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Radial solutions and a local bifurcation result for a singular elliptic problem with Neumann condition

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Abstract. We study the problem $-\Delta u = \lambda u - u^{-1}$ with a Neumann boundary condition; the peculiarity being the presence of the singular term $-u^{-1}$. We point out that the minus sign in front of the negative power of u is particularly challenging, since no convexity argument can be invoked. Using bifurcation techniques we are able to prove the existence of solution (u_λ, λ) with u_λ approaching the trivial constant solution $u = \lambda^{-1/2}$ and λ close to an eigenvalue of a suitable linearized problem. To achieve this we also need to prove a generalization of a classical two-branch bifurcation result for potential operators. Next we study the radial case and show that in this case one of the bifurcation branches is global and we find the asymptotical behavior of such a branch. This results allows to derive the existence of multiple solutions u with λ fixed.

Keywords: singular elliptic equation, positive solutions, radial solutions, variational bifurcation, two branches.

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1 Introduction

In the last decades several authors have studied semilinear elliptic problems with singular nonlinear term (with respect to the unknown function u). The model problem is the following:

$$\begin{cases} -\Delta u = \gamma u^{-q} + f(x,u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $q > 0$ $\gamma \neq 0$ and f is a non linear term with standard growth conditions. Existence and multiplicity of solutions to problem (1.1) are usually investigated in terms of the behavior of f and the sign of γ.

A main aspect to be taken into account is the variational nature of (1.1): formally speaking solutions u of (1.1) are expected to be critical points of the functional

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\gamma}{1-q}\int_\Omega u^{1-q}\,dx - \int_\Omega F(x,u)\,dx,$$

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defined on $W_0^{1,2}(\Omega)$ and restricted to $\{u \geq 0\}$, where $F(x,s)$ is a primitive in s of $f(x,s)$ (if $q = 1$ a logarithm should be introduced). Unfortunately the presence of the singular term makes it problematic to give a rigorous formulation of the above ideas.

The majority of the known results concern the case $\gamma > 0$, where the term $u \mapsto -\frac{\gamma}{1-q}u^{1-q}$ is convex in the interval $]0,+\infty[$. This fact helps a lot, whether one tries to directly deal with I (by using some nonsmooth-critical-point theory) or to use an approximation scheme (by a sequence $I_n \to I$, I_n being $\mathcal{C}^1$ on $W_0^{1,2}(\Omega)$). For instance, if $f = 0$, the problem has a unique weak solution $\bar{u}$ in $W_0^{1,2}(\Omega)$ when $0 < q < 3$ and the solution is a minimizer for I. This result can be extended for all $q > 0$ dropping the request that $\bar{u} \in W_0^{1,2}(\Omega)$ (see [5,9]). For a small non exhaustive list of multiplicity results for solutions of this kind of problems see [1,3,7,11,13–15,17,18,26,27] (and the references therein).

If we turn to $\gamma < 0$ the literature is scarcer: to the author's knowledge the main results are contained in [6,12,22,27,28]. In this situation solutions are "attracted" from the value zero and tend to develop "dead cores", so the formulation (1.1) needs to be modified in order to admit non strictly positive solutions. For instance the only solution for the case $f = 0$ is $u = 0$ (as one can easily see by multiplying the equation by u). Moreover a direct variational approach using the functional I seems difficult for the moment and the usual approach goes by perturbation methods.

We have found particularly interesting the paper [22] by Montenegro and Silva, where the authors use perturbation methods and show that there exist two nontrivial solutions when $\gamma = -1$, $0 < q < 1$, $f(x,u) = \mu u^p$, with $q < p <$ and $\mu > 0$ big enough. If we pass to $q = 1$, simple tests in the radial case suggest that the Dirichlet problem only has the trivial solution. As we said before solutions starting from zero are "forced to stick" at zero and not allowed to "emerge" (in contrast with the case of $q < 1$). On this respect see Remark 4.9.

For this reasons, in the case $q = 1$, we are lead to replace the Dirichlet condition with a Neumann one. In particular we have considered the problem

$$\begin{cases} -\Delta u = \lambda u - \dfrac{1}{u} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ \nabla u \cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

where Ω is a bounded smooth open subset of $\mathbb{R}^N$ and ν denotes the unit normal defined on $\partial\Omega$. This corresponds to the problem of [22] with $q = p = 1$ (with Neumann condition).

In case $N = 1$ (1.2) is closely related to a problem studied by Del Pino, Manásevich, and Montero in 1992 (see [10]) who deal with an ODE, in the periodic case, with a more general, non autonomous, singular term $f(u,x)$ (singular in u and T-periodic in x). Using topological degree arguments they prove for instance that the equation:

$$-\ddot{u} = \lambda u - \frac{1}{u^\alpha}, \qquad u(x) > 0, \qquad u(x+T) = u(x),$$

where $\alpha \geq 1$, has a solution provided $\lambda \neq \frac{\mu_k}{4}$ for all k. Here μ_k denote the eigenvalues of a suitable linearized problem which arises in a natural way from the problem. In this case, which has a variational structure, the results of [10] can be derived from the existence of two global bifurcation branches which originate from trivial solutions of the linearized problem.

