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Nonlinear differential equations of Riccati type on ordered Banach spaces*

V. M. Ungureanu†, V. Dragan

Abstract

In this paper we consider a general class of time-varying nonlinear differential equations on infinite dimensional ordered Banach spaces, which includes as special cases many known differential Riccati equations of optimal control. Using a linear matrix inequalities (LMIs) approach we provide necessary and sufficient conditions for the existence of some global solutions such as maximal, stabilizing and minimal solutions for this class of generalized Riccati equations. The obtained results extend to infinite dimensions and unify corresponding results in the literature. They provide useful tools for solving infinite-time linear quadratic (LQ) control problems for linear differential systems affected by countably-infinite-state Markovian jumps and/or multiplicative noise.

1 Introduction

Differential Riccati equations of stochastic control is a field of intensive research over the last five decades. The systematic development of this research, started with Kalman [20], who investigated the properties of the solutions of matrix differential Riccati equations in connection with the so called linear quadratic optimization problem for deterministic systems. Later on, Wonham [29], [30] extended the results to the framework of stochastic control and introduced the so-called Riccati equations of stochastic control.

Recently, the optimal control problems for linear differential systems subject to Markovian jumps (LDSMJs) have attracted the interest of the researchers due to their new applications in modern queuing network theory [7] or in the study of safety-critical and high integrity systems (as for e.g. aircraft, chemical plants, nuclear power solutions, large scale flexible structures for space stations etc).

In the finite dimensional case there are a lot of works dealing with Riccati differential/difference equations arising in various optimal control problems for both discrete-time and continuous linear systems with Markovian jumps (see

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†Corresponding author.

for e.g. [2], [5], [10], [11], [12], [13], [14]). Also, the infinite dimensional Riccati equations associated with LQ optimal control problems for linear systems with multiplicative noise and without jumps have been intensively studied in the literature (see [8], [1], [23] and the references therein).

However, in the context of infinite dimensional time-varying LDSMJs we are far from a comprehensive theory. It is well known that many time-dependent processes occurring in physics, biology, economy or other sciences are described by such LDSMJs. The infinite dimensional feature of the LDSMJs is due either to the state space of the system or to the one of the Markov process (as it happens in the queueing network theory). First results for the optimal control of LDSMJs with countably infinite state space for the Markov chain were obtained in [5], [16], [17]. The works mentioned above consider only the special case of Riccati equations of stochastic control associated with finite-dimensional autonomous LDSMJs without multiplicative noise and signed quadratic performances. Also in [3] is addressed the problem of existence of maximal solutions to algebraic Riccati equations associated with similar LDSMJs.

In this paper we adopt a LMI approach to study a class of non-autonomous generalized Riccati differential equations (GRDEs) defined on infinite dimensional Banach spaces, sufficiently large to include many of the known Riccati equations of stochastic control.

Unlike the infinite dimensional Riccati equations discussed in [5], [16] or [18], our GDREs also apply to infinite-time LQ control problems for LDSMJs with multiplicative noise and unsigned quadratic criteria. For this class of GDREs we will provide necessary and sufficient conditions for the existence of certain global solutions such as maximal, stabilizing and minimal solutions. The importance of these results comes from the fact that the design of the optimal controller for infinite-horizon LQ control problems is closely related to the existence of such global solutions of GDREs. For example, in the finite dimensional case (see [13]) it is proved that, depending upon the class of admissible controls, the corresponding optimal control may be obtained either with the minimal solution or with the maximal and the stabilizing solutions of GRDEs.

Our results generalize and extend the ones in [10] to infinite dimensional LDSMJ with multiplicative noise and improve the results concerning the Riccati equations of stochastic control from [16] and [3]. Let us recall that [16] employes the classical detectability property to give sufficient conditions for the existence of stabilizing solutions for the Riccati equations of stochastic control associated with finite-dimensional, autonomous LDSMJs with countably-infinite-state Markov chain.

We finally note that all the proofs from this paper are nonstochastic and avoid any connection with the optimization problems. Therefore the obtained results are of interest in their own right for the researchers from the field of differential equations.

2 Notations and statement of the problem

Let H and U be real separable Hilbert spaces. We will denote by $L(H,U)$ the real Banach space of linear and bounded operators from H into U and by $\mathcal{E}_H$ the Banach subspace of $L(H) := L(H,H)$, formed by all self-adjoint operators. As usual, we shall write $\langle .,. \rangle_H$ for the inner product on H and $\|.\|$ for norms of elements and operators, unless indicated otherwise. An operator $A \in L(H)$ is called *nonnegative* and we write $A \geq 0$, if A is self-adjoint and $\langle Ah, h \rangle_H \geq 0$ for all $h \in H$. We will denote by $\mathcal{K}_H$ the cone of all nonnegative operators from H. Let $\mathcal{Z}$ be an interval of integers, which may be finite or infinite and let B be a real Banach space. Then $l_B^{\mathcal{Z}}$ is the space of all $\mathcal{Z}$ -sequences $g = \{g(i) \in B\}_{i \in \mathcal{Z}}$ with the property that $\|g\|_{\mathcal{Z}} := \sup_{i \in \mathcal{Z}} \|g(i)\| < \infty$. It can be shown by using a standard procedure that $l_B^{\mathcal{Z}}$ is a real Banach space when endowed with the usual term-wise addition, the real scalar multiplication and the norm $\|.\|_{\mathcal{Z}}$. If B is $L(U,H)$ or $\mathcal{E}_H$, then $l_B^{\mathcal{Z}}$ will be denoted by $l_{L(U,H)}^{\mathcal{Z}}$ or $l_{\mathcal{E}_H}^{\mathcal{Z}}$. If I_H is the identity operator on H then $\mathcal{I}^H$ will denote the element of $l_{\mathcal{E}_H}^{\mathcal{Z}}$ defined by $\mathcal{I}^H(i) = I_H$, $i \in \mathcal{Z}$.

For any $A \in l_{L(U,H)}^{\mathcal{Z}}, B \in l_{L(H,U)}^{\mathcal{Z}}$, the product AB is defined by $(AB)(i) = A(i)B(i), i \in \mathcal{Z}$. Obviously $AB \in l_{L(H)}^{\mathcal{Z}}$. We use the standard symbol * for the adjoint of a linear and bounded operator. If $P \in l_{L(H,U)}^{\mathcal{Z}}$, then $P^{[*]} := \{P(i)^*, i \in \mathcal{Z}\} \in l_{L(U,H)}^{\mathcal{Z}}$. Also, if $P \in l_{L(H)}^{\mathcal{Z}}$ is such that $P(i)^{-1}$ exists for all $i \in \mathcal{Z}$, then $P^{[-1]} := \{P(i)^{-1}, i \in \mathcal{Z}\} \in l_{L(H)}^{\mathcal{Z}}$. An element $X \in l_{\mathcal{E}_H}^{\mathcal{Z}}$ is said to be *nonnegative* (we write $X \succeq 0$) iff $X(i) \geq 0$ for all $i \in \mathcal{Z}$. The cone $\mathcal{K}_H^{\mathcal{Z}}$ of all nonnegative elements of $l_{\mathcal{E}_H}^{\mathcal{Z}}$ introduces the following order on $l_{\mathcal{E}_H}^{\mathcal{Z}}$:

$$X \succeq Y \text{ iff } X - Y \in \mathcal{K}_H^{\mathcal{Z}}.$$

A linear and bounded operator $\Gamma \in L\left(l_{\mathcal{E}_H}^{\mathcal{Z}}, l_{\mathcal{E}_U}^{\mathcal{Z}}\right)$ is *positive* if and only if (iff) $\Gamma\left(\mathcal{K}_H^{\mathcal{Z}}\right) \subset \mathcal{K}_U^{\mathcal{Z}}$.

Definition 1 *We say that a positive operator $\Gamma \in L\left(l_{\mathcal{E}_H}^{\mathcal{Z}}, l_{\mathcal{E}_U}^{\mathcal{Z}}\right)$ is m-strongly continuous if for any increasing and bounded sequence $\{D_m\}_{m \in \mathbf{N}} \subset l_{\mathcal{E}_H}^{\mathcal{Z}}$ we have*

$$\lim_{m \to \infty} \Gamma(D_m)(i)x = \Gamma(D)(i)x, x \in H, i \in \mathcal{Z},$$

where $D(i)x = \lim_{m \to \infty} D_m(i)x$, for all $i \in \mathcal{Z}$ and $x \in H$.

Remark 2 *If $l_{\mathcal{E}_H}^{\mathcal{Z}}$ is finite-dimensional, then any positive operator $\Gamma \in L\left(l_{\mathcal{E}_H}^{\mathcal{Z}}, l_{\mathcal{E}_U}^{\mathcal{Z}}\right)$ is m-strongly continuous, since the strong and uniform topologies on $l_{\mathcal{E}_H}^{\mathcal{Z}}$ coincide. Obviously $l_{\mathcal{E}_H}^{\mathcal{Z}}$ is finite-dimensional if $\mathcal{Z}$ is finite and H is finite dimensional.*

Let $J \subset \mathbf{R}_+ = [0, \infty)$ be an interval. Any mappings $G : J \to \mathcal{E}_H$, $F : J \to l_{\mathcal{E}_H}^{\mathcal{Z}}$ with the properties $G(J) \subset \mathcal{K}_H, F(J) \subset \mathcal{K}_H^{\mathcal{Z}}$ are called *nonnegative* and we will write $G(t) \geq 0, t \in J$ and $F(t) \succeq 0, t \in J$; if there is $\gamma > 0$ such that, for all $t \in J$, $G(t) \geq \gamma I_H$ and $F(t) \succeq \gamma \mathcal{I}^H$, respectively, then we say that G and F are *uniformly positive* and we write $G(t) > 0, t \in J$ and $F(t) \succ 0, t \in J$, respectively.

