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Normalized solutions to the Schrödinger systems with double critical growth and weakly attractive potentials

Lei Long and **Xiaojing Feng** ✉

School of Mathematical Sciences, Shanxi University, Taiyuan 030006, Shanxi, P.R. China

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Abstract. In this paper, we look for solutions to the following critical Schrödinger system

$$\begin{cases} -\Delta u + (V_1+\lambda_1)u = |u|^{2^*-2}u + |u|^{p_1-2}u + \beta r_1 |u|^{r_1-2}u|v|^{r_2} & \text{in } \mathbb{R}^N, \\ -\Delta v + (V_2+\lambda_2)v = |v|^{2^*-2}v + |v|^{p_2-2}v + \beta r_2 |u|^{r_1}|v|^{r_2-2}v & \text{in } \mathbb{R}^N, \end{cases}$$

having prescribed mass $\int_{\mathbb{R}^N} u^2 = a_1 > 0$ and $\int_{\mathbb{R}^N} v^2 = a_2 > 0$, where $\lambda_1, \lambda_2 \in \mathbb{R}$ will arise as Lagrange multipliers, $N \geqslant 3$, $2^* = 2N/(N-2)$ is the Sobolev critical exponent, $r_1, r_2 > 1$, $p_1, p_2, r_1 + r_2 \in (2 + 4/N, 2^*)$ and $\beta > 0$ is a coupling constant. Under suitable conditions on the potentials V_1 and V_2, $\beta_* > 0$ exists such that the above Schrödinger system admits a positive radial normalized solution when $\beta \geqslant \beta_*$. The proof is based on comparison argument and minmax method.

Keywords: Schrödinger systems, weakly attractive potentials, normalized solutions, positive solutions.

2020 Mathematics Subject Classification: 35J20, 35J60.

1 Introduction and main results

We study the following critical Schrödinger system

$$\begin{cases} -\Delta u + (V_1+\lambda_1)u = |u|^{2^*-2}u + |u|^{p_1-2}u + \beta r_1 |u|^{r_1-2}u|v|^{r_2} & \text{in } \mathbb{R}^N, \\ -\Delta v + (V_2+\lambda_2)v = |v|^{2^*-2}v + |v|^{p_2-2}v + \beta r_2 |u|^{r_1}|v|^{r_2-2}v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.1}$$

with prescribed mass

$$\int_{\mathbb{R}^N} u^2 = a_1 > 0 \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 = a_2 > 0, \tag{1.2}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ will arise as Lagrange multipliers, $N \geqslant 3$, $2^* = 2N/(N-2)$ is the Sobolev critical exponent, $r_1, r_2 > 1$, $p_1, p_2, r_1 + r_2 \in (2 + 4/N, 2^*)$, V_1 and V_2 are the potentials and $\beta > 0$ is a coupling constant. Solutions of (1.1) with prescribed mass (1.2) are called as the normalized solutions in the literature.

✉Corresponding author. Email: fengxj@sxu.edu.cn

The problem (1.1) comes from the research of solitary waves to the following system

$$\begin{cases} -\Delta\Phi_1 + V_1\Phi_1 - i\frac{\partial}{\partial t}\Phi_1 = |\Phi_1|^{2^*-2}\Phi_1 + |\Phi_1|^{p_1-2}\Phi_1 + \beta r_1|\Phi_1|^{r_1-2}\Phi_1|\Phi_2|^{r_2} \\ -\Delta\Phi_2 + V_2\Phi_2 - i\frac{\partial}{\partial t}\Phi_2 = |\Phi_2|^{2^*-2}\Phi_2 + |\Phi_2|^{p_2-2}\Phi_2 + \beta r_2|\Phi_1|^{r_1}|\Phi_2|^{r_2-2}\Phi_2 \\ \Phi_j = \Phi_j(x,t), (x,t) \in \mathbb{R}^N \times \mathbb{R}_+, \quad j = 1,2, \end{cases} \tag{1.3}$$

where t denotes the time, i is imaginary unit, Φ_j is the wave function of the jth component, β is a coupling constant which describes the scattering length of the attractive and repulsive interaction. If $\beta > 0$, then the interaction is attractive; if $\beta < 0$, then the interaction is repulsive. Set $\Phi_1(x,t) = e^{i\lambda_1 t}u(x)$ and $\Phi_2(x,t) = e^{i\lambda_2 t}v(x)$. It is easy to see that a couple (Φ_1, Φ_2) is the solution of (1.3) if and only if (u,v) is the solution of (1.1). The system (1.3) appears in many physical problems, especially in nonlinear optics and the mean-field models for binary mixtures of Bose-Einstein condensation, see [1,13,14] and reference therein for more physical background. An important, of course well known, feature of (1.3) is conservation of mass:

$$\int_{\mathbb{R}^N} |\Phi_j(x,t)|^2 dx = \int_{\mathbb{R}^N} |\Phi_j(x,0)|^2 dx, \qquad t \in \mathbb{R}_+.$$

Physically, the mass represents the number of particles of each component in Bose–Einstein condensates.

The presence of the mass constraint makes some methods developed to deal with unconstrained problems unavailable, and a new critical exponent appears, the mass critical exponent $2 + 4/N \in (2, 2^*)$. In the mass subcritical case, the Schrödinger equation are usually considered by the minimization arguments, we refer the readers to [8,9,29]. As far as we are aware, the mass supercritical case was first considered by Jeanjean in [21], for the Schrödinger equation. The key idea is to obtain mountain pass solution on S_a by constructing the mountain pass structure on a natural constraint related to the Pohozaev identity. Much work has been done extensively on the normalized solutions to the Schrödinger equation in in the last decades by variational methods. Since numerous contributions flourished within this topic and we just mention, among many possible numerous choices, [23,30,31]. For the nonautonomous Schrödinger equations, we refer the readers to [20,33] when mass subcritical case occurs and [5,12,28] when mass supercritical case occurs.

The existence and multiplicity of normalized solutions to the Schrödinger systems also attracted much attention of researchers in recent decades, see [2–4,6,7,10,17,18,22,25–27] and reference therein. In particular, for the Schrödinger system

$$\begin{cases} -\Delta u + \lambda_1 u = \mu_1|u|^{p-2}u + \nu_1|u|^{p_1-2}u + \beta r_1|u|^{r_1-2}u|v|^{r_2} & \text{in } \mathbb{R}^N, \\ -\Delta v + \lambda_2 v = \mu_2|v|^{p-2}v + \nu_2|v|^{p_2-2}v + \beta r_2|u|^{r_1}|v|^{r_2-2}v & \text{in } \mathbb{R}^N, \end{cases} \tag{1.4}$$

when $N \geqslant 3$, $\nu_1 = \nu_2 = 0$, $p = 4$ and $r_1 = r_2 = 2$, the existence and multiplicity of normalized solutions to (1.4) are studied in [4,6,7]; when $N = 3,4$, $\mu_1 = \mu_2 = 0$, $r_1, r_2 > 1$, $p_1, r_1 + r_2 \in (2, 2^*)$ and $p_2 \in (2, 2^*]$, Li and Zou in [22] studied the geometry of the associated Pohozaev manifold and obtained a normalized solution to (1.4); when $N = 4$, $p = 3$, $p_1, p_2 \in (2,4)$ and $r_1 = r_2 = 2$, the coupling terms are the Sobolev critical case, Luo et al. in [27] considered the existence, nonexistence and asymptotic behavior of normalized solutions to (1.4); when $N = 3,4$, $r_1, r_2 > 1$, $p = 2^*$ and $p_1, p_2, r_1 + r_2 \in (2 + 4/N, 2^*]$, recently, Liu and Fang in [26] obtained the existence and nonexistence of normalized solutions to system (1.4).

To the best of our knowledge, a few studies have addressed the existence of normalized solutions to Schrödinger system with potential. We know only [10,25], in which they considered the mass subcritical case. There is no work concerning normalized solutions to Schrödinger systems with mass supercritical, Sobolev critical and potential. This problem is more complicated and stimulating by the fact that both the potential and the critical term are present, which is the focus of this article. Specifically, in this paper, we consider Schrödinger system (1.1) with weakly attractive potentials, that is,

$$V_i(x) \leqslant \limsup_{|x|\to\infty} V_i(x) < \infty, \qquad i = 1,2,$$

and obtain a positive radial normalized solution. For the weakly repulsive potentials, that is,

$$V_i(x) \geqslant \liminf_{|x|\to\infty} V_i(x) > -\infty, \qquad i = 1,2,$$

does the system (1.1) have a normalized solution? This still is an open problem.

Precisely, $V_i \in C^1(\mathbb{R}^N)$ fulfills

(H$_1$) $\lim_{|x|\to\infty} V_i(x) = \sup_{x\in\mathbb{R}^N} V_i(x) = 0$ and there exists $\tau_i \in [0, 1/2)$ such that $|V_i|_{N/2} \leqslant \tau_i S$, where

$$S = \inf_{u\in D^{1,2}(\mathbb{R}^N)\setminus\{0\}} \frac{\int_{\mathbb{R}^N} |\nabla u|^2}{\left(\int_{\mathbb{R}^N} |u|^{2^*}\right)^{2/2^*}}; \tag{1.5}$$

(H$_2$) set $W_i(x) := (\nabla V_i(x)\cdot x)/2$, $W_i \in C^1(\mathbb{R}^N)$, $\lim_{|x|\to\infty} W_i(x) = 0$ and there exists $\theta_i \in [0, 1)$ with $(1-\tau_i)/2 - (1+\theta_i)/(\min\{\gamma_{p_1}p_1, \gamma_{p_2}p_2, \gamma_r r\}) > 0$ such that $|W_i|_{N/2} \leqslant \theta_i S$, where $\gamma_q = N(q-2)/(2q)$.

(H$_3$) set $Y_i(x) := \gamma_{p_i}p_i W_i(x) + Z_i(x)$, where $Z_i(x) := \nabla W_i(x)\cdot x$ and $Z_i \in L^s(\mathbb{R}^N)$ for some $s \in [N/2, \infty]$, there exists $\rho_i \in [0, \gamma_{p_i}p_i - 2)$ such that $|Y_{i,+}|_{N/2} \leqslant \rho_i S$ for any $u \in E_i$, where $Y_{i,+} = \max\{Y_i, 0\}$.

An example satisfying the conditions (H$_1$)–H$_3$) is $V_i(x) = -\frac{b}{|x|^c+1}$, $x \in \mathbb{R}^N$ with constant $c > 2$ and suitable small constant b. Obviously, $V = 0$ also satisfies the conditions (H$_1$)–(H$_3$). Hence, the following theorem includes the autonomous case $V = 0$.

Normalized solutions of (1.1) can be found as critical points of the C^1 functional

$$\begin{aligned} I(u,v) = &\frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + |\nabla v|^2 + V_1u^2 + V_2v^2) - \frac{1}{2^*}\int_{\mathbb{R}^N}(|u|^{2^*} + |v|^{2^*}) \\ &- \frac{1}{p_1}\int_{\mathbb{R}^N}|u|^{p_1} - \frac{1}{p_2}\int_{\mathbb{R}^N}|v|^{p_2} - \beta\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2},\ (u,v) \in E_1\times E_2, \end{aligned}$$

on

$$S_{a_1}\times S_{a_2} := \left\{(u,v) \in E_1\times E_2 : \int_{\mathbb{R}^N} u^2 = a_1,\ \int_{\mathbb{R}^N} v^2 = a_2\right\},$$

with Lagrange multipliers $\lambda_1, \lambda_2 \in \mathbb{R}$. Here

$$E_i := \left\{u \in H_r^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} V_i u^2 < \infty\right\}, \qquad i = 1,2$$

and $H_r^1(\mathbb{R}^N)$ is the usual radial Sobolev space. The norm of E_i is defined by

$$\|u\|_i = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + V_iu^2 + u^2)\right)^{1/2}, \qquad u \in E_i,\ i = 1,2,$$

which is equivalent to the usual norm $\|u\|_{H^1(\mathbb{R}^N)} = \left(\int_{\mathbb{R}^N}(|\nabla u|^2 + u^2)\right)^{1/2}$ due to the condition (H_1). The solution $(u, v) \in S_{a_1} \times S_{a_2}$ is called a positive radial normalized solution of (1.1) if $u > 0$ and $v > 0$.

