.

(c) Relation (2.65), for $q = p - 1 > 0$, becomes

$$\frac{x_2^p(d)}{x_1^{p-1}(d-1)} < \frac{A_2^p}{A_2^{p-1}} = A_2,$$

and so, we have that, (2.66) also holds, and since Lemma 2.3 holds for $q = p - 1 > 0$, the proof of **(c)** is exactly the same with the proof of **I**(c) of Proposition 2.4, and we omit it.

(d) Suppose that $(x_1(n), x_2(n))$ is a solution of (2.75) eventually equal to the equilibrium (a, a), $a > A_2$. Then, there exists a positive integer n_0, such that

$$x_1(n) = a, \qquad x_2(n) = a, \quad \text{for any } n \geq n_0. \tag{2.86}$$

Since $a > A_2$, from (2.75) and (2.86), we have

$$x_1(n_0+1) = \frac{x_1^p(n_0)}{x_2^{p-1}(n_0-1)}, \qquad x_2(n_0+1) = \frac{x_2^p(n_0)}{x_1^{p-1}(n_0-1)},$$

and so, $x_2(n_0 - 1) = a$ and $x_1(n_0 - 1) = a$. Working inductively, we get

$$x_1(n) = a, \qquad x_2(n) = a, \quad \text{for any } \; -1 \leq n \leq n_0 - 1. \tag{2.87}$$

From (2.86) and (2.87), we have that

$$x_1(n) = a, \qquad x_2(n) = a, \quad \text{for any } n \geq -1. \qquad \square$$

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