In this paper we present two types of results concerning problem (1.2). In Theorem (2.1) of Section 2 we prove the existence of two local bifurcation branches $(u_{1,\rho}, \lambda_{1,\rho})$ and $(u_{2,\rho}, \lambda_{2,\rho})$ of solutions of (1.2), such that $(u_{i,\rho}, \lambda_{1,\rho}) \to (\hat{u}, \hat{\lambda})$, as $\rho \to 0$, where $\hat{\lambda}/2$ is an eigenvalue of

$-\Delta$ with Neumann condition and $\hat{u}$ is the constant function: $\hat{u} \equiv \hat{\lambda}^{-1/2}$. The proof of (2.1) heavily relies on a variant of the well known abstract results on the existence of two bifurcation branches in the variational case (see [4, 19, 20, 25]). To the author surprise such a variant (see Theorem (3.1) seems not to be present in the literature so its proof is carried on in Section 3. It has to be said that proving (3.1) requires some additional technicalities compared to the standard version. Indeed in [4, 20] the proof goes by studying a suitable perturbed function f_ρ on the unit sphere S, while in our case S has to be replaced by a sphere-like set S_ρ also varying with ρ. This requires to construct suitable projections to show that all S_ρ's are homeomorphic to $S_0 = S$ (for ρ small). Apart from this the proof of (3.1) follows the ideas of [2].

In Section 4 we study the radial case in dimension $N = 2$ (the same could be probably done for $N \geq 3$) using ODE techniques and a continuation argument for the nodal regions of the solutions. In this way, following the ideas of [24], we are able to prove that one of the two branches (u_ρ, λ_ρ) is global and bounded in λ_ρ. This is done by proving that nodal regions of u_ρ cannot collapse along the branch and that $\lambda_\rho \to \bar{\lambda}$ as $\|u_\rho\| \to +\infty$, where $\bar{\rho}$ is an eigenvalue of another suitable linear problem. In this way – in the radial case – we can find a lower estimate in the number of solutions for a fixed λ, by counting the number of branches that cross λ.

2 A local bifurcation result for the singular problem

Let Ω be a bounded open subset of $\mathbb{R}^N$ with smooth boundary.

Theorem 2.1. *Let $\hat{\mu} > 0$ be an eigenvalue of the following Neumann problem:*

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \nabla u \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

(ν denotes the normal to $\partial\Omega$).

Then there exists $\rho_0 > 0$ such that for all $\rho \in]0, \rho_0[$ there exist two distinct pairs $(u_{1,\rho}, \lambda_{1,\rho})$ and $(u_{2,\rho}, \lambda_{2,\rho})$ such that, for $i = 1, 2$:

$$(u_{i,\rho}, \lambda_{i,\rho}) \text{ are solutions of (1.2)}, \; u_{i,\rho} \to \frac{1}{\sqrt{\mu/2}} \quad (\text{in } W^{1,2}(\Omega)\,)\,, \; \lambda_{i,\rho} \xrightarrow{\rho \to 0} \frac{\hat{\mu}}{2}.$$

Proof. We start by introducing some changes of variables. First of all notice that, for all $\lambda > 0$, Problem (1.2) has the constant solution $u(x) = \frac{1}{\sqrt{\lambda}}$. If we seek for solutions of the form $u = \frac{1}{\sqrt{\lambda}} + z$ we easily end up with the equivalent problem on z:

$$\begin{cases} -\Delta z = 2\lambda z - h_\lambda(z) & \text{in } \Omega, \\ \sqrt{\lambda} z > -1 & \text{in } \Omega, \\ \nabla z \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

where $h_\lambda :] - \frac{1}{\sqrt{\lambda}}, +\infty[\to \mathbb{R}$ is defined by

$$h_\lambda(s) = \frac{\lambda\sqrt{\lambda}s^2}{1 + \sqrt{\lambda}s}.$$

Now we consider another simple transformation: $v := \sqrt{\lambda} z$, so that (2.2) turns out to be equivalent to

$$\begin{cases} -\Delta v = 2\lambda \left(v - \frac{1}{2} h_1(v)\right) & \text{in } \Omega, \\ v > -1 & \text{in } \Omega, \\ \nabla v \cdot \nu = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.3}$$

Now choose s_0 with $0 < s_0 < 1/2$ and a $\mathcal{C}^\infty$ cutoff function $\eta : \mathbb{R} \to [0,1]$ such that $\eta(s) = 1$ for $|s| \le s_0$, $\eta(s) = 0$, for $|s| \ge 2s_0$. Define $\tilde{h}_1 : \mathbb{R} \to \mathbb{R}$ by

$$\tilde{h}_1(s) := \eta(s)\, h_1(s) \tag{2.4}$$

($\tilde{h}_1(s)$ is given the value zero for $s = -1$). Then $\tilde{h}_1 \in \mathcal{C}_0^\infty(\mathbb{R};\mathbb{R})$, $\tilde{h}_1'(0) = h_1''(0) = 0$, $\tilde{h}_1 = h_1$ on $[-s_0, s_0]$. Denote by $\tilde{H}_1 : \mathbb{R} \to \mathbb{R}$ the primitive function for $\tilde{h}_1$ (i.e. $\tilde{H}_1' = \tilde{h}_1$) such that $\tilde{H}_1(0) = 0$.