Now we state our main results.

Theorem 1.1. *Let $N = 3, 4$, $r_1, r_2 > 1$, $p_1, p_2, r_1 + r_2 \in (2 + 4/N, 2^*)$, $\beta > 0$ and (H_1)–(H_3) hold. Then there exists $\beta_* > 0$ such that the system (1.1) has a positive radial normalized solution $(u, v) \in S_{a_1} \times S_{a_2}$ with $\lambda_1, \lambda_2 > 0$ when $\beta \geqslant \beta_*$.*

Remark 1.2.

(i) This seems to be the first study to consider the existence of normalized solutions to Schrödinger system with critical exponent and weakly attractive potentials;

(ii) To simplify, note that $r := r_1 + r_2$. In the proof of Theorem 1.1, we discuss three cases, that is, $p_1 = \min\{p_1, p_2, r\}$, $p_2 = \min\{p_1, p_2, r\}$ and $r = \min\{p_1, p_2, r\}$.

Since the scalar setting will of course be relevant when dealing with system, it is necessary to study firstly some related results of scalar equations. When $\beta = 0$, (1.1) turns to be the scalar equations

$$-\Delta u + (V_i + \lambda_i)u = |u|^{2^*-2}u + |u|^{p_i-2}u \quad \text{in } \mathbb{R}^N,\ i = 1, 2. \tag{1.6}$$

Normalized solutions of (1.6) can be found as critical points of the C^1 functional

$$J_{V_i}(u) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla u|^2 + V_i u^2) - \frac{1}{2^*}\int_{\mathbb{R}^N}|u|^{2^*} - \frac{1}{p_i}\int_{\mathbb{R}^N}|u|^{p_i}, \quad u \in E_i,$$

on

$$S_{a_i} := \left\{u \in E_i : \int_{\mathbb{R}^N} u^2 = a_i\right\}.$$

Moreover, u_{a_i} is a ground state normalized solution to (1.6) on S_{a_i} if $J_{V_i}|'_{S_{a_i}}(u_{a_i}) = 0$ and

$$J_{V_i}(u_{a_i}) = \inf\{J_{V_i}(v) : v \in S_{a_i}, J_{V_i}|'_{S_{a_i}}(v) = 0\}.$$

Here comes our second main result.

Theorem 1.3. *Let $N = 3, 4$, $i = 1$ or $i = 2$, $p_i \in (2 + 4/N, 2^*)$ and (H_1)–(H_3) hold. Then the equation (1.6) has a positive radial ground state normalized solution $u_{a_i} \in S_{a_i}$ with $\lambda_i > 0$.*

Remark 1.4. This is probably the first result to consider the existence of normalized solutions to Schrödinger equation with critical exponent and weakly attractive potentials.

To obtain normalized solution of (1.6), as [12,21,23], we introduce the Pohozaev set

$$\mathcal{P}_{a_i,V_i} = \{u \in S_{a_i} : P_{V_i}(u) = 0\},$$

where

$$P_{V_i}(u) = \int_{\mathbb{R}^N}|\nabla u|^2 - \int_{\mathbb{R}^N} W_i u^2 - \int_{\mathbb{R}^N}|u|^{2^*} - \gamma_{p_i}\int_{\mathbb{R}^N}|u|^{p_i}, \qquad u \in E_i.$$

As a matter of fact, the condition $P_{V_i}(u) = 0$ obtained in Lemma 2.1 is the linear combination of Nehari and Pohozaev identities. Furthermore, J is bounded from below on $\mathcal{P}_{a_i,V_i}$, see Lemma 2.5 (iv). Hence, for $a_i > 0$, define

$$m_{V_i}(a_i) := \inf_{\mathcal{P}_{a_i,V_i}} J_{V_i} \tag{1.7}$$

and consider the reachability of $m_{V_i}(a_i)$. Inspired by [12, 33], we need use the comparison arguments between $m_{V_i}(a_i)$and that to the limit equation

$$-\Delta u + \lambda_i u = |u|^{2^*-2}u + |u|^{p_i-2}u \quad \text{in } \mathbb{R}^N. \tag{1.8}$$

The analogue corresponding (1.8) are denoted by J_∞, P_∞, $\mathcal{P}_{a_i,\infty}$ and $m_\infty(a_i)$. Soave in [31, Theorem 1.1 and Section 6] obtained that $m_\infty(a_i) \in (0, S^{N/2}/N)$ can be reached by u_{a_i} when $N = 3, 4, a_i > 0$ and $p_i \in (2 + 4/N, 2^*)$, furthermore, u_{a_i} is a real-valued, positive and radial.

The Gagliardo–Nirenberg inequality is the key point to study the above problems variationally. For $q \in [1, \infty)$, $|u|_q = \left(\int_{\mathbb{R}^N} |u|^q\right)^{1/q}$ stands for the norm in $L^q(\mathbb{R}^N)$.

Proposition 1.5. *Let $N \geqslant 3$ and $u \in H^1(\mathbb{R}^N)$. Then there exists a constant $C(N, q) > 0$ such that, for any $q \in [2, 2^*]$, we have*

$$|u|_q \leqslant C(N, q)|\nabla u|_2^\theta |u|_2^{1-\theta},$$

where $\theta \in [0, 1]$ satisfies $1/q = \theta/2^* + (1-\theta)/2$. In particular, when $q = 2^*$, $C(N, q) = S^{-1/2}$.

In this article, B_R denotes an open ball at 0 with radius of $R > 0$ and $C, C_1, C_2, \ldots$ denote various positive constants whose exact values are irrelevant.

The paper is organized as follows. In Sections 2 and 4, we give some preliminary results about the scalar equation (1.6) and the system (1.1), respectively. The proofs of Theorems 1.3 and 1.1 are given in Sections 3 and 5, respectively.

2 Preliminaries about the scalar equation

In this section, without loss of generality, we may assume that $i = 1$ and the potential V_1 satisfies (H_1)–(H_3).

Lemma 2.1. *If $u \in E_1$ is a weak solution to (1.6), then $P_{V_1}(u) = 0$.*

Proof. Let $u \in E_1$ be a weak solution of (1.6). We see that the following Nehari and Pohozaev identities hold

$$|\nabla u|_2^2 + \int_{\mathbb{R}^N} (V_1 + \lambda_1)u^2 - |u|_{2^*}^{2^*} - |u|_{p_1}^{p_1} = 0, \tag{2.1}$$

$$\frac{N-2}{2}|\nabla u|_2^2 + \frac{N}{2}\int_{\mathbb{R}^N} (V_1 + \lambda_1)u^2 + \int_{\mathbb{R}^N} W_1 u^2 - \frac{N}{2^*}|u|_{2^*}^{2^*} - \frac{N}{p_1}|u|_{p_1}^{p_1} = 0. \tag{2.2}$$

Combining (2.1) and (2.2), we obtain $P_{V_1}(u) = 0$. □

Lemma 2.2. *Assume that $N = 3, 4$ and $u \in E_1$ is a nonnegative solution of (1.6). Then, $u \geqslant 0$ and $u \neq 0$ implies that $\lambda_1 > 0$.*

Proof. Since $u \neq 0$ satisfies

$$-\Delta u = -(V_1 + \lambda_1)u + |u|^{2^*-2}u + |u|^{p_1-2}u \quad \text{in } \mathbb{R}^N,$$

it follows from $u \geqslant 0$ that the right hand side is nonnegative if $\lambda_1 \leqslant 0$, and by [19, Lemma A.2], we obtain $u = 0$, which contradicts to the assumption $u \neq 0$. Hence, $\lambda_1 > 0$. □

For $u \in E_1$ and $t \in \mathbb{R}$, we introduce the transformation $u^t(x) := e^{Nt/2}u(e^t x), x \in \mathbb{R}^N$, it is easy to check that $|u^t|_2 = |u|_2$. We fix $u \neq 0$ and consider the continuous real valued function $f_u : \mathbb{R} \to \mathbb{R}$ with

$$f_u(t) := J_{V_1}(u^t) = \frac{1}{2}e^{2t}|\nabla u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_1(e^{-t}x)u^2 - \frac{1}{2^*}e^{2^*t}|u|_{2^*}^{2^*} - \frac{1}{p_1}e^{\gamma_{p_1}p_1 t}|u|_{p_1}^{p_1},$$

and

$$P_{V_1}(u^t) = e^{2t}|\nabla u|_2^2 - \int_{\mathbb{R}^N} W_1(e^{-t}x)u^2 - e^{2^*t}|u|_{2^*}^{2^*} - \gamma_{p_1}e^{\gamma_{p_1}p_1 t}|u|_{p_1}^{p_1}.$$

By a simple calculation, we see that $P_{V_1}(u^t) = f_u'(t)$.

Lemma 2.3. *Fix $u \in S_{a_1}$. Then $J_{V_1}(u^t) \to 0^+$ as $t \to -\infty$ and $J_{V_1}(u^t) \to -\infty$ as $t \to \infty$.*

Proof. By the condition (H_1), we have

$$J_{V_1}(u^t) \geqslant \frac{1-\tau_1}{2}e^{2t}|\nabla u|_2^2 - \frac{1}{2^*}e^{2^*t}|u|_{2^*}^{2^*} - \frac{1}{p_1}e^{\gamma_{p_1}p_1 t}|u|_{p_1}^{p_1}$$

and

$$J_{V_1}(u^t) \leqslant \frac{1}{2}e^{2t}|\nabla u|_2^2 - \frac{1}{2^*}e^{2^*t}|u|_{2^*}^{2^*} - \frac{1}{p_1}e^{\gamma_{p_1}p_1 t}|u|_{p_1}^{p_1},$$

it is easy to see that the conclusion holds. □

Lemma 2.4. *Let $D_k := \{u \in S_{a_1} : |\nabla u|_2^2 \leqslant k\}$. Then there exists $k_0 > 0$ such that $J_{V_1}(u) > 0$ and $P_{V_1}(u) > 0$ when $u \in D_{k_0}$.*

Proof. By the conditions (H_1) and (H_2), (1.5) and the Gagliardo–Nirenberg inequalities, we have

$$J_{V_1}(u) \geqslant \frac{1-\tau_1}{2}|\nabla u|_2^2 - \frac{1}{2^*}S^{-2^*/2}|\nabla u|_2^{2^*} - \frac{1}{p_1}C(N,p_1)a^{(1-\gamma_{p_1})p_1/2}|\nabla u|_2^{\gamma_{p_1}p_1}$$

and

$$P_{V_1}(u) \geqslant (1-\tau_2)|\nabla u|_2^2 - S^{-2^*/2}|\nabla u|_2^{2^*} - \gamma_{p_1}C(N,p_1)a^{(1-\gamma_{p_1})p_1/2}|\nabla u|_2^{\gamma_{p_1}p_1},$$

it is easy to see that there exists $k_0 > 0$ small enough such that $J_{V_1}(u) > 0$ and $P_{V_1}(u) > 0$ for all $u \in D_{k_0}$. □

Hence, we can define

$$\bar{m}_{V_1}(a_1) := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J_{V_1}(\gamma(t)) > 0,$$

where $\Gamma = \{\gamma \in C([0,1], S_{a_1}) : \gamma(0) \in D_{k_0}, J_{V_1}(\gamma(1)) \leqslant 0\}$, k_0 is given by Lemma 2.4.