The i - th component of an element of $G(t) \in l^{\mathcal{Z}}_{L(H,U)}, t \in J$ will be denoted shortly $G(t,i)$. For any mappings $M_{11} : J \to l^{\mathcal{Z}}_{\mathcal{E}_H}$, $M_{12} : J \to l^{\mathcal{Z}}_{L(U,H)}$ and $M_{22} : J \to l^{\mathcal{Z}}_{\mathcal{E}_U}$ satisfying $M_{22}(t,i) > 0, i \in \mathcal{Z}$, $t \in J$, we define the function $M : J \to l^{\mathcal{Z}}_{\mathcal{E}_{H\times U}}$ as follows

$$M(t,i) = \begin{pmatrix} M_{11}(t,i) & M_{12}(t,i) \\ M_{12}^{[*]}(t,i) & M_{22}(t,i) \end{pmatrix}, i \in \mathcal{Z}, t \in J. \tag{1}$$

The mapping $M|M_{22} : J \to l^{\mathcal{Z}}_{\mathcal{E}_H}$,

$$M|M_{22}(t,i) = M_{11}(t,i) - M_{12}(t,i) M_{22}^{[-1]}(t,i) M_{12}^{[*]}(t,i), i \in \mathcal{Z}, t \in J$$

is called the *Schur complement* of M_{22} in M. The following result extends a well-known property of Schur complements (see [21], [4]) to families of operators. Although its proof is very similar to the one given in [28] for sequences of operators, we sketch here the outlines of the proof for the reader convenience.

Lemma 3 *Assume that the mapping M defined above is bounded and $M_{22}(t) \succ 0, t \in J$. Then $M(t) \succ 0$ (respectively, $M(t) \succeq 0$), $t \in J$ iff $M(t)|M_{22}(t) \succ 0$ (respectively, $M(t)|M_{22}(t) \succeq 0$), $t \in J$.*

Proof. By hypothesis there are $\mu, \eta > 0$ such that $\|M(t)\|_{\mathcal{Z}} \leq \mu$, $M_{22}(t) \succeq \eta \Phi_U$ and $\left\|M_{22}^{[-1]}(t)\right\|_{\mathcal{Z}} \leq \frac{1}{\eta}$ for all $t \in J$. We consider the mapping $\Omega : J \to l^{\mathcal{Z}}_{L(H\times U)}$, $\Omega(t,i) = \begin{pmatrix} I_H & -M_{12}(t,i) M_{22}^{-1}(t,i) \\ 0 & M_{22}^{-1}(t,i) \end{pmatrix}$, $(t,i) \in J \times \mathcal{Z}$ and we observe that

$$\begin{aligned} \Omega(t) M(t) \Omega(t)^{[*]} &= \begin{pmatrix} M(t)|M_{22}(t) & 0 \\ 0 & M_{22}^{[-1]}(t) \end{pmatrix} \\ \Omega^{-1}(t,i) &= \begin{pmatrix} I_H & M_{12}(t,i) \\ 0 & M_{22}(t,i) \end{pmatrix}, (t,i) \in J \times \mathcal{Z}. \end{aligned} \tag{2}$$

A standard computation shows that, for all $(t,i) \in J \times \mathcal{Z}$,

$$\|\Omega(t,i)\| \leq 1 + 2\mu \max(1, \frac{1}{\eta}) := n_\Omega, \tag{3}$$

$$\left\|\Omega^{-1}(t,i)\right\| \leq 1 + \mu, \tag{4}$$

$$\Omega(t) \Omega(t)^{[*]} \geq \frac{1}{(1+\mu)^2} \Phi_{H\times U}, \tag{5}$$

$$M_{22}^{-1}(t) \succeq \frac{\eta}{\mu^2} \Phi_U. \tag{6}$$

Assume $M(t) \succ\succ 0$. From (5) we obtain $\Omega(t) M(t) \Omega(t)^{[*]} \succeq \gamma \frac{1}{(1+\mu)^2} \Phi_{H\times U}$, where $\gamma > 0$ is such that $M(t) \succeq \gamma \Phi_{H\times U}, t \in J$. Then, by virtue of (2), we conclude that $M(t) | M_{22}(t) \succ\succ 0, t \in J$.

Conversely, suppose that $M(t) | M_{22}(t) \succ\succ 0, t \in J$. Using (6) we can prove that for all $t \in J$ there is $\delta > 0$ such that $\Omega(t) M(t) \Omega(t)^{[*]} \succeq \delta \Phi_{H\times U}$ or, equivalently, $M(t) \succeq \delta \Omega(t)^{[-1]} \left(\Omega(t)^{[-1]}\right)^{[*]}$. Employing (3) we get $M(t) \succeq \delta \left(\frac{1}{n_\Omega}\right)^2 \Phi_{H\times U}$, $t \in J$ and the proof is complete. ∎

Let $T > 0$. If B is an arbitrary Banach space, then we denote by $C([0,T], B)$ the space of all mappings $G : [0,T] \to B$ that are continuous. Also $C^1([0,T], B)$ denotes the subspace of $C([0,T], B)$ of all continuously differentiable mappings G on $(0,T)$(i.e. G is differentiable on $(0,T)$ and G' is continuous on $(0,T)$). Similar notation will be used in the case when the interval $[0,T]$ is replaced by $\mathbf{R}_+$. In addition, $C_b(\mathbf{R}_+, B)$ will denote the subspace of $C(\mathbf{R}_+, B)$ formed by all bounded mappings. If $J = [0,T]$ or $J = \mathbf{R}_+$, then $C_b^1(J, B)$ is the subspace of $C^1(J, B)$ formed by all mappings f with the property that $f(t)$ and $\frac{df(t)}{dt}$ are bounded on J.

Further, the product $t \in J \to G(t)(X(t)) \in l^{\mathcal{Z}}_{L(U,H)}$ of any two functions $G : J \to L\left(l^{\mathcal{Z}}_{\mathcal{E}_H}, l^{\mathcal{Z}}_{L(U,H)}\right)$ and $X : J \to l^{\mathcal{Z}}_{\mathcal{E}_H}$ will be often denoted shortly $G(t, X(t))$. In this case we will write $G(t, X(t))(i)$ for the i-th component of $G(t, X(t))$.

In the sequel we assume the following hypothesis

(H1) $A \in C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{L(H)}\right), M \in C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right), B, D \in C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{L(U,H)}\right)$, $R \in C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_U}\right), \Pi_1(t) \in C_b\left(\mathbf{R}_+, L\left(l^{\mathcal{Z}}_{\mathcal{E}_H}\right)\right), \Pi_{12} \in C_b\left(\mathbf{R}_+, L\left(l^{\mathcal{Z}}_{\mathcal{E}_H}, l^{\mathcal{Z}}_{L(U,H)}\right)\right)$, $\Pi_2 \in C_b\left(\mathbf{R}_+, L\left(l^{\mathcal{Z}}_{\mathcal{E}_H}, l^{\mathcal{Z}}_{\mathcal{E}_U}\right)\right)$.

The goal of this paper is to study the problem of existence of certain global solutions (as maximal, minimal or stabilizing solutions) for the following generalized Riccati differential equations (GRDEs):

$$\begin{gathered}\frac{d}{dt}X(t,i) + A^*(t,i)X(t,i) + X(t,i)A(t,i) + \Pi_1(t, X(t))(i) + M(t,i) - \\ [X(t,i)B(t,i) + \Pi_{12}(t, X(t))(i) + D(t,i)] [R(t,i) + \Pi_2(t, X(t))(i)]^{-1} \cdot \\ [X(t,i)B(t,i) + \Pi_{12}(t, X(t))(i) + D(t,i)]^* = 0, i \in \mathcal{Z}, t \in \mathbf{R}_+,\end{gathered} \tag{7}$$

This class of nonlinear differential Riccati-type equations includes as special cases many of the Riccati equations of the stochastic control. For example, let us consider that $\mathcal{Z}$ is finite and let $\eta(t), t \in \mathbf{R}_+$ be a right continuous homogeneous Markov chain with the state space $\mathcal{Z}$ and the probability transition matrix $P(t) = e^{\Lambda t}, \Lambda = (\lambda_{ij})$ with $\sum_{j\in\mathcal{Z}} \lambda_{ij} = 0$, $\lambda_{ij} \geq 0$ for $i \neq j$. Taking $r \in \mathbf{N}^*$and

$A_k \in C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right), B_k \in C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{L(U,H)}\right), k=\overline{1,r}$ we define

$$\begin{aligned}
A(t,i) &= A_0(t,i) + \frac{\lambda_{ii}}{2} I_H, \\
\Pi_1(t,X)(i) &= \sum_{k=1}^{r} A_k^*(t,i) X(i) A_k(t,i) + \sum_{j\in\mathcal{Z}, j\neq i} \lambda_{ij} X(j) \\
\Pi_{12}(t,X)(i) &= \sum_{k=1}^{r} A_k^*(t,i) X(i) B_k(t,i) \qquad (8)\\
\Pi_2(t,X)(i) &= \sum_{k=1}^{r} B_k^*(t,i) X(t,i) B_k(t,i),
\end{aligned}$$

for all $i \in \mathcal{Z}, t \in \mathbf{R}_+$. If H, U are finite dimensional, we introduce (8) in (7) and we obtain the Riccati equation from [10] associated with the optimal control problem which consist in minimizing the performance

$$\begin{aligned}
J(u) &= E\int_{t_0}^{\infty} [\langle M(t,\eta(t))x(t), x(t)\rangle + 2\langle D(t,\eta(t))u(t), x(t)\rangle \qquad (9)\\
&\quad + \langle R(t,\eta(t))u(t), u(t)\rangle] dt
\end{aligned}$$

subject to

$$\begin{aligned}
dx(t) &= [A_0(t,\eta(t))x(t) + B(t,\eta(t))u(t)]dt + \qquad (10)\\
&\quad \sum_{k=1}^{r} [A_k(t,\eta(t))x(t) + B_k(t,\eta(t))u(t)] dw_k(t),
\end{aligned}$$

within a certain class of admissible control policies. In (9) and (10) $E[.]$ is the expectation and $w(t) = (w_1(t), w_2(t), .., w_r(t)), t \in \mathbf{R}_+$ is a standard r dimensional Wiener process.