Now we apply the bifurcation theorem (3.1) with $\mathbb{H} := W^{1,2}(\Omega)$. $\mathbb{L} = L^2(\Omega)$, $\mathcal{H} = 0$, $\hat{\lambda} = \mu$, $\mathcal{H}_1(v) := \frac{1}{2}\int_\Omega \tilde{H}_1(v)\, dx$. In this way we get that there exists $\rho_0 > 0$ such that for all $\rho \in]0, \rho_0[$ there are two distinct pairs $(v_{1,\rho}, \mu_{1,\rho})$ and $(v_{2,\rho}, \mu_{2,\rho})$ which are *weak* solutions of

$$\begin{cases} -\Delta v = \mu \left(v - \frac{1}{2}\tilde{h}_1(v)\right) & \text{in } \Omega, \\ \nabla v \cdot \nu = 0 & \text{on } \partial\Omega \end{cases} \tag{2.5}$$

and such that

$$v_{i,\rho} \xrightarrow{\rho \to 0} 0 \quad (\text{in } W^{1,2}(\Omega)), \qquad \mu_{i,\rho} \xrightarrow{\rho \to 0} \mu_k, \qquad i = 1,2. \tag{2.6}$$

We claim the there exists a constant K such, for any $\mu \in [\hat{\mu}_k - 1, \hat{\mu}_k + 1]$ and any weak solution v of (2.5), v is bounded and:

$$\|v\|_\infty \le K\mu \left\| v - \frac{1}{2}\tilde{h}_1(v) \right\|_2. \tag{2.7}$$

For this we use a standard bootstrap argument using the fact that the function $k(s) = (s - \frac{1}{2}\tilde{h}_1(s))$, appearing on the right hand side of (2.5), verifies

$$|k(s)| \le M|s| \qquad \forall s \in \mathbb{R} \tag{2.8}$$

for a suitable M (since $\tilde{h}_1'$ is bounded). Assume that v is a solution, i.e. $-\Delta v = \mu k(v)$, and $v \in L^q(\Omega)$ for some $q > 1$ (for sure this is true for $q = 2^*$). Then, by (2.8), $k(v) \in L^q(\Omega)$. From the standard Calderón–Zygmund theory (see e.g. Section 9.6 in [16]), we have $v \in W^{2,q}(\Omega)$. Then, using the Sobolev embedding Theorem, either $v \in L^{q_1}(\Omega)$ with $q_1 \le \frac{Nq}{N-2q}$ (if $2q \le N$) or $v \in \mathcal{C}^{0,\alpha}$ with $\alpha > 0$ (in the case $2q > N$). Iterating this argument a finite number of times we get the conclusion. Notice that we could go further and prove that v is $\mathcal{C}^\infty$ and is a classical solution.

Using (2.6) and (2.7) we get that $v_{i,\rho} \to 0$ in $L^\infty(\Omega)$ as $\rho \to 0$, so $|v_{i,\rho}| < s_0$, $i = 1,2$, for ρ_0 small. This implies that $\tilde{h}_1(v_{i,\rho}) = h_1(v_{i,\rho})$, and $v_{i,\rho}$ actually solve (2.3) with $\lambda_{i,\rho} := \frac{\mu_{i,\rho}}{2}$. Going backwards and setting $u_{i,\rho} := \frac{1}{\sqrt{\lambda_{i,\rho}}} + \sqrt{\lambda_{i,\rho}} v_{i,\rho}$, we find the desired solutions of (1.2).

□

3 A variant for the two bifurcation branches theorem for potential operators

Let $\mathbb{L}$ and $\mathbb{H}$ be two Hilbert spaces such that $\mathbb{H} \subset \mathbb{L}$ with a compact embedding $i : \mathbb{H} \to \mathbb{L}$. We use the notations $\|\cdot\|$, $\langle \cdot, \cdot \rangle$ and $\|\cdot\|_{\mathbb{L}}$, $\langle \cdot, \cdot \rangle_{\mathbb{L}}$ to indicate the norms and inner products in $\mathbb{H}$ and $\mathbb{L}$ respectively. Let $A : \mathbb{H} \to \mathbb{H}$ be a bounded linear symmetric operator such that

$$\langle Au, u \rangle \geq \nu \|u\|^2 - M \|u\|_{\mathbb{L}}^2 \qquad \forall u \in \mathbb{H} \tag{3.1}$$

where $\nu > 0$ and M are two constants. We say that $\lambda \in \mathbb{R}$ is an "eigenvalue for A" if there exists $e \in \mathbb{H} \setminus \{0\}$ with

$$\langle Ae, v \rangle = \lambda \langle e, v \rangle_{\mathbb{L}} \qquad \forall v \in \mathbb{H}$$

which corresponds to say that:

$$Ae = \lambda i^* e.$$

In this case we say that e is an "eigenvector" corresponding to λ.