Consider the decomposition of $\mathcal{P}_{a_1,V_1} = \mathcal{P}^+_{a_1,V_1} \cup \mathcal{P}^0_{a_1 V_1} \cup \mathcal{P}^-_{a_1,V_1}$ and

$$\mathcal{P}^+_{a_1,V_1} := \{u \in \mathcal{P}_{a_1,V_1} : f_u''(0) > 0\},$$
$$\mathcal{P}^0_{a_1,V_1} := \{u \in \mathcal{P}_{a_1,V_1} : f_u''(0) = 0\},$$
$$\mathcal{P}^-_{a_1,V_1} := \{u \in \mathcal{P}_{a_1,V_1} : f_u''(0) < 0\}.$$

Lemma 2.5.

(i) $\mathcal{P}_{a_1,V_1} = \mathcal{P}^-_{a_1,V_1}$;

(ii) for any $u \in S_{a_1}$, there exists a unique $t_u := t(u) \in \mathbb{R}$ such that $u^{t_u} \in \mathcal{P}_{a_1,V_1}$, moreover, $J_{V_1}(u^{t_u}) = \max_{t \in \mathbb{R}} J_{V_1}(u^t)$;

(iii) J_{V_1} *is coercive on* $\mathcal{P}_{a_1,V_1}$*, that is,* $J_{V_1}(u) \to \infty$ *for any* $u \in \mathcal{P}_{a_1,V_1}$ *with* $\|u\| \to \infty$*;*

(iv) there exist constants $\delta, \sigma > 0$ *such that* $|\nabla u|_2 \geqslant \delta$ *and* $J_{V_1}(u) \geqslant \sigma$ *for all* $u \in \mathcal{P}_{a_1,V_1}$*.*

Proof. (i) Using $P_{V_1}(u) = 0$ and the conditions (H_2) and (H_3), we have

$$\begin{aligned} f_u''(0) &= 2|\nabla u|_2^2 + \int_{\mathbb{R}^N} Z_1 u^2 - 2^*|u|_{2^*}^{2^*} - \gamma_{p_1}^2 p_1 |u|_{p_1}^{p_1} \\ &= \int_{\mathbb{R}^N} Y_1 u^2 + (2 - \gamma_{p_1} p_1)|\nabla u|_2^2 + (\gamma_{p_1} p_1 - 2^*)|u|_{2^*}^{2^*} \\ &\leqslant (\rho_1 + 2 - \gamma_{p_1} p_1)|\nabla u|_2^2 < 0. \end{aligned}$$

Hence, $\mathcal{P}_{a_1,V_1}^+ = \mathcal{P}_{a_1,V_1}^0 = \emptyset$, which implies that $\mathcal{P}_{a_1,V_1} = \mathcal{P}_{a_1,V_1}^-$.

(ii) By Lemmas 2.3 and 2.4, we know that $\max_{t\in\mathbb{R}} J_{V_1}(u^t)$ is achieved at $t_u \in \mathbb{R}$ and $J_{V_1}(u^{t_u}) > 0$. In view of $\partial_t J_{V_1}(u^t) = P_{V_1}(u^t)$, we see $P_{V_1}(u^{t_u}) = 0$. Hence, $u^{t_u} \in \mathcal{P}_{a_1,V_1}$. Suppose that there exists another $t_u' \in \mathbb{R}$ such that $u^{t_u'} \in \mathcal{P}_{a_1,V_1}$. Then by Lemma 2.5 (i), we see that t_u and t_u' are strict local maximum points of $f_u(t) := J(u^t)$. Without loss of generality, we assume that $t_u < t_u'$. Hence, there exists $t_u'' \in (t_u, t_u')$ such that $f_u(t_u'') = \min_{t\in[t_u,t_u']} f_u(t)$, and we have $f_u'(t_u'') = 0$ and $f_u''(t_u'') \geqslant 0$. Thus, $u^{t_u''} \in \mathcal{P}_{a_1,V_1}^+ \cup \mathcal{P}_{a_1,V_1}^0$, which contradict to (i).

(iii) For $u \in \mathcal{P}_{a_1,V_1}$, by the conditions ($H_1$) and ($H_2$), we have

$$\begin{aligned} J_{V_1}(u) &= J_{V_1}(u) - \frac{1}{\gamma_{p_1} p_1} P_{V_1}(u) \\ &\geqslant \left(\frac{1}{2} - \frac{1}{\gamma_{p_1} p_1}\right)|\nabla u|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_1 u^2 + \frac{1}{\gamma_{p_1} p_1}\int_{\mathbb{R}^N} W_1 u^2 \\ &\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_{p_1} p_1}\right)|\nabla u|_2^2. \end{aligned} \tag{2.3}$$

Hence, J_{V_1} is coercive on $\mathcal{P}_{a_1,V_1}$.

(iv) If

$$|\nabla u|_2 < \min\left\{\left(\frac{1-\theta_1}{3S^{2^*/2}}\right)^{1/(2^*-2)}, \left(\frac{1-\theta_1}{3\gamma_{p_1} C(N,p_1) a^{(1-\gamma_{p_1})p_1/2}}\right)^{1/(\gamma_{p_1} p_1 - 2)}\right\},$$

using the condition (H_2) and Proposition 1.5, we have

$$\begin{aligned} \Psi(u) &:= \int_{\mathbb{R}^N} W_1 u^2 + |u|_{2^*}^{2^*} + \gamma_{p_1}|u|_{p_1}^{p_1} \\ &\leqslant \left(\theta_1 + S^{-2^*/2}|\nabla u|_2^{2^*-2} + \gamma_{p_1} C(N,p_1) a^{(1-\gamma_{p_1})p_1/2}|\nabla u|_2^{\gamma_{p_1} p_1 - 2}\right)|\nabla u|_2^2 \\ &\leqslant \frac{2+\theta_1}{3}|\nabla u|_2^2. \end{aligned}$$

Now, we prove that there exists $\delta > 0$ such that $|\nabla u|_2 \geqslant \delta$ for all $u \in \mathcal{P}_{a_1,V_1}$. On the contrary, there exists $\{u_n\} \subset \mathcal{P}_{a_1,V_1}$ such that $|\nabla u_n|_2 \to 0$, then, for n large enough, we have

$$0 = P_{V_1}(u_n) = |\nabla u_n|_2^2 - \Psi(u_n) \geqslant \frac{1-\theta_1}{3}|\nabla u_n|_2^2 > 0,$$

which is a contradiction. In view of (2.3), we see that there exists $\sigma > 0$ such that $J_{V_1}(u) \geqslant \sigma$ for all $u \in \mathcal{P}_{a_1,V_1}$. □

Lemma 2.6. *$m_{V_1}(a_1) = \bar{m}_{V_1}(a_1) > 0$. Moreover, there exist $\{v_n\} \subset S_{a_1}$ such that, as $n \to \infty$,*

$$J_{V_1}(v_n) \to m_{V_1}(a_1), \qquad J_{V_1}|'_{S_{a_1}}(v_n) \to 0, \qquad P_{V_1}(v_n) \to 0, \tag{2.4}$$

and $v_n^- \to 0$ a.e. in $\mathbb{R}^N$.

Proof. For any $v \in \mathcal{P}_{a_1,V_1}$, there exist $t_1, t_2 \in \mathbb{R}$ such that $v^{t_1} \in D_{k_0}$ and $J_{V_1}(v^{t_2}) \leqslant 0$. Set

$$\gamma_0(t) := v^{(1-t)t_1+tt_2}, \qquad t \in [0,1],$$

then $\gamma_0 \in \Gamma$ and $\max_{t\in[0,1]} J_{V_1}(\gamma_0(t)) = J_{V_1}(v)$ by Lemma 2.5 (ii), which implies $\bar{m}_{V_1}(a_1) \leqslant m_{V_1}(a_1)$. Now, we prove that any path γ in Γ crosses $\mathcal{P}_{a_1,V_1}$. Using Lemma 2.4, for any $\gamma \in \Gamma$, $P_{V_1}(\gamma(0)) > 0$. On the other hand, by (2.3), $P_{V_1}(\gamma(1)) \leqslant \gamma_{p_1} p_1 J_{V_1}(\gamma(1)) \leqslant 0$. Therefore, there exists $t_0 \in (0,1]$ such that $P_{V_1}(\gamma(t_0)) = 0$, which implies $\bar{m}_{V_1}(a_1) \geqslant m_{V_1}(a_1)$. Thus, $\bar{m}_{V_1}(a_1) = m_{V_1}(a_1)$. In view of Lemma 2.5 (iv), we see that $\bar{m}_{V_1}(a_1) = m_{V_1}(a_1) > 0$.

Now, we recall the stretched functional introduced first in [21]:

$$\tilde{J}_{V_1} : E_1 \times \mathbb{R} \to \mathbb{R}, \qquad (u,t) \mapsto J_{V_1}(u^t)$$

and define

$$\tilde{\Gamma} = \{g \in C([0,1], S_{a_1} \times \mathbb{R}) : g(0) \in D_{k_0} \times \{0\}, g(1) \in J^0 \times \{0\}\},$$

where k is given by Lemma 2.4 and $J^0 := \{u \in E_1 : J_{V_1}(u) \leqslant 0\}$. If $\gamma \in \Gamma$, then $g := (\gamma, 0) \in \tilde{\Gamma}$ and $\tilde{J}_{V_1}(g(t)) = J_{V_1}(\gamma(t)), t \in [0,1]$. And if $g = (g_1, g_2) \in \tilde{\Gamma}$, then $\gamma := g_1^{g_2} \in \Gamma$ and $J_{V_1}(\gamma(t)) = \tilde{J}_{V_1}(g(t)), t \in [0,1]$. Hence, we have

$$\inf_{g\in\tilde{\Gamma}} \max_{t\in[0,1]} \tilde{J}_{V_1}(g(t)) = \bar{m}_{V_1}(a_1) = m_{V_1}(a_1).$$

Thus, using the Ekeland variational principle as in [21, Lemma 2.3], it follows that there exists a sequence $\{(u_n, t_n)\} \subset S_{a_1} \times \mathbb{R}$ such that, as $n \to \infty$,

$$\tilde{J}_{V_1}(u_n, t_n) \to m_{V_1}(a_1), \qquad \tilde{J}_{V_1}|'_{S_{a_1}\times\mathbb{R}}(u_n, t_n) \to 0, \qquad t_n \to 0.$$

Note $v_n := u_n^{t_n}$. For any $w \in \{z \in H^1(\mathbb{R}^N) : \int_{\mathbb{R}^N} v_n z = 0\}$, setting $w_n := w^{-t_n}$, then $(w_n, 0) \in \{(z,t) \in H^1(\mathbb{R}^N) \times \mathbb{R} : \int_{\mathbb{R}^N} u_n z = 0\}$. Hence,

$$J_{V_1}(v_n) \to m_{V_1}(a_1), \qquad \langle J_{V_1}|'_{S_{a_1}}(v_n), w\rangle = \langle \tilde{J}_{V_1}|'_{S_a\times\mathbb{R}}(u_n, t_n), (w_n, 0)\rangle.$$

and by$\|w_n\| \leqslant 2\|w\|$ for n enough large due to $t_n \to 0$, we have $J_{V_1}|'_{S_{a_1}}(v_n) \to 0$. Moreover, by $\langle \tilde{J}_{V_1}|'_{S_{a_1}\times\mathbb{R}}(u_n, t_n), (0,1)\rangle \to 0$, we see $P_{V_1}(v_n) \to 0$. Hence, (2.4) holds. Since $J_{V_1}(v_n) = J_{V_1}(|v_n|)$, $v_n^- \to 0$ a.e. in $\mathbb{R}^N$. □

3 Proof of Theorem 1.3

In this section, the potential $V_1 \neq 0$ and V_1 satisfies (H_1)–(H_3). When $V_1 = 0$, we denote $J_{V_1}, P_{V_1}, \mathcal{P}_{a_1,V_1}$, and $m_{V_1}(a_1)$ by J_∞, P_∞, $\mathcal{P}_{a_1,\infty}$, and $m_\infty(a_1)$, respectively.