In the absence of the Markov perturbations (case $\mathcal{Z} = \{1\}$), (7) and (8) define an infinite dimensional Riccati equation similar to the one in [27], [8]. (We mention here that, unlike our case, the Riccati equations in [27], [8] have unbounded coefficients.)

For another example let $\mathcal{Z}$ be infinite, H, U finite dimensional and let $\Lambda = (\lambda_{ij})_{i,j\in\mathcal{Z}}$, with $\sum_{j\in\mathcal{Z}} \lambda_{ij} = 0$, $\lambda_{ij} \geq 0$ for $i \neq j$ and $0 \leq -\lambda_{ii} \leq c < \infty, i \in \mathcal{Z}$, be the infinitesimal matrix of the transition probability matrix function associated with a stationary standard conservative Markov process. If we consider the autonomous version of (7) with the coefficients

$$A(t,i) = A_0(i) + \frac{\lambda_{ii}}{2} I_H, \Pi_1(t,X)(i) = \sum_{j\in\mathcal{Z}, j\neq i} \lambda_{ij} X(j), i \in \mathcal{Z}, \Pi_{12}(t) \equiv 0,$$

$$\Pi_2(t) \equiv 0, B(t) = B, D(t) \equiv 0, M(t) = M \succeq 0, R(t) = R \succeq 0, t \in \mathbf{R}_+,$$

we get exactly the Riccati equation from [16] associated with the infinite-time LQ optimization problem (9) - (10), where $A_k(t,i) = 0, B_k(t,i) = 0, k = \overline{1,r}$ for all $i \in \mathcal{Z}, t \in \mathbf{R}_+$ and $A(t), B(t), D(t), M(t), R(t), t \in \mathbf{R}_+$ are defined as above.

3 Preliminaries to positive evolutions

In this section we assume that A and Π_1 are defined as in (H1) and that, for all $t \in \mathbf{R}_+$, $\Pi_1(t)$ is a positive and m- strongly continuous operator. First we note that $\mathcal{K}_H^{\mathcal{Z}} \subset l_{\mathcal{E}_H}^{\mathcal{Z}}$ is a closed, solid, normal, convex cone (see [15]) and, consequently, the real Banach space $\left(l_{\mathcal{E}_H}^{\mathcal{Z}}, \|.\|_{\mathcal{Z}}\right)$, ordered by the ordering relation induced by $\mathcal{K}_H^{\mathcal{Z}}$, is a space satisfying the conditions in [15].

Now, let $\Delta = \{(t,s), 0 \leq s \leq t\} \subset \mathbf{R}^2$ and for any $T > 0$, let $\Delta_T = \{(t,s), 0 \leq s \leq t \leq T\} \subset \mathbf{R}^2$.

For a given $\mathcal{L} \in C\left(\mathbf{R}_+, L\left(l_{\mathcal{E}_H}^{\mathcal{Z}}\right)\right)$ and $(t_0, Y_0) \in \mathbf{R}_+ \times l_{\mathcal{E}_H}^{\mathcal{Z}}$, we consider the backward linear equation

$$\frac{dY(t)}{dt} + \mathcal{L}(t)Y(t) = 0, t, t_0 \in \mathbf{R}_+, t \leq t_0 \quad (11)$$

$$Y(t_0) = Y_0. \quad (12)$$

It is known [15] that equation (11), (12) has a unique continuously differentiable solution $Y(t, t_0; Y_0)$. Let us denote by $\widetilde{\Phi}(t, t_0)$ the solution operator of (11) defined by $\widetilde{\Phi}(t, t_0)(Y_0) = Y(t, t_0; Y_0)$, for all $(t, t_0) \in \Delta$ and $Y_0 \in l_{\mathcal{E}_H}^{\mathcal{Z}}$. Following [15], we say that $\mathcal{L}$ defines an *anticausal positive evolution* on $l_{\mathcal{E}_H}^{\mathcal{Z}}$ iff $\widetilde{\Phi}(t, t_0)$ is a positive operator for all $(t, t_0) \in \Delta$. In what follows we will consider the mapping

$$\mathcal{L}(t)(X) := A(t)^{[*]} X + XA(t), X \in l_{\mathcal{E}_H}^{\mathcal{Z}}, t \in \mathbf{R}_+, \quad (13)$$

which belongs to $C_b\left(\mathbf{R}_+, L\left(l_{\mathcal{E}_H}^{\mathcal{Z}}\right)\right)$, by (H1). We will show that it defines an anticausal positive evolution on $l_{\mathcal{E}_H}^{\mathcal{Z}}$.

From Theorem 5.5.1 in [22], we see that, for any $i \in \mathcal{Z}$, $A(.,i) \in C_b(\mathbf{R}_+, H)$ generates an evolution operator $U(t,s;i)$ on H with the following properties:

u_0) $U(s,s;i) = I_H, U(t,s;i)U(s,r;i) = U(t,r;i)$, for all $0 \leq r \leq s \leq t$;

u_1) $\|U(t,s;i)\| \leq e^{\alpha(t-s)}$ where $\alpha = |A|_{\mathbf{R}_+} := \sup_{t \in \mathbf{R}_+} \|A(t)\|_{\mathcal{Z}}$, and $\|U(t,s;i) - I_H\| \leq \alpha e^{\alpha T}(t-s)$ for all $0 \leq s \leq t \leq T$.

u_2) the mapping $(t,s) \to U(t,s;i)$ is $\|.\|$ continuous on Δ, uniformly with respect to i (by u_1));

u_3) $\frac{\partial U(t,s;i)}{\partial t} = A(t,i)U(t,s;i)$ uniformly with respect to i for $t \geq s$;

u_4) $\frac{\partial U(t,s;i)}{\partial s} = -U(t,s;i)A(s,i)$ uniformly with respect to i for $s \leq t$;

Further, for all $(t,s) \in \Delta$, we define $U(t,s) := \{U(t,s;i), i \in \mathcal{Z}\}$, and $\Phi(t,s)(X) := U(t,s)^{[*]} XU(t,s), X \in l_{\mathcal{E}_H}^{\mathcal{Z}}$. We observe that u_3) and u_4) imply that $U(t,s)$ is continuously differentiable with respect to t and s, respectively.

Moreover $\frac{\partial U(t,s)}{\partial s} = -U(t,s)A(s)$ and $\frac{\partial U^{[*]}(t,s)}{\partial s} = -A^{[*]}(s)\,U^{[*]}(t,s)$. Obviously $\Phi(t,s)$ is a positive and *m-strongly continuous* operator on $l^{\mathcal{Z}}_{\mathcal{E}_H}$. The above properties of $U(t,s)$ imply that $\Phi(t,s)$ is continuously differentiable in s and

$$\frac{\partial \Phi(t,s)}{\partial s} + \mathcal{L}(s)\,\Phi(t,s) = 0,$$
$$\Phi(t,t) = I_{l^{\mathcal{Z}}_{\mathcal{E}_H}}.$$

Therefore $\Phi(t,t_0)\,Y_0$ is a continuously differentiable solution of (11), (12). From the uniqueness of the solution it follows that $\Phi(t,s)$ is the solution operator of (11) with $\mathcal{L}$ defined by (13). Because the operator $\Phi(t,s)$ is positive for all $(t,s) \in \Delta$, we conclude that $\mathcal{L}$ generates a positive anticausal evolution on $l^{\mathcal{Z}}_{\mathcal{E}_H}$.

Let $Q \in C([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H})$ and $R \in l^{\mathcal{Z}}_{\mathcal{E}_H}$. We consider the following Lyapunov-type equation $\{\mathcal{L}, \Pi_1; Q\}$

$$\frac{dX(s)}{ds} + \mathcal{L}(s, X(s)) + \Pi_1(s, X(s)) + Q(s) = 0, s \leq T \tag{14}$$

$$X(T) = R \in l^{\mathcal{Z}}_{\mathcal{E}_H}. \tag{15}$$

It is well-known that (14)-(15) has a unique solution $X \in C^1_b([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H})$ (see for e.g. [6] and [9]). Moreover, this solution, often denoted by $X(T,s;R)$, is also the unique solution in $C^1_b([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H})$ of the integral equation

$$X(s) = \Phi(T,s)(R) + \int_s^T \Phi(r,s)\left(\Pi_1(r, X(r)) + Q(r)\right) dr, \tag{16}$$

for all $s \leq T$. A mapping $X \in C^1(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H})$ which satisfies (14) for all $t \geq 0$ is called *a global solution* of (14). It is not difficult to see that a *global solution* X of (14) is also a solution of

$$X(s) = \Phi(t,s)(R) + \int_s^t \Phi(r,s)\left(\Pi_1(r, X(r)) + Q(r)\right) dr, 0 \leq s \leq t. \tag{17}$$

In the case when $Q = 0$, the Lyapunov equation $\{\mathcal{L}, \Pi_1; Q\}$ is denoted by $\{\mathcal{L}, \Pi_1; 0\}$. Since $\mathcal{L} + \Pi_1 \in C(\mathbf{R}_+, L(l^{\mathcal{Z}}_{\mathcal{E}_H}))$, we apply Proposition 3.3 from [15] to deduce that $\mathcal{L} + \Pi_1$ generates a positive anticausal evolution on $l^{\mathcal{Z}}_{\mathcal{E}_H}$; the associated solution operator will be denoted by $\Phi_{\Pi_!}(t,s)$. (By definition, $\Phi_{\Pi_!}(T,s)(R)$ is the unique solution of $\{\mathcal{L}, \Pi_1; 0\}$ with the final condition (15) and it satisfies (16).)