It is well known that there exists a sequence (λ_n) of eigenvalues of A with $\lambda_n \leq \lambda_{n+1}$, $\lambda_n \to +\infty$, such that the corresponding eigenvectors generate $\mathbb{H}$. It is convenient to agreee that $\lambda_0 = -\infty$. We can suppose that for any $k \geq 1$ we are given an eigenvector e_k relative to λ_k with $\|e_n\|_{\mathbb{L}} = 1$, and

$$\langle e_n, e_m \rangle = \langle e_n, e_m \rangle_{\mathbb{L}} = 0 \quad \text{if } n \neq m.$$

If $\lambda \in \mathbb{R}$ we define

$$E_\lambda^- := \text{span}\{e_i : \lambda_i < \lambda\}, \quad E_\lambda^0 := \text{span}\{e_i : \lambda_i = \lambda\}, \quad E_\lambda^+ := \overline{\text{span}\{e_i : \lambda_i > \lambda\}}^{(\mathbb{H})}$$

($E_\lambda^0 = \{0\}$ if λ is not an eigenvalue). If $\lambda_n \leq \lambda \leq \lambda_{n+1}$ it is clear that

$$\sup_{\left\{u \in E_\lambda^- : \|u\|_{\mathbb{L}} = 1\right\}} \langle Au, u \rangle \leq \lambda_n, \qquad \inf_{\{u \in E^+\lambda : \|u\|_{\mathbb{L}} = 1\}} \langle Au, u \rangle \geq \lambda_{n+1},$$

while $\langle Au, u \rangle = \lambda$, if $u \in E_\lambda^0$.

Theorem 3.1 (Bifurcation). *Let $\mathcal{H} \in \mathcal{C}^1(\mathbb{H}; \mathbb{R})$, $\mathcal{H}_1 \in \mathcal{C}^1(\mathbb{L}; \mathbb{R})$ be such that*

$$\begin{aligned} &\mathcal{H}(0) = 0, \qquad \nabla \mathcal{H}(0) = 0, \qquad \lim_{u \to 0} \frac{\|\nabla \mathcal{H}(u)\|_{\mathbb{L}}}{\|u\|_{\mathbb{L}}} = 0 \\ &\mathcal{H}_1(0) = 0, \qquad \nabla_{\mathbb{L}} \mathcal{H}_1(0) = 0, \qquad \lim_{u \to 0} \frac{\|\nabla_{\mathbb{L}} \mathcal{H}_1(u)\|_{\mathbb{L}}}{\|u\|_{\mathbb{L}}} = 0. \end{aligned} \tag{3.2}$$

Notice that we are using the symbol ∇ to denote the gradient with respect to the inner product in $\mathbb{H}$ and $\nabla_{\mathbb{L}}$ for the corresponding gradient in $\mathbb{L}$.

Let $\hat{\lambda}$ be an eigenvalue for A. Then, for any $\rho > 0$ small, there exist $(u_{1,\rho}, \lambda_{1,\rho})$ and $(u_{2,\rho}, \lambda_{2,\rho})$ which solve the problem

$$Au + \nabla \mathcal{H}(u) = \lambda i^* \left(u + \nabla_{\mathbb{L}} \mathcal{H}_1(u)\right), \qquad u \neq 0, \tag{3.3}$$

such that $u_{1,\rho} \neq u_{2,\rho}$ and

$$u_{1,\rho} \xrightarrow{\mathbb{H}} 0, \qquad u_{2,\rho} \xrightarrow{\mathbb{H}} 0, \qquad \lambda_{1,\rho} \to \hat{\lambda}, \qquad \lambda_{2,\rho} \to \hat{\lambda} \quad \text{as } \rho \to 0. \tag{3.4}$$

Proof. We adapt the proof of Lemma 3.4 in [2]. Let $\hat{\lambda} = \lambda_i = \lambda_k$ with $\lambda_{i-1} < \lambda_i$ and $\lambda_k < \lambda_{k+1}$. We define $f : \mathbb{H} \to \mathbb{R}$ and $g : \mathbb{L} \to \mathbb{R}$ by

$$f(u) := \frac{1}{2}\langle Au, u\rangle + \mathcal{H}(u), \qquad g(u) := \frac{1}{2}\|u\|_{\mathbb{L}}^2 + \mathcal{H}_1(u).$$

Let $\mathcal{C} := \{u \in \mathbb{L} : 1 < \|u\|_{\mathbb{L}} < 2\}$. Moreover, if $0 < \rho < 1$ we define

$$\begin{aligned} f_\rho(u) &:= \frac{1}{\rho^2} f(\rho u), & g_\rho(u) &:= \frac{1}{\rho^2} g(\rho u), \\ \mathcal{H}_\rho(u) &:= \frac{1}{\rho^2}\mathcal{H}(\rho u), & \mathcal{H}_{1,\rho}(u) &:= \frac{1}{\rho^2}\mathcal{H}_1(\rho u), \\ \mathcal{S}_\rho &:= \{u \in \mathcal{C} : g_\rho(u) = 1\}. \end{aligned}$$

In fact $f_\rho(u) = \frac{1}{2}\langle Au, u\rangle + \mathcal{H}_\rho(u)$ and $g_\rho(u) = \frac{1}{2}\|u\|_{\mathbb{L}}^2 + \mathcal{H}_{1,\rho}(u)$.