Before proving Theorem 1.1, we first consider the monotonicity of $m_\infty(\cdot)$.

Lemma 3.1. *The map $m_\infty(\cdot)$ is decreasing on $\mathbb{R}_+ \setminus \{0\}$.*

Proof. Fix $a > a_1 > 0$. By [31, Theorem 1.1 and Section 6], there exists $u \in \mathcal{P}_{a_1,\infty}$ such that $J_\infty(u) = m_\infty(a_1)$. Set $v := (a_1/a)^{(N-2)/4}u((a_1/a)^{1/2}\cdot)$. Then $|v|_2^2 = a$, and by Lemma 2.5 (ii), there exists $t_v \in \mathbb{R}$ such that $v^{t_v} \in \mathcal{P}_{a,\infty}$. Moreover,

$$|\nabla v^{t_v}|_2^2 = e^{2t_v}|\nabla v|_2^2 = e^{2t_v}|\nabla u|_2^2 = |\nabla u^{t_v}|_2^2,$$

$$|v^{t_v}|_{2^*}^{2^*} = e^{2^*t_v}|v|_{2^*}^{2^*} = e^{2^*t_v}|u|_{2^*}^{2^*} = |u^{t_v}|_{2^*}^{2^*},$$

$$|v^{t_v}|_{p_1}^{p_1} = e^{\gamma_{p_1}p_1t_v}|v|_{p_1}^{p_1} = e^{\gamma_{p_1}p_1t_v}(a_1/a)^{p_1(\gamma_{p_1}-1)/2}|u|_{p_1}^{p_1} = (a_1/a)^{p_1(\gamma_{p_1}-1)/2}|u^{t_v}|_{p_1}^{p_1}.$$

Let

$$\Psi(u,t_v) := \frac{1}{p_1}e^{\gamma_{p_1}p_1t_v}\left(1-(a_1/a)^{p_1(\gamma_{p_1}-1)/2}\right)|u|_{p_1}^{p_1} < 0.$$

Then, we can deduce that

$$m_\infty(a) \leqslant J_\infty(v^{t_v}) = J_\infty(u^{t_v}) + \Psi(u,t_v) < J_\infty(u) = m_\infty(a_1),$$

which indicate $m_\infty(\cdot)$ is decreasing on $\mathbb{R}_+ \setminus \{0\}$. □

Now, we present a key estimate for $m_{V_1}(a_1)$.

Lemma 3.2. *One has that $m_{V_1}(a_1) < m_\infty(a_1)$.*

Proof. By [31, Theorem 1.1 and Section 6], there exists a positive radial $v_{a_1} \in \mathcal{P}_{a_1,\infty}$ such that $J_\infty(v_{a_1}) = m_\infty(a_1)$. Using Lemma 2.5 (ii), there exists $t_{v_{a_1}} := t(v_{a_1}) > 0$ such that $v_{a_1}^{t_{v_{a_1}}} \in \mathcal{P}_{a_1,V_1}$. Since $V_1 \leqslant 0$ and $V_1 \neq 0$, it is easy to check that

$$m_{V_1}(a_1) \leqslant J(v_{a_1}^{t_{v_{a_1}}}) < J_\infty(v_{a_1}^{t_{v_{a_1}}}) \leqslant \max_{t>0} J_\infty(v_{a_1}^t) = J_\infty(v_{a_1}) = m_\infty(a_1). \qquad □$$

Proof of Theorem 1.3. In view of Lemma 2.6, we can obtain a sequence $\{u_n\} \subset S_{a_1}$ satisfying

$$J_{V_1}(u_n) \to m(a_1), \qquad J_{V_1}|_{S_{a_1}}'(u_n) \to 0, \qquad P_{V_1}(u_n) \to 0, \qquad n \to \infty,$$

and $u_n^- \to 0$ a.e. in $\mathbb{R}^N$, and by Lemma 2.5 (iii), it is easy to see that $\{u_n\}$ is bounded in E_1. Up to a subsequence, we assume that $u_n \rightharpoonup u_{a_1}$ in E_1, $u_n \to u_{a_1}$ in $L^s(\mathbb{R}^N)$, $s \in (2, 2^*)$, a.e. in $\mathbb{R}^N$ and $u_{a_1} \geqslant 0$ a.e. in $\mathbb{R}^N$. Moreover, since $J_{V_1}|_{S_{a_1}}'(u_n) \to 0$, by [32, Proposition 5.12], there exists $\lambda_n \in \mathbb{R}$ such that, for any $\varphi \in H^1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N}[\nabla u_n \cdot \nabla\varphi + (V_1+\lambda_n)u_n\varphi - |u_n|^{2^*-2}u_n\varphi - |u_n|^{p_1-2}u_n\varphi] = o_n(1)\|\varphi\|. \tag{3.1}$$

Choosing $\varphi = u_n$, we deduce that $\{\lambda_n\}$ is bounded in $\mathbb{R}$, and hence up to a subsequence, $\lambda_n \to \lambda_1 \in \mathbb{R}$. Now, we prove $u_{a_1} \neq 0$. If not, then $u_n \rightharpoonup 0$ in $H_r^1(\mathbb{R}^N)$ and $u_n \to 0$ in $L^s(\mathbb{R}^N)$, $s \in (2, 2^*)$. By Lemma 2.5 (ii), there exists $t_n := t(u_n) \in \mathbb{R}$ such that $P_\infty(u_n^{t_n}) = 0$ and $u_n^{t_n} \in \mathcal{P}_{a_1,\infty}$. By $P_{V_1}(u_n) \to 0$ and $J_{V_1}(u_n) \to m(a_1)$, we see that there exists $\delta > 0$ such that $|\nabla u_n|_2 \geqslant \delta$ for sufficient large n. Using $P_{V_1}(u_n) \to 0$ again, we can assume that $|u_n|_{2^*}^{2^*} \geqslant \delta^2$ for sufficient large n. In view of Lemma 2.5 (iv), we see that $\liminf_{n\to\infty} e^{t_n} > 0$. If $t_n \to \infty$, then,

$$\begin{aligned} 0 &\leqslant e^{-2t_n}J_\infty(u_n^{t_n}) \\ &= \frac{1}{2}|\nabla u_n|_2^2 - \frac{1}{2^*}e^{(2^*-2)t_n}|u_n|_{2^*}^{2^*} - \frac{1}{p_1}e^{(\gamma_{p_1}p_1-2)t_n}|u_n|_{p_1}^{p_1} \\ &\leqslant \frac{1}{2}C - \frac{1}{2^*}e^{(2^*-2)t_n}\delta^2 \to -\infty, \end{aligned} \tag{3.2}$$

which is a contradiction. Hence, $\{t_n\}$ is bounded in $\mathbb{R}$ and we can assume that $t_n \to t_* \in (-\infty, \infty)$. Since $u_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} W_1(x) = 0$, we can obtain that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W_1 u_n^2 = 0,$$

and by $P_{V_1}(u_n) \to 0$, we have

$$\begin{aligned} 0 &= P_\infty(u_n^{t_n}) \\ &= e^{2t_n} \int_{\mathbb{R}^N} W_1 u_n^2 + (e^{2t_n} - e^{2^* t_n})|u_n|_{2^*}^{2^*} + \gamma_{p_1}(e^{2t_n} - e^{\gamma_{p_1} p_1 t_n})|u_n|_{p_1}^{p_1} + o_n(1) \\ &= (e^{2t_n} - e^{2^* t_n})|u_n|_{2^*}^{2^*} + o_n(1) \end{aligned} \tag{3.3}$$

which implies $t_* = 0$. Therefore,

$$m_\infty(a_1) \leqslant J_\infty(u_n^{t_n}) = J_{V_1}(u_n) + o_n(1) = m_{V_1}(a_1) + o_n(1),$$

that is, $m_\infty(a_1) \leqslant m_{V_1}(a_1)$, this is impossible, and thus $u_{a_1} \neq 0$. Moreover, passing to the limit in (3.1) by the weak convergence, we infer that u_{a_1} solves (1.6) with $\lambda = \lambda_1$, and by Lemma 2.2, we see that $\lambda_1 > 0$. Hence, $\langle J'_{V_1}(u_{a_1}), u_{a_1}\rangle + \lambda_1 |u_{a_1}|_2^2 = 0$ and $P_{V_1}(u_{a_1}) = 0$, and by (2.3), we have $J_{V_1}(u_{a_1}) > 0$.

Set $a := |u_{a_1}|_2^2$. We claim that $a = a_1$. If not, then $b := a_1 - a \in (0, a_1)$ due to $a \leqslant a_1$. Let $v_n := u_n - u_{a_1}$, then $v_n \rightharpoonup 0$ in E_1 and $v_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, and by $\lim_{|x|\to\infty} V_1(x) = \lim_{|x|\to\infty} W_1(x) = 0$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V_1 v_n^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} W_1 v_n^2 = 0.$$

From the Brezis–Lieb lemma and (3.1), one have $|v_n|_2^2 = b + o_n(1)$ and

$$J_\infty(v_n) = J_{V_1}(v_n) + o_n(1) = J_{V_1}(u_n) - J_{V_1}(u_{a_1}) + o_n(1) = m_{V_1}(a_1) - J_{V_1}(u_{a_1}) + o_n(1), \tag{3.4}$$

$$\begin{aligned} \langle J'_\infty(v_n), v_n\rangle &= \langle J'_{V_1}(v_n), v_n\rangle + o_n(1) \\ &= \langle J'_{V_1}(u_n), u_n\rangle - \langle J'_{V_1}(u_{a_1}), u_{a_1}\rangle + o_n(1) \\ &= -\lambda_1 a_1 - \langle J'_{V_1}(u_{a_1}), u_{a_1}\rangle + o_n(1) \\ &= -\lambda_1 a_1 + \lambda_1 a + o_n(1) = -\lambda_1 b + o_n(1), \end{aligned} \tag{3.5}$$

$$P_\infty(v_n) = P_{V_1}(v_n) + o_n(1) = P_{V_1}(u_n) - P_{V_1}(u_{a_1}) + o_n(1) = o_n(1). \tag{3.6}$$