The perturbation theory (see [6] and Proposition 7 in [25]) ensures that the unique solution of (14)-(15) is given by

$$X(s) = \Phi_{\Pi_1}(T,s)(R) + \int_s^T \Phi_{\Pi_1}(r,s)(Q(r))\, dr. \tag{18}$$

Using (18) and the positivity of the anticausal evolution operator $\Phi_{\Pi_1}(t,s)$ we get the next result.

Lemma 4 *Assume that $Q \in C([0,T],\mathcal{K}_H^{\mathcal{Z}})$ and $R \in \mathcal{K}_H^{\mathcal{Z}}$. Then (14)-(15) has a unique solution $X(T,.;R)$ in $C_b^1([0,T],\mathcal{K}_H^{\mathcal{Z}})$. Moreover, $X(T,.;R_1) \succeq X(T,.;R_2)$ for all $R_1 \succeq R_2 \succeq 0$.*

We say that $\mathcal{L}+\Pi_1$ generates an *anticausal uniformly exponentially stable* evolution if there are $\alpha \in (0,1), \beta > 1$ such that

$$\|\Phi_{\Pi_!}(t,s)\| \leq \beta\alpha^{(t-s)}, \tag{19}$$

for all $0 \leq s \leq t$. Lemma 6 from [26] implies that (19) is equivalent to

$$\Phi_{\Pi_!}(t,s)\left(\mathcal{I}^H\right) \preceq \beta\alpha^{(t-s)}\mathcal{I}^H. \tag{20}$$

The following theorem is a direct consequence of Theorems 4.4 and 4.5 from [15].

Theorem 5 *The following statements are equivalent*

a) $\mathcal{L}+\Pi_1$ generates an anticausal uniformly exponentially stable evolution;

b) there is $M>0$ such that $\int\limits_s^\infty \Phi_{\Pi_1}(r,s)\left(\mathcal{I}^H\right)dr \leq M\mathcal{I}^H$ for all $s \geq 0$;

c) there is a uniformly positive mapping $Q \in C_b(\mathbf{R}_+,\mathcal{K}_H^{\mathcal{Z}})$, such that the Lyapunov equation $\{\mathcal{L},\Pi_1;Q\}$ has a global solution $X \in C_b^1(\mathbf{R}_+,\mathcal{K}_H^{\mathcal{Z}})$.

d) for any $Q \in C_b(\mathbf{R}_+,l_{\mathcal{E}_H}^{\mathcal{Z}})$, the Lyapunov equation $\{\mathcal{L},\Pi_1;Q\}$ has a unique global solution $X \in C_b^1(\mathbf{R}_+,l_{\mathcal{E}_H}^{\mathcal{Z}})$ given by $X(s) = \int\limits_s^\infty \Phi_{\Pi_1}(r,s)(Q(r))\,dr$.

4 Generalized Riccati equations

Assume that (H1) holds and let us denote

$$\Pi(t,X) = \begin{pmatrix} \Pi_1(t,X) & \Pi_{12}(t,X) \\ \Pi_{12}(t,X)^{[*]} & \Pi_2(t,X) \end{pmatrix} \tag{21}$$

and

$$\mathcal{Q}(t) = \begin{pmatrix} M(t) & D(t) \\ D(t)^{[*]} & R(t) \end{pmatrix}.$$

It follows that $\Pi \in C_b\left(\mathbf{R}_+, L\left(l_{\mathcal{E}_H}^{\mathcal{Z}}, l_{\mathcal{E}_{H\times U}}^{\mathcal{Z}}\right)\right)$, $\mathcal{Q} \in C_b\left(\mathbf{R}_+, l_{\mathcal{E}_{H\times U}}^{\mathcal{Z}}\right)$. In the rest of the paper we need the additional hypothesis:

(H2) $\Pi(t)$ is a positive (i.e. $\Pi(t)\left(\mathcal{K}_H^{\mathcal{Z}}\right) \subset \mathcal{K}_{H\times U}^{\mathcal{Z}}$) and m-strongly continuous operator for each $t \in \mathbf{R}_+$.

The equation (7) can be written in a compact form as

$$\frac{d}{dt}X(t,i)+\mathcal{R}(t,X(t))(i)=0, i\in\mathcal{Z},\ t\in\mathbf{R}_+, \tag{22}$$

where $\mathcal{R}:Dom\mathcal{R}\rightarrow l_{\mathcal{E}_H}^{\mathcal{Z}}$ is defined by

$$\mathcal{R}(t,X)=\mathcal{L}(t)(X)+\Pi_1(t,X)+M(t)- \tag{23}$$
$$[XB(t)+\Pi_{12}(t,X)+D(t)][R(t)+\Pi_2(t,X)]^{[-1]}[XB(t)+\Pi_{12}(t,X)+D(t)]^{[*]},$$

$Dom\mathcal{R}=\left\{(t,X)\in\mathbf{R}_+\times l_{\mathcal{E}_H}^{\mathcal{Z}}|R(t,i)+\Pi_2(t,X)(i)\text{ is invertible for all } i\in\mathcal{Z}\right\}$ and $\mathcal{L}(t)$ is defined by (13). Let $Y\in l_{\mathcal{E}_H}^{\mathcal{Z}}$ and $T\in\mathbf{R}_+$ be fixed. We first consider the problem of solving (22) with the final condition

$$X(T)=Y, \tag{24}$$

on $[0,T]$. In other words, we look for a continuously differentiable mapping $X:[0,T]\rightarrow l_{\mathcal{E}_H}^{\mathcal{Z}}$ with the property $\{(t,X(t)),t\in[0,T]\}\subset Dom\mathcal{R}$ that verifies (22)-(24). This mapping, if it exists, will be denoted by $X(T,t;Y)$ and will be called a *solution* of (22)-(24).

A mapping $X\in C^1\left(\mathbf{R}_+,l_{\mathcal{E}_H}^{\mathcal{Z}}\right)$ satisfying $\{(t,X(t)),t\in\mathbf{R}_+\}\subset Dom\mathcal{R}$ and (22) for all $t\in\mathbf{R}_+$ is called a global solution of (22); if X is bounded we say that (22) has a *bounded solution.*

Let $\Phi(t,s)$ be the solution operator of (11) with $\mathcal{L}$ defined by (13). For all $X\in\mathcal{E}_H,(t,s)\in\Delta$ and $i\in\mathcal{Z}$ we define $\Phi(t,s;i)(X)=U(t,s;i)^*XU(t,s;i)$ and we observe that $\Phi(t,s)(X)(i)=\Phi(t,s;i)X(i)$. As in the case of the Lyapunov type equations, a solution of (22)-(24) also verifies the interconnected integral equations

$$X(t,i)=\Phi(T,t)(Y)(i)+\int_t^T\Phi(u,t;i)[\mathcal{R}(u,X(u))- \tag{25}$$
$$\mathcal{L}(u)(X(u))](i)\,du, i\in\mathcal{Z},t\in[0,T]$$

and, conversely, any mapping $X\in C\left([0,T],l_{\mathcal{E}_H}^{\mathcal{Z}}\right)$ satisfying $\{(t,X(t)),t\in[0,T]\}\subset Dom\mathcal{R}$ and (25)(which we called a solution of (25)) is also a solution of (22)-(24). We observe that if X is a solution of (25), then the Bochner integral $Z(t)=\int_t^T\Phi(u,t)[\mathcal{R}(u,X(u))-\mathcal{L}(u)(X(u))]\,du$ exists and the operators $P_i\in L\left(l_{\mathcal{E}_H}^{\mathcal{Z}},\mathcal{E}_H\right)$ defined by $P_i(X)=X(i),i\in\mathcal{Z}$ commute with it. We conclude, via (25), that $P_i(X(t)-\Phi(T,t)(Y)-Z(t))=0$ for all $i\in\mathcal{Z}$. Hence $X(t)-\Phi(T,t)(Y)-Z(t)=0$ and X satisfies the integral equation

$$X(t)=\Phi(T,t)(Y)+\int_t^T\Phi(u,t)[\mathcal{R}(u,X(u))-\mathcal{L}(u)(X(u))]\,du. \tag{26}$$

Now it is clear that $X \in C^1\left([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H}\right), \{(t, X(t)), t \in [0,T]\} \subset Dom\mathcal{R}$, $X(T) = Y$ and

$$\frac{d}{dt}X(t) + \mathcal{R}(t, X(t)) = 0. \tag{27}$$

We just have proved that a solution of (22)-(24) is also a solution of (27)-(24). The converse is obviously true.