Since the result we are proving only involves the behaviour of $\mathcal{H}, \mathcal{H}_1$ near zero, we are allowed to modify $\mathcal{H}$ and $\mathcal{H}_1$ outside of a small ball around the origin. More precisely using (3.2) we can find R in $]0, 1/3[$ such that

$$\|\nabla\mathcal{H}(u)\| \le \frac{\nu}{8}\|u\| \ \forall u \text{ with } \|u\| < 3R, \quad \|\nabla_{\mathbb{L}}\mathcal{H}_1(u)\| \le \frac{1}{2}\|u\|_1 \ \forall u \text{ with } \|u\|_1 < 3R, \tag{3.5}$$

and define $\tilde{\mathcal{H}}(u) := \eta(\|u\|)\mathcal{H}(u)$, $\tilde{\mathcal{H}}_1(u) := \eta(\|u\|_1)\mathcal{H}_1(u)$, where $\eta : [0, +\infty[\to [0, 1]$ is a cutoff function with $\eta(s) = 1$ for $0 \le s \le R$, $\eta(s) = 0$ for $s \ge 3R$, and $\eta'(s) \le 1$. Now since

$$\tilde{\mathcal{H}}(u) = \mathcal{H}(u) \quad \forall u \text{ with } \|u\| < R, \qquad \tilde{\mathcal{H}}_1(u) = \mathcal{H}_1(u) \quad \forall u \text{ with } \|u\|_{\mathbb{L}} < R,$$

then the conclusion of Theorem 3.1 holds for $\mathcal{H}, \mathcal{H}_1$ if and only if it holds for $\tilde{\mathcal{H}}, \tilde{\mathcal{H}}_1$. Indeed the first component u_ρ of a bifucation branch (for any of the two problems) eventually verifies $\|u_\rho\| < R$ and $\|u_\rho\|_1 < R$. So from now on we replace $\mathcal{H}$ with $\tilde{\mathcal{H}}$ and $\mathcal{H}_1$ with $\tilde{\mathcal{H}}_1$, maintaining the same notation. With simple computations we can deduce from (3.5) that the redefined functions verify:

$$\text{(a)} \quad |\nabla\mathcal{H}(u)| \le \frac{\nu}{4}\|u\| \ \forall u \in \mathbb{H}, \qquad \text{(b)} \quad |\nabla_{\mathbb{L}}\mathcal{H}_1(u)| \le \|u\|_{\mathbb{L}} \ \forall u \in \mathbb{L}. \tag{3.6}$$

From (a) in (3.6) we get

$$|\mathcal{H}(u)| \le \frac{\nu}{4}\|u\|^2 \Rightarrow |\mathcal{H}_\rho(u)| \le \frac{\nu}{4}\|u\|^2 \qquad \forall u \in \mathbb{H}, \forall \rho \in [0, 1]. \tag{3.7}$$

Using (3.1) and (3.7) we get that:

$$\|u\|^2 \le \frac{4}{\nu}\left(f_\rho(u) + M\|u\|_{\mathbb{L}}^2\right). \tag{3.8}$$

From (3.2) and (3.8) we easily get that, if $c \in \mathbb{R}$ and $\rho \to 0$:

$$\begin{aligned} &\sup_{u \in \mathcal{C}, f_\rho(u) \le c} |\mathcal{H}_\rho(u)| \to 0, & &\sup_{u \in \mathcal{C}} |\mathcal{H}_{1,\rho}(u)| \to 0, \\ &\sup_{u \in \mathcal{C}, f_\rho(u) \le c} \|\nabla\mathcal{H}_\rho(u)\| \to 0, & &\sup_{u \in \mathcal{C}} \|\nabla_{\mathbb{L}}\mathcal{H}_{1,\rho}(u)\|_{\mathbb{L}} \to 0. \end{aligned} \tag{3.9}$$

So if we extend the definition to $\rho = 0$ by letting $f_0(u) := \frac{1}{2}\langle Au, u\rangle$ and $g_0(u) := \frac{1}{2}\|u\|_{\mathbb{L}}^2$, then $(\rho, u) \mapsto f_\rho(u)$ is continuous on $[0, +\infty[\times\mathbb{H}$ and $(\rho, u) \mapsto g_\rho(u)$ is continuous on $[0, +\infty[\times\mathbb{L}$. We also define $\mathcal{S}_\rho$ for $\rho = 0$:

$$\mathcal{S}_0 := \{u \in \mathbb{L} : g_0(u) = 1\} = \left\{u \in \mathbb{L} : \|u\|_{\mathbb{L}}^2 = 2\right\}.$$

Notice that the critical values of f_0 on $\mathcal{S}_\rho$ are precisely the eigenvalues λ_n.