We claim that

$$\liminf_{n\to\infty} |\nabla v_n|_2^2 > 0. \tag{3.7}$$

As a matter of fact, if not, then we may assume that $v_n \to 0$ in $D^{1,2}(\mathbb{R}^N)$ and hence in $L^{2^*}(\mathbb{R}^N)$ by the Sobolev inequality. We also have $|v_n|_{p_1} \to 0$ by the Gagliardo–Nirenberg inequality. Therefore, $\langle J'_\infty(v_n), v_n\rangle \to 0$, and by (3.5), we have $b = 0$, this is a contradiction. Thus, (3.7) holds. Using Lemma 2.5 (ii) again, there exists $t_n := t(v_n) \in \mathbb{R}$ such that $P_\infty(v_n^{t_n}) = 0$ and $v_n^{t_n} \in \mathcal{P}_{|v_n|_2^2, \infty}$. By Lemma 2.5 (iv) and (3.7), it is easy to see that $\liminf_{n\to\infty} e^{t_n} > 0$. Since (3.6) and $P_\infty(v_n^{t_n}) = 0$, by a similar proof as (3.2) and (3.3), we know that $\{t_n\}$ is bounded and $t_n \to 0$. Hence, by (3.4), we have

$$m_\infty(|v_n|_2^2) \leqslant J_\infty(v_n^{t_n}) = J_\infty(v_n) + o_n(1) = m_{V_1}(a_1) - J_{V_1}(u_{a_1}) + o_n(1).$$

Noting that $m_\infty(\cdot)$ is decreasing in $\mathbb{R}_+ \setminus \{0\}$ by Lemma 3.1, we have, for n large enough,

$$m_\infty(a_1) < m_\infty(|v_n|_2^2) \leqslant m_{V_1}(a_1) - J_{V_1}(u_a) + o_n(1) < m_\infty(a_1) - J_{V_1}(u_{a_1}) + o_n(1),$$

which implies that $J_{V_1}(u_{a_1}) \leqslant 0$ contradicting to $J_{V_1}(u_{a_1}) > 0$. Hence, $|u_{a_1}|_2^2 = a = a_1$. Using $u_n \to u_{a_1}$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ and $\lim_{|x|\to\infty} V_1(x) = \lim_{|x|\to\infty} W_1(x) = 0$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V_1 u_n^2 = \int_{\mathbb{R}^N} V_1 u_{a_1}^2, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} W_1 u_n^2 = \int_{\mathbb{R}^N} W_1 u_{a_1}^2,$$

and by $P_{V_1}(u_{a_1}) = 0$, we deduce that

$$\begin{aligned}
&J_{V_1}(u_{a_1})\\
&\quad = J_{V_1}(u_{a_1}) - \frac{1}{\gamma_{p_1} p_1} P_{V_1}(u_{a_1})\\
&\quad = \left(\frac{1}{2} - \frac{1}{\gamma_{p_1} p_1}\right) |\nabla u_{a_1}|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_1 u_{a_1}^2 + \frac{1}{\gamma_{p_1} p_1}\int_{\mathbb{R}^N} W_1 u_{a_1}^2 + \left(\frac{1}{\gamma_{p_1} p_1} - \frac{1}{2^*}\right) |u_{a_1}|_{2^*}^{2^*}\\
&\quad \leqslant \liminf_{n\to\infty} \left[\left(\frac{1}{2} - \frac{1}{\gamma_{p_1} p_1}\right) |\nabla u_n|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} V_1 u_n^2 + \frac{1}{\gamma_{p_1} p_1}\int_{\mathbb{R}^N} W_1 u_n^2 + \left(\frac{1}{\gamma_{p_1} p_1} - \frac{1}{2^*}\right) |u_n|_{2^*}^{2^*}\right]\\
&\quad = \lim_{n\to\infty} J_{V_1}(u_n) = m(a_1),
\end{aligned}$$

in view of $m_{V_1}(a_1) \leqslant J_{V_1}(u_{a_1})$, consequently, $J_{V_1}(u_{a_1}) = m_{V_1}(a_1)$. Using the strong maximum principle [16, Theorem 8.19], we see that $u_{a_1} > 0$. Therefore, u_{a_1} is a positive radial ground state normalized solution of (1.6). □

4 Preliminaries about the system

In this section, we may assume that the potentials V_i, $i = 1, 2$ satisfy (H_1)–(H_3).

First, we prove the following monotonicity result.

Lemma 4.1. *The map $m_{V_i}(\cdot)$ is nonincreasing on $\mathbb{R}_+ \setminus \{0\}$, where $m_{V_i}(a)$ is defined in (1.7), i=1,2.*

Proof. Here, we only consider the case $i = 1$. The case $i = 2$ is similar to the case $i = 1$. Fix $a > a_1 > 0$. By the definition of $m_{V_1}(a_1)$, there exists $u_0 \in \mathcal{P}_{a_1,V_1}$ such that

$$J_{V_1}(u_0) \leqslant m_{V_1}(a_1) + \varepsilon/3. \tag{4.1}$$

Let $\phi \in C_0^\infty(\mathbb{R}^N)$ be a radial cut off function such that $\phi(x) = 1$ when $x \in B_1$, $\phi(x) = 0$ when $x \in B_2^c$. Set $u_\delta(x) := \phi(\delta x) u_0(x)$, $x \in \mathbb{R}^N, \delta > 0$. Then $u_\delta \in E_1 \setminus \{0\}$ and $u_\delta \to u_0$ in E_1 as $\delta \to 0^+$. It follows from Lemma 2.5 (ii) that, for any $u \in S_{a_1}$, there exists a unique $t_u := t(u) \in \mathbb{R}$ such that $u^{t_u} \in \mathcal{P}_{a_1,V_1}$. Moreover, the map $u \mapsto t_u$ is C^1 by the Implicit Function Theorem. Hence, $t(u_\delta) \to t(u_0) = 0$ in $\mathbb{R}$ and $u_\delta^{t(u_\delta)} \to u_0$ in E_1 as $\delta \to 0^+$. Take a fixed $\delta > 0$ small enough such that

$$J_{V_1}(u_\delta^{t(u_\delta)}) \leqslant J_{V_1}(u_0) + \varepsilon/3 \tag{4.2}$$

and take $\zeta \in C_0^\infty(\mathbb{R}^N)$ such that $\mathrm{supp}(\zeta) \subset B_{1+4/\delta} \setminus B_{4/\delta}$. Set $\bar{\zeta} := (a - |u_\delta|_2^2)/|\zeta|_2^2 \zeta$. Then $|\bar{\zeta}|_2^2 = a - |u_\delta|_2^2$ and $\mathrm{supp}(\bar{\zeta}) \cap \mathrm{supp}(u_\delta) = \emptyset$. For every $s \leqslant 0$, let $w_s := u_\delta + \bar{\zeta}^s$, then $w_s \in S_a$ and there exists $t(w_s) \in \mathbb{R}$ such that $w_s^{t(w_s)} \in \mathcal{P}_{a,V_1}$. We claim that $t(w_s)$ is bounded from above as $s \to -\infty$. Suppose by contradiction that $t(w_s) \to \infty$ as $s \to -\infty$, and by $w_s \to u_\delta \neq 0$ a.e.

in $\mathbb{R}^N$, we deduce that $J_{V_1}(w_s^{t(w_s)}) \to -\infty$ as $s \to -\infty$. However, $J_{V_1}(w_s^{t(w_s)}) > 0$ by Lemma 2.5 (ii). This is absurd. Hence, the claim holds. Since $s + t(w_s) \to -\infty$ as $s \to -\infty$, we have, as $s \to -\infty$,

$$|\nabla \bar{\zeta}^{s+t(w_s)}|_2 \to 0, \qquad \int_{\mathbb{R}^N} V_1(e^{-(s+t(w_s))})\bar{\zeta}^2 \to 0,$$
$$|\bar{\zeta}^{s+t(w_s)}|_{2^*} \to 0, \qquad |\bar{\zeta}^{s+t(w_s)}|_{p_1} \to 0.$$

Consequently, $J_{V_1}(\bar{\zeta}^{s+t(w_s)}) \leqslant \varepsilon/3$ when $s < 0$ small enough. Thus, by (4.2) and (4.1),

$$\begin{aligned} m_{V_1}(a) &\leqslant J_{V_1}(w_s^{t(w_s)}) \\ &= J_{V_1}(u_\delta^{t(w_s)}) + J_{V_1}(\bar{\zeta}^{s+t(w_s)}) \\ &\leqslant J_{V_1}(u_\delta^{t(u_\delta)}) + J_{V_1}(\bar{\zeta}^{s+t(w_s)}) \\ &\leqslant J_{V_1}(u_0) + 2\varepsilon/3 \leqslant m_{V_1}(a_1) + \varepsilon, \end{aligned}$$

which implies $m_{V_1}(a) \leqslant m_{V_1}(a_1)$. Hence, the conclusion holds. □

Lemma 4.2. *Assume that $N = 3, 4$ and $(u, v) \in E_1 \times E_2$ is a nonnegative solution of (1.1). Then, $u \geqslant 0$ and $u \neq 0$ imply that $\lambda_1 > 0$; $v \geqslant 0$ and $v \neq 0$ imply that $\lambda_2 > 0$.*

Proof. Since $u \neq 0$ satisfies

$$-\Delta u = -(V_1 + \lambda_1)u + |u|^{2^*-2}u + |u|^{p_1-2}u + \beta r_1 |u|^{r_1-2}u|v|^{r_2} \quad \text{in } \mathbb{R}^N,$$

it follows from $u \geqslant 0$ that the right hand side is nonnegative if $\lambda_1 \leqslant 0$, and by [19, Lemma A.2], we obtain $u = 0$, which contradicts to the assumption $u \neq 0$. Hence, $\lambda_1 > 0$. Similarly, we also can obtain that $v \geqslant 0$ and $v \neq 0$ implies that $\lambda_2 > 0$. □

The following lemma is a version of the Brezis–Lieb lemma.

Lemma 4.3. *Suppose that $N \geqslant 3$, $r_1, r_2 > 1$ and $r \in (2, 2^*]$. If $(u_n, v_n) \rightharpoonup (u, v)$ in $E_1 \times E_2$, then, up to a subsequence if you need,*

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (|u_n|^{r_1}|v_n|^{r_2} - |u_n - u|^{r_1}|v_n - v|^{r_2} - |u|^{r_1}|v|^{r_2}) = 0.$$

Proof. See [11, Lemma 2.3] for the proof of the lemma. □

Let $\eta : \mathbb{R} \times E_1 \times E_2 \to E_1 \times E_2$,

$$\eta(t, u, v) := (u^t, v^t) = (e^{Nt/2}u(e^t\cdot), e^{Nt/2}v(e^t\cdot)).$$

Then

$$\begin{aligned} I(\eta(t,u,v)) = \frac{e^{2t}}{2}\left(|\nabla u|_2^2 + |\nabla v|_2^2\right) + \frac{1}{2}\int_{\mathbb{R}^N}\left(V_1(e^{-t}x)u^2 + V_2(e^{-t}x)v^2\right) - \frac{e^{2^*t}}{2^*}\left(|u|_{2^*}^{2^*} + |v|_{2^*}^{2^*}\right) \\ - \frac{e^{\gamma_{p_1}p_1 t}}{p_1}|u|_{p_1}^{p_1} - \frac{e^{\gamma_{p_2}p_2 t}}{p_2}|v|_{p_2}^{p_2} - \beta e^{\gamma_r r t}\int_{\mathbb{R}^N}|u|^{r_1}|v|^{r_2}. \end{aligned}$$

Lemma 4.4. *Fix $(u, v) \in S_{a_1} \times S_{a_2}$. Then $I(\eta(t, u, v)) \to 0^+$ as $t \to -\infty$ and $I(\eta(t, u, v)) \to -\infty$ as $t \to \infty$.*