In what follows we will say that the operator $\mathcal{R}$ and the equation (22) are defined by the quadruple $\Sigma = (A, B, \Pi, \mathcal{Q})$. Now we associate with Σ the dissipation operator D^{Σ} defined by

$$D^{\Sigma}(t, X(t)) = \begin{pmatrix} d_1^{\Sigma}(t, X(t)) & d_2^{\Sigma}(t, X(t)) \\ d_2^{\Sigma}(t, X(t))^{[*]} & R(t) + \Pi_2(t)(X(t)) \end{pmatrix}. \tag{28}$$

where

$$\begin{aligned} d_1^{\Sigma}(t, X(t)) &= \frac{d}{dt}X(t) + \mathcal{L}(t)(X(t)) + \Pi_1(t, X(t)) + M(t), \\ d_2^{\Sigma}(t, X(t)) &= X(t)B(t) + \Pi_{12}(t, X(t)) + D(t). \end{aligned}$$

As in [12], we define the subsets Γ^{Σ} and $\widetilde{\Gamma}^{\Sigma}$ of $C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$:

$$\begin{aligned} \Gamma^{\Sigma} &= \{X \in C_b^1\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right) | R(t) + \Pi_2(t, X(t)) \succ 0, D^{\Sigma}(t, X(t)) \succeq 0, t \in \mathbf{R}_+\} \\ \widetilde{\Gamma}^{\Sigma} &= \{X \in C_b^1\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right) | D^{\Sigma}(t, X(t)) \succ 0, t \in \mathbf{R}_+\}. \end{aligned}$$

Lemma 6 *The following two statements hold true.*

a) $\widetilde{\Gamma}^{\Sigma} \subset \Gamma^{\Sigma}$.

b) Γ^{Σ} *contains all global and bounded solutions* $X \in C^1\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$ *of (7) which verify*

$$R(t) + \Pi_2(t, X(t)) \succ 0, t \in \mathbf{R}_+. \tag{29}$$

Proof. a) Since $D^{\Sigma}(t, X(t)) \succ 0, t \in \mathbf{R}_+$ implies $R(t) + \Pi_2(t, X(t)) \succ 0, t \in \mathbf{R}_+$, it follows that $\widetilde{\Gamma}^{\Sigma} \subset \Gamma^{\Sigma}$ and the proof is complete.

b) If $X(t)$ is a bounded solution of (7) satisfying (29), then $\frac{d}{dt}X(t) + \mathcal{R}(t, X(t)) = 0$ is exactly the Schur complement of $R(t) + \Pi_2(t, X(t))$ in $D^{\Sigma}(t, X(t))$ for all $t \in \mathbf{R}_+$. Lemma 3 implies that $D^{\Sigma}(t, X(t)) \succeq 0, t \in \mathbf{R}_+$.

Furthermore, from the boundedness of X and (29), it follows that $[R(t) + \Pi_2(t, X(t))]^{-1}$, $t \in \mathbf{R}_+$ and $\mathcal{R}(t, X(t)), t \in \mathbf{R}_+$ are bounded. Hence $X \in C_b^1\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$. We deduce that $X \in \Gamma^{\Sigma}$ and the proof is complete. ■

The above lemma shows that a global and bounded solution of (7) satisfying (29) belongs to $C_b^1\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$.

Let us consider a mapping $W : \mathbf{R}_+ \to l^{\mathcal{Z}}_{L(H,U)}$ and $X \in l^{\mathcal{Z}}_{\mathcal{E}_H}$. For notational convenience let

$$\Pi_W(t, X) = \begin{pmatrix} \mathcal{I}^H & W(t)^{[*]} \end{pmatrix} \Pi(t, X) \begin{pmatrix} \mathcal{I}^H \\ W(t) \end{pmatrix}, \tag{30}$$

$$Q_W(t) = \begin{pmatrix} \mathcal{I}^H & W(t)^{[*]} \end{pmatrix} \mathcal{Q}(t) \begin{pmatrix} \mathcal{I}^H \\ W(t) \end{pmatrix}, t \in \mathbf{R}_+. \tag{31}$$

By a direct computation (see also [12]) we get the following useful formula

$$\mathcal{R}(t,X) = [(A+BW)(t)]^{[*]} X + X(A+BW)(t) + \Pi_W(t,X) \tag{32}$$
$$+Q_W(t) - \left(W(t) - F^X(t)\right)^{[*]} [R(t) + \Pi_2(t,X)] \left(W(t) - F^X(t)\right),$$

where

$$F^X(t) = -[R(t) + \Pi_2(t,X)]^{[-1]} d_2^{\Sigma}(t,X)^{[*]}. \tag{33}$$

The following theorem gives a sufficient condition for the existence of the solution $X(T,t;Y)$, $0 \le t \le T$, of (7)-(24) or, equivalently, of (22)-(24).

Theorem 7 *Let $\widehat{X} \in \Gamma^{\Sigma}$. If $Y \in l^{\mathcal{Z}}_{\mathcal{E}_H}$ is such that $Y \succeq \widehat{X}(T)$, then (22)-(24) has a unique solution $X(T,.;Y) \in C_b^1\left([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$.*

Proof. *Existence. Let $k \in \mathbf{N}$* and $F_k \in C([0,T], l^{\mathcal{Z}}_{L(H,U)})$ be given. We consider the Lyapunov equation

$$d_1^{\Sigma}(t, X_{k+1}(t)) + F_k(t)^{[*]} d_2^{\Sigma}(t, X_{k+1}(t))^{[*]} \tag{34}$$
$$+d_2^{\Sigma}(t, X_{k+1}(t)) F_k(t) + F_k(t)^{[*]} [R(t) + \Pi_2(t, X_{k+1}(t))] F_k(t) = 0$$
$$X_{k+1}(T) = Y, k \in \mathbf{N}^*. \tag{35}$$

By virtue of (32), it follows that (34) can be equivalently rewritten as

$$\frac{d}{dt} X_{k+1}(t) + [A(t) + B(t) F_k(t)]^{[*]} X_{k+1}(t) + X_{k+1}(t) [A(t) + B(t) F_k(t)]$$
$$+\Pi_{F_k}(t, X_{k+1}(t)) + Q_{F_k}(t) = 0, \tag{36}$$

Since $\Pi_{F_k}(t) \in C([0,T], L\left(l^{\mathcal{Z}}_{\mathcal{E}_H}\right))$ and $Q_{F_k}(t) \in C([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H})$ we deduce from the results of the previous section that, the Lyapunov equation (36) - (35) has a unique solution $X_{k+1} \in C_b^1([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H})$.

On the other hand, $\widehat{X} \in \Gamma^{\Sigma}$ implies $\frac{d}{dt}\widehat{X}(t) + \mathcal{R}\left(t, \widehat{X}(t)\right) \succeq 0$, $t \in \mathbf{R}_+$. Taking $\widehat{Q}(t) = \frac{d}{dt}\widehat{X}(t) + \mathcal{R}\left(t, \widehat{X}(t)\right) \succeq 0, t \in \mathbf{R}_+$, we see that $\widehat{X}$ satisfies the Riccati equation

$$\frac{d}{dt}\widehat{X}(t) + \mathcal{R}\left(t, \widehat{X}(t)\right) - \widehat{Q}(t) = 0, t \in \mathbf{R}_+. \tag{37}$$

Applying (32) with $X(t)$ and $W(t)$ replaced by $\widehat{X}(t)$ and $F_k(t)$, respectively, we may rewrite (37) in a form similar to (36). We subtract the obtained equation from (36) and we see that $\Delta_{k+1} = X_{k+1} - \widehat{X}$ is the unique solution of the equation

$$\frac{d}{dt}\Delta_{k+1}(t) + [A(t) + B(t) F_k(t)]^{[*]} \Delta_{k+1}(t) + \Delta_{k+1}(t) [A(t) + B(t) F_k(t)]$$
$$+\Pi_{F_k}(t, \Delta_{k+1}) + \widehat{Q}(t) \tag{38}$$
$$+\left[F^{\widehat{X}}(t) - F_k(t)\right]^{[*]} \left[R(t) + \Pi_2\left(t, \widehat{X}(t)\right)\right] \left[F^{\widehat{X}}(t) - F_k(t)\right] = 0$$
$$\Delta_{k+1}(T) = Y - \widehat{X}(T) \succeq 0.$$

The hypotheses of Lemma 4 hold and we get $X_{k+1}(t) - \widehat{X}(t) \succeq 0, t \in [0,T]$. Hence $R(t) + \Pi_2(t, X_{k+1}(t)) \succeq R(t) + \Pi_2\left(t, \widehat{X}\right) \succ 0, t \in [0,T]$, $F_{k+1}(t) = -[R(t) + \Pi_2(t, X_{k+1}(t))]^{[-1]} d_2^{\Sigma}(t, X_{k+1}(t))^{[*]}$ is well defined and $F_{k+1} \in C([0,T], l^{\mathcal{Z}}_{L(H,U)})$. Repeating the above reasoning for k replaced by $k+1, k+2, .., m, ..$ we get a sequence $\{X_m\}_{m \geq k+1} \subset C_b^1([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H})$ of solutions of the equations (36) - (35) with the property

$$X_m(t) - \widehat{X}(t) \succeq 0, t \in [0,T], m \geq k+1. \tag{39}$$

Therefore, if we start with $k = 0$ and $F_0 = 0$, we obtain inductively a sequence $\{X_m\}_{m \geq 1}$ of solutions of (36) - (35) satisfying (39). Now we observe that, for any $k \geq 1$, $\Lambda_{k+1} = X_k - X_{k+1}$ verifies the Lyapunov equation

$$\begin{gathered}\frac{d}{dt}\Lambda_{k+1}(t) + [A(t) + B(t)F_k(t)]^{[*]}\Lambda_{k+1}(t) + \Lambda_{k+1}(t)[A(t) + B(t)F_k(t)] \\ +\Pi_{F_k}(t, \Lambda_{k+1}) \\ + [F_{k-1}(t) - F_k(t)]^{[*]}[R(t) + \Pi_2(t, X_k(t))][F_{k-1}(t) - F_k(t)] = 0 \\ \Lambda_{k+1}(T) = 0.\end{gathered} \tag{40}$$

Lemma 4 implies that $X_k(t) - X_{k+1}(t) \succeq 0, t \in [0,T]$. Taking into account (39), we get

$$\widehat{X}(t) \preceq .. \preceq X_{k+1}(t) \preceq X_k(t) \preceq .. \preceq X_1(t), t \in [0,T].$$

It follows that $\widehat{X}(t,i) \leq ... \leq X_{k+1}(t,i) \leq X_k(t,i) \leq ... \leq X_1(t,i), t \in [0,T], i \in \mathcal{Z}$. Since a monotone and bounded sequence of linear operators is strongly convergent, we deduce that there is $S(t,i) \in \mathcal{E}_H$,

$$\widehat{X}(t,i) \leq S(t,i) \leq X_1(t,i) \tag{41}$$

such that $X_k(t,i)$ converges strongly, as $k \to \infty$, to $S(t,i)$. Evidently $S(t) = \{S(t,i)\}_{i \in \mathcal{Z}} \in l^{\mathcal{Z}}_{\mathcal{E}_H}$ for all $t \in [0,T]$.