We claim that there exist $\bar{\rho} >$ such that the $\mathbb{L}$-closure of $\mathcal{S}_\rho$ is contained in $\mathcal{C}$ for all $\rho \in]0, \bar{\rho}]$ in other terms $\mathcal{S}_\rho$ is closed for $\rho > 0$ small. Indeed if the claim were false there would exist two sequences (ρ_n) and (u_n) such that $\rho_n \to 0$, $\rho_n \to 0$, $g_{\rho_n}(u_n) = \frac{\|u_n\|_{\mathbb{L}}^2}{2} + \frac{\mathcal{H}_1(\rho_n u_n)}{\rho^2} = 1$, and $\|u_n\|_{\mathbb{L}} \in \{1, 2\}$. From (3.2) we would have $\frac{\mathcal{H}_1(\rho_n u_n)}{\rho_n^2} = \frac{\mathcal{H}_1(\rho_n u_n)}{\|\rho_n u_n\|_{\mathbb{L}}^2}\|u_n\|_{\mathbb{L}}^2 \to 0$, so $\|u_n\|_{\mathbb{L}} \to \sqrt{2}$ which yields a contradiction for n large.

Let us split $\mathbb{H}$ as $\mathbb{H} = \mathbb{X}_1 \oplus \mathbb{X}_2 \oplus \mathbb{X}_3$, where

$$\mathbb{X}_1 := E_{\hat{\lambda}}^-, \qquad \mathbb{X}_2 := E_{\hat{\lambda}}^0, \qquad \mathbb{X}_3 := E_{\hat{\lambda}}^+$$

and consider the orthogonal projections $\Pi_i : \mathbb{H} \to \mathbb{X}_i$. $i = 1, 2, 3$. We also denote $\Pi_{13} := \Pi_1 + \Pi_3$. Given $\rho \in [0, \bar{\rho}]$ and $\delta \in]0, 1[$, we set

$$\mathcal{C}_\delta := \{u \in \mathcal{C} : \|\Pi_2(u)\|_{\mathbb{L}} \geq \delta\}, \qquad \mathcal{S}_{\rho,\delta} := \mathcal{S}_\rho \cap \mathcal{C}_\delta.$$

Since $\mathcal{S}_\rho$ is closed, then $S_{\rho,\delta}$ is a smooth manifold with boundary, the boundary being

$$\Sigma_{\rho,\delta} := \left\{u \in \mathcal{S}_\rho : \|\Pi_2(u)\|_{\mathbb{L}} = \delta\right\}.$$

Notice that $\mathcal{S}_{0,\delta} \neq \emptyset$ ($\delta < 1$). Let us indicate by $\bar{f}_\rho$ the restriction of f_ρ on $\mathcal{S}_{\rho,\delta}$.

We will use the notion of lower critical point for $\bar{f}_\rho$ (see [2,21] and the references therein): u is (lower) critical for $\bar{f}_\rho$ if and only there exist $\lambda, \mu \in \mathbb{R}$ such that $\mu \geq 0$, $\mu = 0$ if $\|\Pi_2(u)\|_{\mathbb{L}} > \delta$, and

$$\langle Au, v\rangle + \langle \nabla\mathcal{H}_\rho(u), v\rangle = \lambda\left\langle u + \nabla_{\mathbb{L}}\mathcal{H}_{1,\rho}(u), v\right\rangle_{\mathbb{L}} + \mu\left\langle \Pi_2(u), v\right\rangle_{\mathbb{L}} \quad \forall v \in \mathbb{H}. \tag{3.10}$$

Define $\Gamma : \mathcal{C}_\delta \times [1/2, 2] \to \mathcal{C}_\delta$ and $\varphi : [0, \bar{\rho}] \times \mathcal{C}_\delta \times [1/2, 2] \to \mathbb{R}$ by

$$\begin{aligned}
\Gamma(u, t) &:= \frac{\delta\Pi_2(u)}{\|\Pi_2(u)\|_{\mathbb{L}}} + t\left(u - \frac{\delta\Pi_2(u)}{\|\Pi_2(u)\|_{\mathbb{L}}}\right) \\
&= t\Pi_{13}(u) + (\delta + t(\|\Pi_2(u)\|_{\mathbb{L}} - \delta))\frac{\Pi_2(u)}{\|\Pi_2(u)\|_{\mathbb{L}}}, \\
\varphi(\rho, u, t) &:= g_\rho(\Gamma(u, t)).
\end{aligned}$$

With easy computations:

$$\begin{aligned}
\varphi(0, u, t) &= \frac{1}{2}\left(t^2\|\Pi_{13}(u)\|_{\mathbb{L}}^2 + (\delta + t(\|\Pi_2(u)\|_{\mathbb{L}} - \delta))^2\right) \\
&= \frac{1}{2}\left(t^2(\|u\|_{\mathbb{L}}^2 - 2\delta\|\Pi_2(u)\|_{\mathbb{L}} + \delta^2) + 2\delta t(\|\Pi_2(u)\|_{\mathbb{L}} - \delta) + \delta^2\right).
\end{aligned}$$