Proof. The proof is standard, therefore it is omitted here. □

Lemma 4.5. *Let $D_k := \{(u,v) \in S_{a_1} \times S_{a_2} : |\nabla u|_2^2 + |\nabla v|_2^2 \leqslant k\}$. Then there exists $k_0 > 0$ sufficiently small such that*

$$0 < \sup_{(u,v)\in D_{k_0}} I < \inf_{(u,v)\in \partial D_{2k_0}} I.$$

Proof. For any $(u,v) \in S_{a_1} \times S_{a_2}$, using the condition ($H_1$), (1.5), the Gagliardo–Nirenberg and Hölder inequalities, we have

$$\begin{aligned}
\int_{\mathbb{R}^N} (V_1 u^2 + V_2 v^2) &\geqslant -\max\{\tau_1, \tau_2\} \left(|\nabla u|_2^2 + |\nabla v|_2^2\right), \\
\frac{1}{2^*}\left(|u|_{2^*}^{2^*} + |v|_{2^*}^{2^*}\right) &\leqslant \frac{1}{2^* S^{2^*/2}} \left(|\nabla u|_2^2 + |\nabla v|_2^2\right)^{2^*/2}, \\
\frac{1}{p_1}|u|_{p_1}^{p_1} \leqslant C_1 |\nabla u|_2^{\gamma_{p_1} p_1} &\leqslant C_1 \left(|\nabla u|_2^2 + |\nabla v|_2^2\right)^{\gamma_{p_1} p_1/2}, \\
\frac{1}{p_2}|v|_{p_2}^{p_2} \leqslant C_2 |\nabla v|_2^{\gamma_{p_2} p_2} &\leqslant C_2 \left(|\nabla u|_2^2 + |\nabla v|_2^2\right)^{\gamma_{p_2} p_2/2}
\end{aligned}$$

and

$$\beta \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} \leqslant \beta |u|_r^{r_1} |v|_r^{r_2} \leqslant \beta C_3 \left(|\nabla u|_2^2 + |\nabla v|_2^2\right)^{\gamma_r r/2}, \tag{4.3}$$

where $C_1 = C(N, p_1, a_1)$, $C_2 = C(N, p_2, a_2)$ and $C_3 = C(N, r_1, r_2, a_1, a_2)$. Set $d := |\nabla u|_2^2 + |\nabla v|_2^2$. Then

$$I(u,v) \geqslant \frac{1}{2}\left(1 - \max\{\tau_1,\tau_2\}\right) d - \frac{1}{2^* S^{2^*/2}} d^{2^*/2} - C_1 d^{\gamma_{p_1} p_1/2} - C_2 d^{\gamma_{p_2} p_2/2} - \beta C_3 d^{\gamma_r r/2}.$$

Since $2^*, \gamma_{p_1} p_1, \gamma_{p_2} p_2, \gamma_r r > 2$, it is easy to see that there exists $k_0 > 0$ small enough such that $I(u,v) > 0$ for all $(u,v) \in D_{2k_0}$. Fixing $(u_1, v_1) \in D_{k_0}$ and $(u_2, v_2) \in \partial D_{2k_0}$, we have

$$\begin{aligned}
&I(u_2,v_2) - I(u_1,v_1) \\
&\geqslant \frac{1}{2}(|\nabla u_2|_2^2 + |\nabla v_2|_2^2) + \frac{1}{2}\int_{\mathbb{R}^N} (V_1 u_2^2 + V_2 v_2^2) - \frac{1}{2^*}(|u_2|_2^{2^*} + |v_2|_2^{2^*}) \\
&\quad - \frac{1}{p_1}\int_{\mathbb{R}^N} |u_2|^{p_1} - \frac{1}{p_2}\int_{\mathbb{R}^N} |v_2|^{p_2} - \beta \int_{\mathbb{R}^N} |u_2|^{r_1}|v_2|^{r_2} - \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u_1|^2 + |\nabla v_1|^2) \\
&\geqslant \left(\frac{1}{2} - \max\{\tau_1,\tau_2\}\right) k_0 - \frac{1}{2^* S^{2^*/2}} (2k_0)^{2^*/2} - C_1 (2k_0)^{\gamma_{p_1} p_1/2} - C_2 (2k_0)^{\gamma_{p_2} p_2/2} - \beta C_3 (2k_0)^{\gamma_r r/2} \\
&\geqslant \frac{1}{4}\left(\frac{1}{2} - \max\{\tau_1,\tau_2\}\right) k_0,
\end{aligned}$$

for $k_0 > 0$ small enough. Thus, we can choose a sufficient small $k_0 > 0$ to satisfy the desired result. □

Let $\tilde{u} \in S_{a_1}$ be the positive radial ground state normalized solution of (1.6) with $i = 1$ and $\tilde{v} \in S_{a_2}$ be the positive radial ground state normalized solution of (1.6) with $i = 2$. By Lemmas 4.4 and 4.5, there exist $t_1, t_2 \in \mathbb{R}$ with $t_1 < -1 < 1 < t_2$ such that

$$e^{2t_1}\left(|\nabla \tilde{u}|_2^2 + |\nabla \tilde{v}|_2^2\right) < k, \qquad I(\eta(t_1, \tilde{u}, \tilde{v})) > 0,$$

and

$$e^{2t_2}\left(|\nabla \tilde{u}|_2^2 + |\nabla \tilde{v}|_2^2\right) > 2k, \qquad I(\eta(t_1, \tilde{u}, \tilde{v})) \leqslant 0.$$

Set

$$\Gamma_0 := \{h \in C([0,1], S_{a_1} \times S_{a_2}) : h(0) = \eta(t_1, \tilde{u}, \tilde{v}),\ h(1) = \eta(t_2, \tilde{u}, \tilde{v})\}.$$

Then $\Gamma_0 \neq \emptyset$. In fact, set $h_0(t) = \eta((1-t)t_1 + tt_2, \tilde{u}, \tilde{v})$, then $h_0 \in \Gamma_0$. Thus, we can define

$$c_\beta(a_1, a_2) := \inf_{h \in \Gamma_0} \max_{t \in [0,1]} I(h(t)).$$

Clearly, $c_\beta(a_1, a_2) > 0$.

Lemma 4.6. $\lim_{\beta \to \infty} c_\beta(a_1, a_2) = 0$.

Proof. Since $h_0 \in \Gamma_0$, we have

$$\begin{aligned} c_\beta(a_1, a_2) &\leqslant \max_{t \in [0,1]} I(h_0(t)) \\ &\leqslant \max_{t \geqslant 0} \left(\frac{1}{2} t^2 (|\nabla \tilde{u}|_2^2 + |\nabla \tilde{v}|_2^2) - \beta t^{\gamma_r r} \int_{\mathbb{R}^N} |\tilde{u}|^{r_1} |\tilde{v}|^{r_2} \right) \\ &= C\beta^{-2/(\gamma_r r - 2)} \to 0, \quad \beta \to \infty, \end{aligned}$$

where C is a positive constant independent of β. □

5 Proof of Theorem 1.1

In order to construct a bounded PS sequence of I at the level $c_\beta(a_1, a_2)$. Adapting the approach from [21], we introduce the C^1-functional $\Phi : E_1 \times E_2 \times \mathbb{R} \to \mathbb{R}$ with $\Phi(u, v, t) := I(\eta(t, u, v))$ and define

$$\tilde{c}_\beta(a_1, a_2) := \inf_{\tilde{h} \in \tilde{\Gamma}_0} \max_{t \in [0,1]} \Phi(\tilde{h}(t)),$$

where $\tilde{\Gamma}_0 = \{\tilde{h} \in C([0,1], S_{a_1} \times S_{a_2} \times \mathbb{R}) : \tilde{h}(0) = (\eta(t_1, \tilde{u}, \tilde{v}), 0), \tilde{h}(1) = (\eta(t_2, \tilde{u}, \tilde{v}), 0)\}$. It is easy to prove that $c_\beta(a_1, a_2) = \tilde{c}_\beta(a_1, a_2)$. The next lemma is special case of [15, Theorem 4.5].

Lemma 5.1. *Let X be a Hilbert manifold, $F \in C^1(X, \mathbb{R})$ be a given functional, $K \subset X$ be compact and consider a subset*

$$\mathcal{D} \subset \{E \subset X : E \text{ is compact}, \ K \subset E\},$$

which is homotopy-stable, that is, it is invariant with respect to deformations leaving K fixed. Assume that

$$\max_{u \in K} F(u) < c := \inf_{E \in \mathcal{D}} \max_{u \in E} F(u) \in \mathbb{R}.$$

Let $\varepsilon_n \in \mathbb{R}$, $\varepsilon_n \to 0$ and $E_n \in \mathcal{D}$ be a sequence such that

$$0 \leqslant \max_{u \in E_n} F(u) - c \leqslant \varepsilon_n.$$

Then there exists a sequence $u_n \in X$ such that, for some constant $C > 0$,

$$|F(u_n) - c| \leqslant \varepsilon_n, \quad \|F|_X'(u_n)\| \leqslant C\sqrt{\varepsilon_n}, \quad \text{dist}(u_n, E_n) \leqslant C\sqrt{\varepsilon_n}.$$

Lemma 5.2. *Let $\{\tilde{h}_n\} \subset \tilde{\Gamma}_0$ be a sequence such that*

$$\max_{t \in [0,1]} \Phi(\tilde{h}_n(t)) \leqslant c_\beta(a_1, a_2) + \frac{1}{n}.$$

Then there exist a sequence $(u_n, v_n, t_n) \in S_{a_1} \times S_{a_2} \times \mathbb{R}$ such that, as $n \to \infty$,

$$\Phi(u_n, v_n, t_n) \to c_\beta(a_1, a_2), \qquad \Phi|_{S_{a_1} \times S_{a_2} \times \mathbb{R}}'(u_n, v_n, t_n) \to 0, \tag{5.1}$$

and

$$\min_{t \in [0,1]} \|(u_n, v_n, t_n) - \tilde{h}_n(t)\|_{H^1(\mathbb{R}^N) \times \mathbb{R}} \to 0. \tag{5.2}$$

Proof. This lemma follows directly from Lemma 5.1 applied to Φ with

$$X := S_{a_1} \times S_{a_2} \times \mathbb{R}, \qquad K := \{(\eta(t_1, \tilde{u}, \tilde{v}), 0), (\eta(t_2, \tilde{u}, \tilde{v}), 0)\},$$

$$\mathcal{D} := \{\tilde{h}([0,1]) : \tilde{h} \in \tilde{\Gamma}_0\}, \qquad E_n := \{\tilde{h}_n(t) : t \in [0,1]\}.$$

Indeed, $c := \inf_{E \in \mathcal{D}} \max_{(u,v,t) \in E} \Phi(u,v,t) = \inf_{E \in \mathcal{D}} \max_{(u,v,t) \in E} I(\eta(t,u,v)) = c_\beta(a_1, a_2)$. On the one hand, for any $h \in \Gamma_0$, $\tilde{h}([0,1]) = (h([0,1]), 0) \in \mathcal{D}$. Hence,

$$c \leqslant \max_{(u,v,t) \in \tilde{h}([0,1])} I(\eta(t,u,v)) = \max_{(u,v) \in h([0,1])} I(u,v) = \max_{t \in [0,1]} I(h(t)).$$

Thus, $c \leqslant c_\beta(a_1, a_2)$. On the other hand, we show that $c_\beta(a_1, a_2) \leqslant c$. Suppose by contradiction that $c < c_\beta(a_1, a_2)$. Then $\max_{(u,v,t) \in E} I(\eta(t,u,v)) < c_\beta(a_1, a_2)$ for some $E \in \mathcal{D}$, hence $\sup_{(u,v,t) \in B_\delta(E)} I(\eta(t,u,v)) < c_\beta(a_1, a_2)$ for some $\delta > 0$, where $B_\delta(E)$ is the δ neighborhood of E. Moreover, $B_\delta(E)$ is open and connected, so it is path connected. Therefore, there exists a path $\tilde{h}_0 \in \tilde{\Gamma}_0$ such that $\max_{t \in [0,1]} \Phi(\tilde{h}_0(t)) < c_\beta(a_1, a_2)$. This is impossible. □

Lemma 5.3. *There exists a bounded sequence $\{(w_n, z_n)\} \subset S_{a_1} \times S_{a_2}$ such that, as $n \to \infty$,*

$$I(w_n, z_n) \to c_\beta(a_1, a_2), \qquad I|'_{S_{a_1} \times S_{a_2}}(w_n, z_n) \to 0, \tag{5.3}$$

$$\begin{aligned} P(w_n, z_n) := |\nabla w_n|_2^2 + |\nabla z_n|_2^2 - \int_{\mathbb{R}^N} (W_1 w_n^2 + W_2 z_n^2) - |w_n|_{2^*}^{2^*} - |z_n|_{2^*}^{2^*} \\ - \gamma_{p_1} |w_n|_{p_1}^{p_1} - \gamma_{p_2} |z_n|_{p_2}^{p_2} - \beta \gamma_r r \int_{\mathbb{R}^N} |w_n|^{r_1} |z_n|^{r_2} \to 0, \end{aligned} \tag{5.4}$$

$w_n^- \to 0$ a.e. in $\mathbb{R}^N$ and $z_n^- \to 0$ a.e. in $\mathbb{R}^N$.