On the other hand, in view of (H2), Π_2 is *m-strongly continuous* and positive and the sequence $\{R(t,i) + \Pi_2(t, X_k(t))(i)\}_{k \in \mathbf{N}^*}$ is strongly convergent to $R(t,i) + \Pi_2(t, S(t))(i) \geq R(t,i) + \Pi_2\left(t, \widehat{X}(t)\right)(i), i \in \mathcal{Z}, t \in [0,T]$. It follows that $R(t) + \Pi_2(t, S(t)) \succ 0, t \in [0,T]$ and $(t, S(t)) \in Dom\mathcal{R}$, $t \in [0,T]$. Using again (H2) we deduce that

$$F_k(t,i)x \underset{k \to \infty}{\to} -[R(t,i) + \Pi_2(t, S(t))(i)]^{-1}\left[d_2^{\Sigma}(t, S(t))(i)\right]^* x$$

for all $i \in \mathcal{Z}$, $t \in [0,T]$ and $x \in H$. Letting $k \to \infty$, componentwise, in the integral equation (see (16)) satisfied by the solution of (34) - (35) and applying the Lebesgue Dominated Convergence Theorem we see that

$$S(t,i)x = \Phi(T,t)(Y)(i)x + \int_t^T \Phi(u,t)[\mathcal{R}(u, S(u)) - \mathcal{L}(u)(S(u))](i)x du \tag{42}$$

for all $x \in H$. Further, we note that

$$t \to \Phi(u,t)\left[\mathcal{R}(u,S(u)) - \mathcal{L}(u)(S(u))\right](i)$$

is $\|.\|$- continuous, uniformly with respect to u and i and it follows that $t \to S(t,i)$ is $\|.\|$- continuous uniformly with respect to i. Hence $S \in C([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H})$ and $\int\limits_t^T \Phi(u,t)[\mathcal{R}(u,S(u)) - A^{[*]}(u)S(u) - S(u)A(u)]du$ is well defined. Now it is clear that S satisfies (26). On the other hand (41) and the properties of $\widehat{X}(t)$ imply that $u \to \mathcal{R}(u,S(u)) - \mathcal{L}(u)(S(u))$ is bounded on $[0,T]$. Thus, differentiating (26) with respect to t, we deduce that $S(t)$ is a solution of (22)-(24) in $C^1_b([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H})$.

Uniqueness: Assume that $X \in C^1_b([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H})$ is another solution of (22)-(24) and let $Z(t) = S(t) - X(t)$. Applying (32) with W replaced by F^S for both $\mathcal{R}(t,S)$ and $\mathcal{R}(t,X)$, we rewrite the two differential equations satisfied by X and S and, subtracting them, we obtain

$$\frac{d}{dt}Z(t) + \left[A(t) + B(t)F^S(t)\right]^{[*]} Z(t) + Z(t)\left[A(t) + B(t)F^S(t)\right] +$$

$$\Pi_{F^S}(t, Z(t)) + \left[F^X(t) - F^S(t)\right]^{[*]}\left[R(t) + \Pi_2(t, X(t))\right]\left[F^X(t) - F^S(t)\right] = 0$$

$$Z(T) = 0.$$

This equation may be viewed as a Lyapunov equation in Z. It is not difficult to see that it satisfies the hypotheses of Lemma 4 and, consequently, $Z(t) \succeq 0, t \in [0,T]$. Hence $S(t) \succeq X(t)$ for all $t \in [0,T]$. By interchanging the roles of X and S, we also obtain $X(t) \succeq S(t), t \in [0,T]$ and the conclusion follows. ■

A direct consequence of the Theorem 7 is the following.

Corollary 8 *Assume $0 \in \Gamma^{\Sigma}$. For any $Y \in \mathcal{K}^{\mathcal{Z}}_H$ and $T > 0$, the Riccati equation (22)-(24) has a unique solution $X(T,.;Y) \in C^1_b\left([0,T], \mathcal{K}^{\mathcal{Z}}_H\right)$. Moreover, if $Y_1, Y_2 \in \mathcal{K}^{\mathcal{Z}}_H$ are such that $Y_1 \succeq Y_2$, then $X(T,t;Y_1) \succeq X(T,t;Y_2)$.*

Proof. Note that $0 \in \Gamma^{\Sigma}$ is equivalent with $\mathcal{Q} \succeq 0$, $R(t) \succ 0, t \in \mathbf{R}_+$. From the above theorem, (22)-(24) has a unique solution $X(T,.;Y) \in C^1_b\left([0,T], l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$. By (32), $X(T,.;Y)$ is the unique solution of the Lyapunov equation

$$\frac{dX(t)}{dt} + \left[(A + BF^X)(t)\right]^{[*]} X(t) + X(t)\left(A + BF^X\right)(t) + \Pi_{F^X}(t, X(t))$$
$$+ Q_{F^X}(t) = 0.$$

Since $Q_{F^X} \in C\left([0,T], \mathcal{K}^{\mathcal{Z}}_H\right)$ and $\Pi_{F^X}(t)$ is positive for all $t \in [0,T]$, we deduce from Lemma 4 that $X(t) \succeq 0$ for all $t \in [0,T]$ and, therefore, $X(T,.;Y) \in C^1_b\left([0,T], \mathcal{K}^{\mathcal{Z}}_H\right)$. It remains to prove the last statement of the corollary. If we denote $X_i(t) = X(T,t;Y_i)$, $i = 1,2$, we deduce from (32) that $Y = X_1 - X_2$

satisfies

$$\frac{d}{dt}Y(t) + \mathcal{L}_{F^{X_1}}(t, Y(t)) + \Pi_{F^{X_1}}(t, Y(t)) +$$
$$\left(F^{X_1}(t) - F^{X_2}(t)\right)^{[*]}\left[R(t) + \Pi_2(t, X_1(t))\right]\left(F^{X_1}(t) - F^{X_2}(t)\right) = 0$$
$$Y(T) = R_1 - R_2 \succeq 0.$$

Arguing as above we see that the hypotheses of Lemma 4 fulfilled and $Y(t) \succeq 0, t \in [0, T]$. We get the conclusion. ■

Definition 9 *We say that a global and bounded solution $X \in C^1\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$ of (22) is maximal if $X(t) \succeq \widehat{X}(t), t \in \mathbf{R}_+$ for arbitrary $\widehat{X} \in \Gamma^{\Sigma}$.*

Definition 10 *A global solution $X_{\min} \in C^1\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$ of (22) is minimal in the class of all nonnegative global solutions of (22) if $0 \preceq X_{\min}(t) \preceq X(t), t \in \mathbf{R}_+$ for any solution $X(t) \in C^1\left(\mathbf{R}_+, \mathcal{K}^{\mathcal{Z}}_H\right)$ of (22).*

Let A, B be the mappings defined by (H1) and $F \in C(\mathbf{R}_+, l^{\mathcal{Z}}_{L(H,U)})$. For all $t \in \mathbf{R}_+$, we introduce the operator $\mathcal{L}_F(t) \in L\left(l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$ defined by

$$\mathcal{L}_F(t)(Y) = \left[A(t) + B(t)F(t)\right]^{[*]} Y + Y\left[A(t) + B(t)F(t)\right], Y \in l^{\mathcal{Z}}_{\mathcal{E}_H}. \quad (43)$$

Definition 11 *The triple $\{A, B, \Pi\}$ is stabilizable if there is $F \in C_b(\mathbf{R}_+, l^{\mathcal{Z}}_{L(H,U)})$ such that $\mathcal{L}_F + \Pi_F$ generates an anticausal uniformly exponentially stable evolution; F is called a stabilizing feedback gain. Here Π_F is defined by (30) with W replaced by F.*

Definition 12 *Let $\mathbb{X} \in C^1\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$ be a global solution of (22) and let $F^{\mathbb{X}}$ be defined by (33) with X replaced by $\mathbb{X}$. We say that a $\mathbb{X}$ is a stabilizing solution of (22) if $F^{\mathbb{X}}$ is a stabilizing feedback gain for the triple $\{A, B, \Pi\}$.*

5 Global solutions of generalized Riccati equations

In this section we give necessary and sufficient conditions for the existence of the maximal, stabilizing and minimal solutions of (22). To simplify the notations,let us denote by $\Theta(t, X(t))$ the mapping $t \to R(t) + \Pi_2(t, X(t))$, where $X \in \Gamma^{\Sigma}$.

In view of the results of Sections 3 and 4, the next theorems can be proved in a similar manner to the ones in [12] and [10]. Therefore we only sketch the proofs in pointing out the differences. The next result is a generalization of Theorem 4.7 in [12] and provides necessary and sufficient conditions for the existence of the maximal solution.

Theorem 13 *Assume that $\{A,B,\Pi\}$ is stabilizable. Then the following are equivalent:*

(i) $\Gamma^{\Sigma}\neq\varnothing$;

(ii) Equation (22) has a maximal solution $\widetilde{X}\in C_b^1\left(\mathbf{R}_+,l_{\mathcal{E}_H}^{\mathcal{Z}}\right)$ with the property $R(t)+\Pi_2(t)\widetilde{X}(t)\succ 0, t\in\mathbf{R}_+$.