Since $1 < \|u\|_{\mathbb{L}} < 2$ and $\|\Pi_2(u)\|_{\mathbb{L}} \geq \delta$, we have

$$\frac{t^2}{2} - 2\delta t^2 \leq \varphi(0, u, t) \leq \left(2 + \frac{\delta^2}{2}\right)t^2 + 2\delta t + \frac{\delta^2}{2}.$$

In particular:

$$2-2\delta \le \varphi(0,u,2), \qquad \varphi\,(0,u,1/2) \le \frac{1}{2}+\frac{\delta^2}{8}+\delta+\frac{\delta^2}{2} < \frac{1}{2}+2\delta.$$

We can choose $\delta_0 > 0$ so that $2-2\delta > 3/2$ and $\frac{1}{2}+2\delta < 3/4$ for all $\delta \in]0,\delta_0]$. From now on we consider $0<\delta\le\delta_0$. By (3.9), up to shrinking $\bar\rho$, we have

$$\sup_{u\in\mathcal{C}_\delta} \varphi(\rho,u,1/2) < 1, \qquad \inf_{u\in\mathcal{C}_\delta} \varphi(\rho,u,2) > 1 \qquad \forall \rho \in [0,\bar\rho].$$

Moreover,

$$\frac{\partial}{\partial t}\varphi(0,u,t) = t(\|u\|_{\mathbb{L}}^2 - 2\delta\,\|\Pi_2(u)\|_{\mathbb{L}} + \delta^2) + \delta(\|\Pi_2(u)\|_{\mathbb{L}} - \delta) \ge t(1-4\delta)$$

so, up to shrinking δ_0, we have $\frac{\partial}{\partial t}\varphi(0,u,t) \ge \frac{1}{4}$ for all $t \ge \frac{1}{2}$. Up to further shrinking $\bar\rho > 0$ (again we use (3.9)), we have that $\rho\in[0,\bar\rho]$, $\delta \in]0,\delta_0]$, $u \in \mathcal{C}_\delta$ imply

$$\varphi(\rho,u,1/2) < 1, \quad \varphi(\rho,u,2) > 1, \quad \frac{\partial}{\partial t}\varphi(\rho,u,t) \ge \frac{1}{8} \quad \forall t\in[1/2,2].$$

We can therefore conclude that for all $u \in [0,\bar\rho]$ and $u\in\mathcal{C}_\delta$ there exists a unique $\bar t = \bar t(\rho,u)$ in $[1/2,2]$ such that $\varphi(\rho,u,\bar t(u,\rho)) = 1$, that is $\Gamma(u,\bar t(\rho,u)) \in \mathcal{S}_{\rho,\delta}$. It is easy to check that $\bar t : [0,\bar\rho]\times\mathcal{C}_\delta \to [1/2,2]$ is continuous and so is $\Phi : [0,\bar\rho]\times\mathcal{C}_\delta \to \mathcal{S}_{\rho,\delta}$ defined by $\Phi(\rho,u) := \Gamma(u,\bar t(\rho,u))$. Notice that

$$t\in[1/2,2],\ u\in\mathcal{C}_\delta,\ \|\Pi_2(u)\|_{\mathbb{L}} = \delta \ \Rightarrow\ \|\Pi_2(\Gamma(u,t))\|_{\mathbb{L}} = \delta.$$

Therefore $\Phi(\rho,\cdot)$ maps $\{u\in\mathcal{C}_\delta, \|\Pi_2(u)\|_{\mathbb{L}} = \delta\}$ into $\Sigma_{\rho,\delta}$. Also notice that $\Phi(\rho,u)\circ\Phi(0,u) = u$ whenever $u\in\mathcal{S}_{\rho,\delta}$ and $\Phi(0,u)\circ\Phi(\rho,u) = u$ whenever $u\in\mathcal{S}_{0,\delta}$. We have thus proven that $\Phi(\rho,\cdot)|_{S_{0,\delta}}$ is a homeomorphism from $(S_{0,\delta},\Sigma_{0,\delta})$ to $(S_{\rho,\delta},\Sigma_{\rho,\delta})$ whose inverse is $\Phi(0,\cdot)|_{S_{\rho,\delta}}$.

Now let

$$a'_\rho := \sup_{(\mathbb{X}_1\oplus\mathbb{X}_2)\cap\Sigma_{\rho,\delta}} f_\rho \qquad\qquad a''_\rho := \inf_{(\mathbb{X}_2\oplus\mathbb{X}_3)\cap\mathcal{S}_{\rho,\delta}} f_\rho \tag{3.11}$$

$$b'_\rho := \sup_{(\mathbb{X}_1\oplus\mathbb{X}_2)\cap\mathcal{S}_{\rho,\delta}} f_\rho \qquad\qquad b''_\rho := \inf_{(\mathbb{X}_2\oplus\mathbb{X}_3)\cap\Sigma_{\rho,\delta}} f_\rho. \tag{3.12}$$

Notice that, by definition, $a''_\rho \le b'_\rho$. For $\rho = 0$ it is easy to see that

$$a'_0 = \lambda_{i-1} + \frac{\delta^2}{2}(\hat\lambda - \lambda_{i-1}) < \hat\lambda = a''_0 = b'_0 = \hat\lambda < \lambda_{k+1} - \frac{\delta^2}{2}(\lambda_{k+1}-\hat\lambda) = b''_0$$