Proof. First, by the definition of $c_\beta(a_1, a_2)$, there exists a sequence $\{h_n\} \subset \Gamma_0$ such that

$$\max_{t \in [0,1]} I(h_n(t)) \leqslant c_\beta(a_1, a_2) + \frac{1}{n}.$$

We observe that, since $I(u,v) = I(|u|, |v|)$ for any $(u,v) \in E_1 \times E_2$, we can take $h_n(t) \geqslant 0$ a.e. in $\mathbb{R}^N$ for every $t \in [0,1]$ and $n \in \mathbb{N}$. Applying Lemma 5.2 to $\tilde{h}_n := (h_n, 0) \in \tilde{\Gamma}_0$, we see that there exists a sequence $\{(u_n, v_n, t_n)\} \subset S_{a_1} \times S_{a_2} \times \mathbb{R}$ such that (5.1) and (5.2) hold. Note $(w_n, z_n) := (u_n^{t_n}, v_n^{t_n})$. By $h_n(t) \geqslant 0$ a.e. in $\mathbb{R}^N$ and (5.2), we see that, up to a subsequence, $u_n^- \to 0$ a.e. and $v_n^- \to 0$ a.e.. Hence, $w_n^- \to 0$ a.e. and $z_n^- \to 0$ a.e.. For any

$$(w_1, w_2) \in \{(u,v) \in E_1 \times E_2 : \int_{\mathbb{R}^N} w_n u = \int_{\mathbb{R}^N} z_n v = 0\},$$

setting $(w_1^n, w_2^n) := (w_1^{-t_n}, w_2^{-t_n})$, then

$$(w_1^n, w_2^n, 0) \in \left\{(u,v,t) \in E_1 \times E_2 \times \mathbb{R} : \int_{\mathbb{R}^N} u_n u = \int_{\mathbb{R}^N} v_n v = 0\right\}.$$

Hence,

$$I(w_n, z_n) \to c_\beta(a_1, a_2), \qquad t_n \to 0$$

and

$$\langle I|'_{S_{a_1} \times S_{a_2}}(w_n, z_n), (w_1, w_2) \rangle = \langle \Phi|'_{S_{a_1} \times S_{a_2} \times \mathbb{R}}(u_n, v_n, t_n), (w_1^n, w_2^n, 0) \rangle.$$

Since $\|(w_1^n, w_2^n)\| \leqslant 4\|(w_1, w_2)\|$ for n enough large, we have $I|'_{S_{a_1}\times S_{a_2}}(w_n, z_n) \to 0$. Therefore, (5.3) hold. Moreover, by $\langle \Phi|'_{S_{a_1}\times S_{a_2}\times\mathbb{R}}(u_n, v_n, t_n), , (0,0,1)\rangle \to 0$, we see $P(w_n, z_n) \to 0$. Hence, (5.4) hold.

Now, we prove that $\{(w_n, z_n)\} \subset S_{a_1} \times S_{a_2}$ is bounded in $E_1 \times E_2$. By (H_1) and (H_2), if $r = \min\{p_1, p_2, r\}$, then, for sufficiently large n,

$$\begin{aligned}
&c_\beta(a_1, a_2) + 1\\
&\quad\geqslant I(w_n, z_n) - \frac{1}{\gamma_r r}P(w_n, z_n)\\
&\quad\geqslant \left(\frac{1}{2} - \frac{1}{\gamma_r r}\right)\left(|\nabla w_n|_2^2 + |\nabla z_n|_2^2\right) + \frac{1}{2}\int_{\mathbb{R}^N}(V_1 w_n^2 + V_2 z_n^2) + \frac{1}{\gamma_r r}\int_{\mathbb{R}^N}(W_1 w_n^2 + W_2 z_n^2)\\
&\quad\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right)|\nabla w_n|_2^2 + \left(\frac{1-\tau_2}{2} - \frac{1+\theta_2}{\gamma_r r}\right)|\nabla z_n|_2^2;
\end{aligned}$$

if $p_1 = \min\{p_1, p_2, r\}$, then, for sufficiently large n,

$$\begin{aligned}
&c_\beta(a_1, a_2) + 1\\
&\quad\geqslant I(w_n, z_n) - \frac{1}{\gamma_{p_1} p_1}P(w_n, z_n)\\
&\quad\geqslant \left(\frac{1}{2} - \frac{1}{\gamma_{p_1} p_1}\right)\left(|\nabla w_n|_2^2 + |\nabla z_n|_2^2\right) + \frac{1}{2}\int_{\mathbb{R}^N}(V_1 w_n^2 + V_2 z_n^2) + \frac{1}{\gamma_{p_1} p_1}\int_{\mathbb{R}^N}(W_1 w_n^2 + W_2 z_n^2)\\
&\quad\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_{p_1} p_1}\right)|\nabla w_n|_2^2 + \left(\frac{1-\tau_2}{2} - \frac{1+\theta_2}{\gamma_{p_1} p_1}\right)|\nabla z_n|_2^2;
\end{aligned}$$

if $p_2 = \min\{p_1, p_2, r\}$, then, for sufficiently large n,

$$\begin{aligned}
&c_\beta(a_1, a_2) + 1\\
&\quad\geqslant I(w_n, z_n) - \frac{1}{\gamma_{p_2} p_2}P(w_n, z_n)\\
&\quad\geqslant \left(\frac{1}{2} - \frac{1}{\gamma_{p_2} p_2}\right)\left(|\nabla w_n|_2^2 + |\nabla z_n|_2^2\right) + \frac{1}{2}\int_{\mathbb{R}^N}(V_1 w_n^2 + V_2 z_n^2) + \frac{1}{\gamma_{p_2} p_2}\int_{\mathbb{R}^N}(W_1 w_n^2 + W_2 z_n^2)\\
&\quad\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_{p_2} p_2}\right)|\nabla w_n|_2^2 + \left(\frac{1-\tau_2}{2} - \frac{1+\theta_2}{\gamma_{p_2} p_2}\right)|\nabla z_n|_2^2.
\end{aligned}$$

In these three cases, we conclude that $\{(w_, z_n)\}$ is bounded in $E_1 \times E_2$. □

It follows from Lemma 5.2 that there exists a nonnegative $(w_0, z_0) \in E_1 \times E_2$ such that, up to a subsequence,

$$\begin{cases}
(w_n, z_n) \rightharpoonup (w_0, z_0) & \text{in } E_1 \times E_2,\\
(w_n, z_n) \rightharpoonup (w_0, z_0) & \text{in } L^{q_1}(\mathbb{R}^N) \times L^{q_2}(\mathbb{R}^N),\ q_1, q_1 \in [2, 2^*],\\
(w_n, z_n) \to (w_0, z_0) & \text{in } L^{q_1}(\mathbb{R}^N) \times L^{q_2}(\mathbb{R}^N),\ q_1, q_1 \in (2, 2^*),\\
(w_n, z_n) \to (w_0, z_0) & \text{a.e. in } \mathbb{R}^N.
\end{cases} \tag{5.5}$$

Since $I|'_{S_{a_1}\times S_{a_2}}(w_n, z_n) \to 0$, by the Lagrange multipliers rule, there exists a sequence $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R} \times \mathbb{R}$ such that

$$I'(w_n, z_n) + \lambda_1^n(w_n, 0) + \lambda_2^n(0, z_n) \to 0, \quad \text{in } (E_1 \times E_2)^*. \tag{5.6}$$

Take $(w_n, 0)$ and $(0, z_n)$ as test functions in (5.6), we see that $\{(\lambda_1^n, \lambda_2^n)\}$ is bounded in $\mathbb{R} \times \mathbb{R}$. Then there exists $(\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}$ such that, up to a subsequence, $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$.

Lemma 5.4. *There exists $\beta_* > 0$ sufficiently large such that $(w_n, z_n) \to (w_0, z_0)$ in $L^{2^*}(\mathbb{R}^N) \times L^{2^*}(\mathbb{R}^N)$ when $\beta \geqslant \beta_*$, moreover, $(w_0, z_0) \neq 0$.*

Proof. We firstly prove that $w_n \to w_0$ in $L^{2^*}(\mathbb{R}^N)$. Using the concentration-compactness principle [24], we see that there exist finite nonnegative measure μ and ν, and a most countable index set Λ such that $|\nabla w_n|^2 \rightharpoonup \mu$ in sense of measure, $|w_n|^{2^*} \rightharpoonup \nu$ in sense of measure and

$$\begin{cases} \mu \geqslant |\nabla w_0|^2 + \sum_{j\in\Lambda} \mu_j \delta_{x_j} & \mu_j \geqslant 0, \\ \nu = |w_0|^{2^*} + \sum_{j\in\Lambda} \nu_j \delta_{x_j} & \nu_j \geqslant 0, \\ \nu_j \leqslant S^{-2^*/2} \mu_j^{2^*/2} & j \in \Lambda, \end{cases} \tag{5.7}$$

where $x_j \in \mathbb{R}^N$ and δ_{x_j} is the Dirac measure at x_j. Let $\chi_R \in C_0^\infty(\mathbb{R}^N)$ be a cut off function satisfying $\chi_R(x) = 1$ in $B_R(x_j)$, $\chi_R(x) = 0$ in $B_{2R}^c(x_j)$ and $|\nabla \chi_R| \leqslant 2/R$. It follows from Lemma 5.2 that $\{\chi_R w_n\}$ is bounded in E_1. Now, take $(\chi_R w_n, 0)$ as a test function in (5.6), then

$$\lim_{n\to\infty} \langle I'(w_n, z_n) + \lambda_1^n(w_n, 0) + \lambda_2^n(0, z_n), (\chi_R w_n, 0)\rangle = 0. \tag{5.8}$$