Proof (sketch). The implication (ii) $\Longrightarrow$ (i) is obvious, since $\widetilde{X}\in\Gamma^{\Sigma}$ by Lemma 6. Now let us prove that (i) $\Longrightarrow$ (ii). If $\{A,B,\Pi\}$ is stabilizable, then there exists $F_0\in C_b(\mathbf{R}_+,l_{L(H,U)}^{\mathcal{Z}})$ such that $\mathcal{L}_{F_0}+\Pi_{F_0}$ generates an anticausal uniformly exponentially stable evolution.

Let $\varepsilon>0$ be fixed. By Theorem 5 (see the implication (a) $\Longrightarrow$ (d)), the equation

$$\frac{d}{dt}X_1(t)+\mathcal{L}_{F_0}(t)[X_1(t)]+\Pi_{F_0}(t,X_1(t))+Q_{F_0}(t)+\varepsilon\mathcal{I}^H=0, \tag{44}$$

has a unique global solution $X_1\in C_b^1(\mathbf{R}_+,l_{\mathcal{E}_H}^{\mathcal{Z}})$. Recall that $\mathcal{L}_{F_0}$ is obtained by replacing F with F_0 in (43), whereas $\Pi_{F_0}(t)$ and $Q_{F_0}(t)$ are defined by (30) and (31), respectively, with W replaced by F_0.

We shall prove that $X_1(t)\succeq\widehat{X}(t), t\in\mathbf{R}_+$ for all $\widehat{X}\in\Gamma^{\Sigma}$. First we note that, applying (32), the equation (37) can be equivalently rewritten as

$$\begin{gathered}\frac{d}{dt}\widehat{X}(t)+\mathcal{L}_{F_0}\left(t,\widehat{X}(t)\right)+\Pi_{F_0}\left(t,\widehat{X}(t)\right)+Q_{F_0}(t)\\ -\left(F_0(t)-F^{\widehat{X}}(t)\right)^{[*]}\Theta\left(t,\widehat{X}(t)\right)\left(F_0(t)-F^{\widehat{X}}(t)\right)-\widehat{Q}(t)=0,\end{gathered} \tag{45}$$

$t\in\mathbf{R}_+$. From (44) and (45) it follows that the mapping $t\rightarrow X_1(t)-\widehat{X}(t)$ belongs to $C_b^1(\mathbf{R}_+,l_{\mathcal{E}_H}^{\mathcal{Z}})$ and verifies the Lyapunov equation

$$\frac{d}{dt}Y(t)+\mathcal{L}_{F_0}(t,Y(t))+\Pi_{F_0}(t,Y(t))+H_1(t)=0, \tag{46}$$

where $H_1(t)=\varepsilon\mathcal{I}^H+\left(F_0(t)-F^{\widehat{X}}(t)\right)^{[*]}\Theta\left(t,\widehat{X}(t)\right)\left(F_0(t)-F^{\widehat{X}}(t)\right)+\widehat{Q}(t)\succeq\varepsilon\mathcal{I}^H, t\in\mathbf{R}_+$. Since $\mathcal{L}_{F_0}+\Pi_{F_0}$ generates a positive anticausal evolution, we use again Theorem 5 (see the implication (a) $\Longrightarrow$ (d)) to deduce that $X_1(t)-\widehat{X}(t)$ is the unique global nonnegative and bounded solution of (46). Therefore $X_1(t)-\widehat{X}(t)\succeq 0, t\in\mathbf{R}_+$, which implies that $F_1(t):=F^{X_1}(t)$ is well defined. Moreover, $F_1(t)\in C_b(\mathbf{R}_+,l_{L(H,U)}^{\mathcal{Z}})$. Now, let us prove that F_1 is a stabilizing feedback gain for the triple $\{A,B,\Pi\}$. Using (32), we rewrite equation (44) as

$$\begin{gathered}\frac{d}{dt}X_1(t)+\mathcal{L}_{F_1}(t,X_1(t))+\Pi_{F_1}(t,X_1(t))+Q_{F_1}(t)\\ +\varepsilon\mathcal{I}^H+(F_0(t)-F_1(t))^{[*]}\Theta(t,X_1(t))(F_0(t)-F_1(t))=0.\end{gathered} \tag{47}$$

Equation (45) can be rewritten in a similar manner and we get

$$\frac{d}{dt}\widehat{X}(t)+\mathcal{L}_{F_1}\left(t,\widehat{X}(t)\right)+\Pi_{F_1}\left(t,\widehat{X}(t)\right)+Q_{F_1}(t)$$
$$-\widehat{Q}(t)-\left(F^{\widehat{X}}(t)-F_1(t)\right)^{[*]}\Theta\left(t,\widehat{X}(t)\right)\left(F^{\widehat{X}}(t)-F_1(t)\right)=0.$$

Subtracting the last equation from (47) we see that $X_1(t)-\widehat{X}(t)$ is a bounded and nonnegative solution of the Lyapunov equation $\{\mathcal{L}_{F_1},\Pi_{F_1};Q_1\}$, where

$$Q_1(t)=\varepsilon\mathcal{I}^H+\widehat{Q}(t)+(F_0(t)-F_1(t))^{[*]}\Theta(t,X_1(t))(F_0(t)-F_1(t))$$
$$+\left(F^{\widehat{X}}(t)-F_1(t)\right)^{[*]}\Theta\left(t,\widehat{X}(t)\right)\left(F^{\widehat{X}}(t)-F_1(t)\right)\succ 0, t\in\mathbf{R}_+.$$

Clearly $Q_1\in C_b(\mathbf{R}_+,\mathcal{K}_H^{\mathcal{Z}})$ and the statement (c) of Theorem 5 holds. Therefore $\mathcal{L}_{F_1}+\Pi_{F_1}$ generates an anticausal uniformly exponentially stable evolution and F_1 is a stabilizing feedback gain for $\{A,B,\Pi\}$. Starting from $X_1(t)$ and $F_1(t)$, we construct two sequences $X_k(t)$ and $F_k(t), k\in\mathbf{N}, k>1$ such as X_k is the unique global, bounded and nonnegative solution of the Lyapunov equation $\{\mathcal{L}_{F_{k-1}},\Pi_{F_{k-1}};Q_{F_{k-1}}+\frac{\varepsilon}{k}\mathcal{I}^H\}$ and $F_k(t)=F^{X_k}(t)$. We can establish inductively that:

a_k) $X_k(t)-\widehat{X}(t)\succeq 0, t\in\mathbf{R}_+$ for any $\widehat{X}\in\Gamma^{\Sigma}$;

b_k) F_k is a stabilizing feedback gain for the triple $\{A,B,\Pi\}$ and

c_k) $X_k(t)-X_{k+1}(t)\succeq 0, t\in\mathbf{R}_+$ for all $k\in\mathbf{N}, k>1$.

The proof of a_k) - c_k) is very similar to the one in [12] and is omitted. From a_k) and c_k) we deduce that the sequence $\{X_k(t)\}_{k\in\mathbf{N}^*}$ is bounded and decreasing. Therefore $\{X_k(t,i)\}_{k\in\mathbf{N}}$ converges strongly to $S(t,i)\in l_{\mathcal{E}_H}^{\mathcal{Z}}$ for all $i\in\mathcal{Z}$ and $t\in\mathbf{R}_+$. Letting $k\to\infty$, componentwise, in the integral equation (of the type 17) satisfied by the global solution of the Lyapunov equation $\{\mathcal{L}_{F_{k-1}},\Pi_{F_{k-1}};Q_{F_{k-1}}+\frac{\varepsilon}{k}\mathcal{I}^H\}$ and arguing as in the proof of the Theorem 7 we see that $S\in C_b(\mathbf{R}_+,l_{\mathcal{E}_H}^{\mathcal{Z}})$ and satisfies

$$S(t)=\Phi(\tau,t)(S(\tau))+\int_t^{\tau}\Phi(u,t)(\mathcal{R}(u,S(u))-\mathcal{L}(u,S(u)))du. \tag{48}$$

Differentiating (48) with respect to t, we see that S is a global and bounded solution of (22). An appeal to a_k) shows that S is just the maximal solution of equation (22). The proof is complete. ■

Now let us investigate the existence of the stabilizing solution.

Theorem 14 *The following assertions are equivalent:*

(i) $\{A,B,\Pi\}$ *is stabilizable and the set* $\widetilde{\Gamma}^{\Sigma}\neq\varnothing$*;*

(ii) Equation (22) has a stabilizing solution $\mathbb{X}\in C_b^1\left(\mathbf{R}_+,l_{\mathcal{E}_H}^{\mathcal{Z}}\right)$ *satisfying* $R(t)+\Pi_2(t)\mathbb{X}(t)\succ 0, t\in\mathbf{R}_+$.

Proof. (i) $\Longrightarrow$ (ii). Assume (i). Then, by virtue of Theorem 13, the Riccati equation (22) has a maximal solution $\widetilde{X}$. Following [12] we can show that $\widetilde{X}$

is just the stabilizing solution of (22). This proof is very similar to the one of Theorem 5.8. from [12] and will be omitted.