By (5.5), the absolute continuity of integral and the Hölder inequality, we can deduce that

$$\lim_{R\to0}\lim_{n\to\infty} \int_{\mathbb{R}^N} V_1 w_n^2 \chi_R = \lim_{R\to0} \int_{\mathbb{R}^N} V_1 w_0^2 \chi_R = 0, \tag{5.9}$$

$$\lim_{R\to0}\lim_{n\to\infty} \int_{\mathbb{R}^N} \lambda_1^n w_n^2 \chi_R = \lambda_1 \lim_{R\to0} \int_{\mathbb{R}^N} w_0^2 \chi_R = 0, \tag{5.10}$$

$$\lim_{R\to0}\lim_{n\to\infty} \int_{\mathbb{R}^N} w_n \nabla w_n \cdot \nabla \chi_R = \lim_{R\to0} \int_{\mathbb{R}^N} w_0 \nabla w_0 \cdot \nabla \chi_R = 0, \tag{5.11}$$

$$\lim_{R\to0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |w_n|^{p_1} \chi_R = \lim_{R\to0} \int_{\mathbb{R}^N} |w_0|^{p_1} \chi_R = 0, \tag{5.12}$$

and

$$\lim_{R\to0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |w_n|^{r_1} |z_n|^{r_2} \chi_R = \lim_{R\to0} \int_{\mathbb{R}^N} |w_0|^{r_1} |z_0|^{r_2} \chi_R = 0. \tag{5.13}$$

It follows from (5.8) and (5.9)–(5.13) that

$$\lim_{R\to0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |\nabla w_n|^2 \chi_R = \lim_{R\to0}\lim_{n\to\infty} \int_{\mathbb{R}^N} |w_n|^{2^*} \chi_R,$$

that is,

$$\lim_{R\to0} \int_{\mathbb{R}^N} \chi_R d\mu = \lim_{R\to0} \int_{\mathbb{R}^N} \chi_R d\nu. \tag{5.14}$$

Using (5.7) and (5.14), we can obtain $\nu_j \geqslant \mu_j$, furthermore, either $\mu_j = 0$ or $\mu_j \geqslant S^{N/2}$ for $j \in \Lambda$. Observe that, for any $j \in \Lambda$, $\mu_j = 0$ if and only if $\nu_j = 0$. If $\mu_j = 0$, then $\nu_j = 0$ and $|w_n|_{2^*}^{2^*} \to |w_0|_{2^*}^{2^*}$ by (5.7), combining $w_n \rightharpoonup w_0$ in $L^{2^*}(\mathbb{R}^N)$, we conclude that $w_n \to w_0$ in $L^{2^*}(\mathbb{R}^N)$. If $\mu_j \geqslant S^{N/2}$, then we split three cases.

If $r = \min\{r, p_1, p_2\}$, then, by Lemma 4.6, there exists $\beta_1 > 0$ sufficiently large such that, for $\beta \geqslant \beta_1$,

$$c_\beta(a_1, a_2) < \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right) S^{N/2}. \tag{5.15}$$

It follows from (5.7) that

$$\begin{aligned} c_\beta(a_1,a_2) &= \lim_{n\to\infty} I(w_n,z_n) - \frac{1}{\gamma_r r} P(w_n,z_n) \\ &\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right) \int_{\mathbb{R}^N} |\nabla w_n|^2 \chi_R dx \\ &= \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right) \int_{\mathbb{R}^N} \chi_R d\mu \\ &\geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right) \mu_j \geqslant \left(\frac{1-\tau_1}{2} - \frac{1+\theta_1}{\gamma_r r}\right) S^{N/2}, \end{aligned}$$

which contradicts to (5.15). If $p_1 = \min\{r, p_1, p_2\}$ or $p_2 = \min\{r, p_1, p_2\}$, similarly as the case $r = \min\{r, p_1, p_2\}$, them also yields a contradiction.

In summary, going if necessary to replace a larger β_*, we obtain $\mu_j = \nu_j = 0$ for all $j \in \Lambda$ and $\beta \geqslant \beta_*$. Consequently, $w_n \to w_0$ in $L^{2^*}(\mathbb{R}^N)$ when $\beta \geqslant \beta_*$. $z_n \to z_0$ in $L^{2^*}(\mathbb{R}^N)$ can be obtained in the similar way.

By Lemma 5.3, we know that (w_0, z_0) is a nonnegative solution of (1.1). Suppose that by contradiction $(w_0, z_0) = 0$, and by (4.3), $\int_{\mathbb{R}^N} W_1 w_n^2 \to 0$, $\int_{\mathbb{R}^N} W_2 z_n^2 \to 0$, the strong convergence of $L^{2^*}, L^{p_1}, L^{p_2}, L^r$ and $P(w_n, z_n) \to 0$, we see that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} (|\nabla w_n|^2 + |\nabla z_n|^2) = 0.$$

Hence, by $\int_{\mathbb{R}^N} V_1 w_n^2 \to 0$, $\int_{\mathbb{R}^N} V_2 z_n^2 \to 0$, we have $c_\beta(a_1,a_2) = \lim_{n\to\infty} I(w_n, z_n) = 0$, which contradicts to $c_\beta(a_1, a_2) > 0$. Hence, $(w_0, z_0) \neq 0$. □

Lemma 5.5. *If $c_\beta(a_1,a_2) < \min\{m_{V_1}(a_1), m_{V_2}(a_2)\}$, then $(w_n, z_n) \to (w_0, z_0)$ in $E_1 \times E_2$. Moreover, $(u_0, v_0) \in S_{a_1} \times S_{a_2}$ is a positive radial normalized solution of (1.1) with $\lambda_1 > 0$ and $\lambda_2 > 0$.*

Proof. We know from Lemmas 5.3 and 5.4 that (w_0, z_0) is nonnegative and $(w_0, z_0) \neq 0$.

If $w_0 \neq 0$ and $z_0 = 0$, then w_0 is a nontrivial radial solutions of (1.6) with $i = 1$ and $w_0 > 0$ by the maximum principle, where $|w_0|_2^2 = a \leqslant a_1$. By Lemma 4.1 and Theorem 1.3, we see that $m_{V_1}(a_1) \leqslant m_{V_1}(a) \leqslant J_{V_1}(w_0) = I(w_0, 0)$. It follows from the conditions (H_1) and (H_2) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} V_1 \left[w_n^2 - (w_n - w_0)^2 - w_0^2\right] = 0, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} V_1 (w_n - w_0)^2 = 0 \tag{5.16}$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} W_1 \left[w_n^2 - (w_n - w_0)^2 - w_0^2\right] = 0, \qquad \lim_{n\to\infty} \int_{\mathbb{R}^N} W_1 (w_n - w_0)^2 = 0. \tag{5.17}$$

Applying the Brezis–Lieb lemma, Lemma 4.3, (5.17), (5.16) and the $L^{p_1}, L^{p_2}, L^{2^*}, L^r$ strong convergence, we deduce that

$$\begin{aligned} o_n(1) &= P(w_n, z_n) \\ &= P(w_n - w_0, z_n) + P(w_0, 0) + o_n(1) \\ &= \int_{\mathbb{R}^N} (|\nabla (w_n - w_0)|^2 + |\nabla z_n|^2) + o_n(1) \end{aligned} \tag{5.18}$$

and

$$\begin{aligned} c_\beta(a_1,a_2) &= \lim_{n\to\infty} I(w_n,z_n) \\ &= \lim_{n\to\infty} I(w_n-w_0,z_n)+I(w_0,0)+o_n(1) \\ &\geqslant \frac{1}{2}\lim_{n\to\infty}\int_{\mathbb{R}^N}(|\nabla(w_n-w_0)|^2+|\nabla z_n|^2)+m_{V_1}(a_1)\geqslant m_{V_1}(a_1), \end{aligned} \tag{5.19}$$

which contradicts to $c_\beta(a_1,a_2)<m_{V_1}(a_1)$.

If $w_0=0$ and $z_0\neq 0$, then z_0 is a nontrivial radial solutions of (1.6) with $i=2$ and $z_0>0$ by the maximum principle, where $b=|z_0|_2^2\leqslant a_2$ and $m_{V_2}(a_2)\leqslant m_{V_2}(b)\leqslant J_{V_2}(z_0)=I(0,z_0)$. Similarly as (5.18) and (5.19), we also can derive a contradiction.

Hence, (w_0,z_0) is nonnegative, $w_0\neq 0$ and $z_0\neq 0$, and by Lemma 4.2, we can obtain $\lambda_1>0$ and $\lambda_2>0$. By the Pohozaev and Nehari identities, it is easy to see that

$$\begin{aligned} \lambda_1|w_0|_2^2+\lambda_2|z_0|_2^2 = &-\int_{\mathbb{R}^N}(V_1w_0^2+V_2z_0^2)-\int_{\mathbb{R}^N}(W_1w_0^2+W_2z_0^2) \\ &+(1-\gamma_{p_1})|w_0|_{p_1}^{p_1}+(1-\gamma_{p_2})|z_0|_{p_2}^{p_2}+\beta r(1-\gamma_r)\int_{\mathbb{R}^N}|w_0|^{r_1}|z_0|^{r_2}, \end{aligned}$$

and combining $P(w_n,z_n)\to 0$, we have

$$\begin{aligned} \lambda_1a_1+\lambda_2a_2 &= \lim_{n\to\infty}\left(\lambda_1^n|w_n|_2^2+\lambda_2^n|z_n|_2^2\right) \\ &= \lim_{n\to\infty}\Big[-\int_{\mathbb{R}^N}(V_1w_n^2+V_2z_n^2)-\int_{\mathbb{R}^N}(W_1w_n^2+W_2z_n^2) \\ &\quad +(1-\gamma_{p_1})|w_n|_{p_1}^{p_1}+(1-\gamma_{p_2})|z_n|_{p_2}^{p_2}+\beta r(1-\gamma_r)\int_{\mathbb{R}^N}|w_n|^{r_1}|z_n|^{r_2}\Big] \\ &= -\int_{\mathbb{R}^N}(V_1w_0^2+V_2z_0^2)-\int_{\mathbb{R}^N}(W_1w_0^2+W_2z_0^2) \\ &\quad +(1-\gamma_{p_1})|w_0|_{p_1}^{p_1}+(1-\gamma_{p_2})|z_0|_{p_2}^{p_2}+\beta r(1-\gamma_r)\int_{\mathbb{R}^N}|w_0|^{r_1}|z_0|^{r_2} \\ &= \lambda_1|w_0|_2^2+\lambda_2|z_0|_2^2, \end{aligned}$$

which implies that $|w_0|_2^2=a_1$ and $|z_0|_2^2=a_2$, that is, $(w_n,z_n)\to(w_0,z_0)$ in $L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$. Therefore, from (5.5), (5.6) and Lemma 5.4, we know that

$$\begin{aligned} \lim_{n\to\infty}(|\nabla w_n|_2^2+\lambda_1|w_n|_2^2) &= \lim_{n\to\infty}\left(-\int_{\mathbb{R}^N}V_1w_n^2+|w_n|_{2^*}^{2^*}+|w_n|_{p_1}^{p_1}+\beta r_1\int_{\mathbb{R}^N}|w_n|^{r_1}|z_n|^{r_2}\right) \\ &= -\int_{\mathbb{R}^N}V_1w_0^2+|w_0|_{2^*}^{2^*}+|w_0|_{p_1}^{p_1}+\beta r_1\int_{\mathbb{R}^N}|w_0|^{r_1}|z_0|^{r_2} \\ &= |\nabla w_0|_2^2+\lambda_1|w_0|_2^2, \end{aligned}$$

that is $\|w_n\|_1\to\|w_0\|_1$ as $n\to\infty$. Similarly, we also have $\|z_n\|_2\to\|z_0\|_2$. Hence, it is easy to see that $(w_n,z_n)\to(w_0,z_0)$ in $E_1\times E_2$. This completes the proof. □

Proof of Theorem 1.1. By Lemmas 5.3, 5.4 and 5.5, we complete the proof of Theorem 1.1. □

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