(ii) $\Longrightarrow$ (i) By definition, the triple $\{A, B, \Pi\}$ is stabilizable with the stabilizing gain $F^{\mathbb{X}} \in C_b(\mathbf{R}_+, l^{\mathcal{Z}}_{L(H,U)})$. It remains to prove that $\widetilde{\Gamma}^{\Sigma} \neq \varnothing$. We first note $C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$ is a Banach space when endowed with the norm $|X|_{\mathbf{R}_+} = \sup_{t\in\mathbf{R}_+} \|X(t)\|_{\mathcal{Z}}$. For any $X, X_0 \in C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right), \delta > 0$ we have $|X - X_0|_{\mathbf{R}_+} \leq \delta \Leftrightarrow$"$X_0(t) - \delta\mathcal{I}^H \preceq X(t) \preceq X_0(t) + \delta\mathcal{I}^H$ for all $t \in \mathbf{R}_+$". Now it is easy to see that

$$\mathcal{U} = \{X \in C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right) | R(t) + \Pi_3(t, X(t)) \succ 0, t \in \mathbf{R}_+\}$$

is an open subset of $C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$, which contains the stabilizing and bounded solution $\mathbb{X}$. Let us consider the Riccati equation

$$\frac{d}{dt}X(t) + \mathcal{R}(t, X(t)) - \delta\mathcal{I}^H = 0, t \in \mathbf{R}_+. \tag{49}$$

We shall prove that for an appropriate $\delta \in \mathbf{R}$, (49) has a global solution in $\mathcal{U}$. Let us denote $F^{\mathbb{X}}$ by $\mathbb{F}$. A direct computation and (32)(see [12]) shows that (49) can be equivalently rewritten as

$$\begin{gathered}\frac{d}{dt}X(t) + \left[\mathcal{L}_{\mathbb{F}}(t) + \Pi_{\mathbb{F}}(t)\right](X(t)) + Q_{\mathbb{F}}(t) - \delta\mathcal{I}^H \\ -\mathcal{P}_{\mathbb{F}}(t, X(t))^{[*]}\Theta(t, X(t))^{[-1]}\mathcal{P}_{\mathbb{F}}(t, X(t)) = 0,\end{gathered} \tag{50}$$

where $\mathcal{P}_{\mathbb{F}}(t, X) := \left(\mathbb{F}(t) - F^X\right)\Theta(t, X) = d^{\Sigma}_2(t, X) + \mathbb{F}(t)^{[*]}\Theta(t, X)$. Denoting by $\Phi_{\mathbb{F}}(u, t)$ the solution operator associated with the Lyapunov equation $\{\mathcal{L}_{\mathbb{F}}, \Pi_{\mathbb{F}}; 0\}$, we introduce the mapping $\Psi : \mathbf{R} \times \mathcal{U} \to C_b\left(\mathbf{R}_+, l^{\mathcal{Z}}_{\mathcal{E}_H}\right)$,

$$\begin{gathered}\Psi(\delta, X)(t) = -X(t) + \int_t^{\infty} \Phi_{\mathbb{F}}(u, t)\left(Q_{\mathbb{F}}(u) - \delta\mathcal{I}^H\right. \\ \left.-\mathcal{P}_{\mathbb{F}}(u, X(u))^{[*]}\Theta(u, X(u))^{[-1]}\mathcal{P}_{\mathbb{F}}(u, X(u))\right)du.\end{gathered}$$

From (32) and (18) we get $\mathbb{X}(t) = \int_t^{\infty} \Phi_{\mathbb{F}}(u, t)(Q_{\mathbb{F}}(u))du$ and, since $\mathcal{P}_{\mathbb{F}}(u, \mathbb{X}(u)) \equiv 0$, it follows easily that $\Psi(0, \mathbb{X}) \equiv 0$. It is not difficult to see that the function Ψ is continuously Gateaux differentiable on $\mathbf{R} \times \mathcal{U}$. Hence it is Frechet differentiable. The Gateaux derivative of $\Psi(t, X)$ with respect to X is

$$\begin{gathered}D^G_2\Psi(\delta, X)(Y)(t) = -Y(t) + \\ \int_t^{\infty} \Phi_{\mathbb{F}}(u, t)\left[\mathcal{V}_{\mathbb{F}}(u, Y(u))^{[*]}\Theta(u, X(u))^{[-1]}\mathcal{P}_{\mathbb{F}}(u, X(u))\right) \\ +\mathcal{P}_{\mathbb{F}}(u, X(u))^{[*]}\Theta(u, X(u))^{[-1]}\mathcal{V}_{\mathbb{F}}(u, Y(u))) \\ -\mathcal{P}_{\mathbb{F}}(u, X(u))^{[*]}\Theta(u, Y(u))^{[-1]}\Pi_2(u)(Y(u))\Theta(u, Y(u))^{[-1]}\mathcal{P}_{\mathbb{F}}(u, X(u))]du,\end{gathered}$$

where $\mathcal{V}_{\mathbb{F}}(t, Y(t)) = Y(t) B(t) + \Pi_{12}(t, Y(t)) + \mathbb{F}(t)^{[*]} \Pi_2(t, Y(t))$. We see that $D_2^G \Psi(0, \mathbb{X})(Y)(t) = -Y(t)$ and $Y \to D_2^G \Psi(0, \mathbb{X})(Y)(t)$ is an isomorphism on $C_b\left(\mathbf{R}_+, l_H^{\mathcal{Z}}\right)$. Applying the implicit function theorem, we deduce that there are the neighborhoods $\left(-\overline{\delta}, \overline{\delta}\right)$ and $\mathcal{U}_{\mathbb{X}} \subset \mathcal{U}$ of 0 and $\mathbb{X}$, respectively, and a unique continuously differentiable function $X : \left(-\overline{\delta}, \overline{\delta}\right) \to \mathcal{U}_{\mathbb{X}}$, $\delta \to X_\delta$ such that $\Psi(\delta, X_\delta) = 0$ for all $\delta \in \left(-\overline{\delta}, \overline{\delta}\right)$. It is easy to see that if we set $\delta_0 \in \left(0, \overline{\delta}\right)$ then X_{δ_0} is a global solution of (50) from $\mathcal{U}$. An appeal to Lemma 3 shows that $X_{\delta_0} \in \widetilde{\Gamma}^{\Sigma}$. The proof is complete. ■

Finally we provide necessary and sufficient conditions for the existence of the minimal solution.

Theorem 15 *Assume $0 \in \Gamma^{\Sigma}$. Then the following statements are equivalent:*
(i) Equation (22) has a global solution $X_0 \in C_b^1\left(\mathbf{R}_+, \mathcal{K}_H^{\mathcal{Z}}\right)$.
(ii) Equation (22) has a minimal solution $\widetilde{\widetilde{X}} \in C_b^1\left(\mathbf{R}_+, \mathcal{K}_H^{\mathcal{Z}}\right)$.

Proof. The implication (ii) $\Rightarrow$ (i) is obviously true. It remains to prove (i) $\Rightarrow$ (ii). For each $\tau > 0$ let $X(t) = X(\tau, t; 0)$ be the solution of (22) with the final condition $X(\tau) = 0$. From Corollary 8 it follows that X is well defined on $[0, \tau]$ and $X(t) \succeq 0$ for all $t \in [0, \tau]$. Moreover, if $\tau_1 \geq \tau_2$, then

$$X(\tau_1, t; 0) \succeq X(\tau_2, t; 0) \tag{51}$$

for all $0 \leq t \leq \tau_2 \leq \tau_1$. Indeed, $X(\tau_1, t; 0) = X(\tau_2, t; X(\tau_1, \tau_2; 0))$ from Theorem 7. By the last statement of the Corollary 8, it follows that $X(\tau_2, t; X(\tau_1, \tau_2; 0)) \succeq X(\tau_2, t; 0)$ and hence (51). Reasoning as above and using again Corollary 8 we see that

$$0 \preceq X(\tau, t, 0) \preceq \widetilde{Y}(t), t \leq \tau, \tag{52}$$

where $\widetilde{Y} \in C^1\left(\mathbf{R}_+, \mathcal{K}_H^{\mathcal{Z}}\right)$ is any global solution of (22). Particularly, (52) holds for $\widetilde{Y} = X_0 \in C_b^1\left(\mathbf{R}_+, \mathcal{K}_H^{\mathcal{Z}}\right)$. From (51) and (52) we deduce that, the sequence $\{X(\tau, t; 0)\}_{\tau \geq t}$ is increasing and bounded. Therefore, for any $i \in \mathcal{Z}$ and $t \in [0, \tau]$, $X(\tau, t; 0)(i)$ is strongly convergent, as $\tau \to \infty$, to an element $\widetilde{\widetilde{X}}(t, i) \in \mathcal{K}_H$ satisfying $\widetilde{\widetilde{X}}(t, i) \leq X_0(t, i)$. Passing to the limit as $\tau \to \infty$, componentwise, in the integral equation verified by $X(\tau, t; 0)$ we see that $\widetilde{\widetilde{X}}(t) = \{\widetilde{\widetilde{X}}(t, i)\}_{i \in \mathcal{Z}}$ satisfies the following integral equation

$$\begin{aligned}
\widetilde{\widetilde{X}}(s, i) \quad = \quad & \Phi(t, s)\left(\widetilde{\widetilde{X}}(t)\right)(i) + \int_s^t \Phi(u, t)\left(\mathcal{R}\left(u, \widetilde{\widetilde{X}}(u)\right)\right. \\
& -A^{[*]}(u)\widetilde{\widetilde{X}}(u) - \widetilde{\widetilde{X}}(u) A(u))(i)\, du.
\end{aligned}$$

Arguing as in the proof of Theorem 13 we can show that $\widetilde{\widetilde{X}} \in C_b^1\left(\mathbf{R}_+, l_{\mathcal{E}_H}^{\mathcal{Z}}\right)$, and therefore it is a global and bounded solution of (22). Invoking (52) we conclude that $\widetilde{\widetilde{X}}$ is the minimal solution of (22). ■

Remark 16 *If in addition the coefficients A, B, Π, Q of the Riccati equation (22) are periodic functions with the same period θ, then a similar reasoning as in [12] leads to the conclusion that the maximal, minimal and stabilizing solutions of (22) are θ-periodic functions. Hence, in the time invariant case, the main results of this section apply to algebraic Riccati equations (see [3] and [12] for example).*

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Department of Mathematics, “Constantin Brancusi”University,
Tg. Jiu, Bulevardul Republicii, nr. 1, jud. Gorj, Romaina,
e-mail vio@utgjiu.ro.

Institute of Mathematics of the Romanian Academy,
PO Box 1-764, RO-014700 Bucharest, Romania,
e-mail Vasile.Dragan@imar.ro.