(recall that $0<\delta<1$). Let $\varepsilon_0 > 0$ with $\varepsilon_0 < \hat\lambda - \lambda_{i-1}$. We claim that, if $\delta^2(\hat\lambda-\lambda_{i-1}) < 2\varepsilon_0$, then

$$\text{there exists no } u\in\Sigma_{0,\delta} \text{ with } u \text{ lower critical for } \bar f_0 \text{ and } \ \lambda_{i-1}+\varepsilon_0 \le f_0(u). \tag{3.13}$$

By contradiction assume that such a u exists; then there exist $\lambda\in\mathbb{R}$ and $\mu\ge 0$ such that (3.10) holds. Let $u_i = \Pi_i(u)$, $i = 1,2,3$. Taking $v = u_2$ in (3.10) (with $\rho = 0$) yields

$$\hat\lambda\,\|u_2\|_{\mathbb{L}}^2 = \langle Au_2,u_2\rangle = \langle Au,u_2\rangle = \lambda\,\langle u,u_2\rangle_{\mathbb{L}} + \mu\,\langle u_2,u_2\rangle_{\mathbb{L}} = (\lambda+\mu)\,\|u_2\|_{\mathbb{L}}^2\,.$$

Since $\|u_2\|_{\mathbb{L}} = \delta > 0$, we have $\lambda + \nu = \hat{\lambda}$, so $\lambda = \hat{\lambda} - \nu \leq \hat{\lambda}$. Taking $v = u_3$:

$$\lambda_{k+1} \|u_3\|^2 \leq \langle Au_3, u_3 \rangle = \langle Au, u_3 \rangle = \langle \lambda u + \mu u_2, u_3 \rangle = \lambda \|u_3\|^2_{\mathbb{L}} \leq \hat{\lambda} \|u_3\|^2_{\mathbb{L}}.$$

Since $\lambda_{k+1} < \hat{\lambda}$, we have $u_3 = 0$. Then $u \in \mathbb{X}_1 \oplus \mathbb{X}_2 \cap \Sigma_{0,\delta}$, which implies

$$f_0(u) \leq a'_0 = \lambda_{i-1} + \frac{\delta^2}{2}(\hat{\lambda} - \lambda_{i-1}) < \lambda_{i-1} + \varepsilon_0$$

which gives a contradiction. Hence the claim is proven. Notice that (3.13) implies that the only critical value λ_0 of $\bar{f}_0$, with $\lambda_{i-1} + \varepsilon_0 \leq \lambda_0 \leq \lambda_{k+1} - \varepsilon_0$, is $\lambda_0 = \hat{\lambda}$. Indeed assume u_0 to be a critical point with $\bar{f}_0(u_0) = \lambda_0$: then, by (3.13), $u_0 \notin \Sigma_{0,\delta}$ so (3.10) holds with $\mu = 0$ which easily implies $\lambda_0 = \hat{\lambda}$.

From now on we fix $\varepsilon_0 > 0$ such that $5\varepsilon_0 < \min(\hat{\lambda} - \lambda_{i-1}, \lambda_{k+1} - \hat{\lambda})$ and $\delta > 0$ such that $\delta^2(\hat{\lambda} - \lambda_{i-1}) \leq \varepsilon_0$ (so (3.13) holds with $\varepsilon_0/2$). Using (3.9) we can derive that, given $\varepsilon \in]0, \varepsilon_0]$ there exists $\rho(\varepsilon) \in]0, \bar{\rho}]$ such that, if $\rho \in]0, \rho(\varepsilon)]$:

$$\begin{aligned} a'_\rho &\leq \lambda_{i-1} + \varepsilon_0 < \hat{\lambda} - 4\varepsilon < \hat{\lambda} - \varepsilon \leq a''_\rho \leq \inf_{\mathbb{X}_2 \cap \mathcal{S}_{\rho,\delta}} f_\rho \\ &\leq \sup_{\mathbb{X}_2 \cap \mathcal{S}_{\rho,\delta}} f_\rho \leq b'_\rho \leq \hat{\lambda} + \varepsilon < \hat{\lambda} + 4\varepsilon < \lambda_{k+1} - \varepsilon_0 \leq b''_\rho; \end{aligned} \tag{3.14}$$

$$\begin{gathered} \text{there are no } u \in \Sigma_{\rho,\delta} \text{ with } u \text{ lower critical for } \bar{f}_\rho \text{ and} \\ f_\rho(u) \in [\lambda_{i-1} + \varepsilon_0, \lambda_{k+1} - \varepsilon_0]\,; \end{gathered} \tag{3.15}$$

$$\begin{gathered} \text{there are no } u \in \mathcal{S}_{\rho,\delta} \text{ with } u \text{ lower critical for } \bar{f}_\rho \text{ and} \\ f_\rho(u) \in \left[\lambda_{i-1} + \varepsilon_0, \hat{\lambda} - \varepsilon\right] \cup \left[\hat{\lambda} + \varepsilon, \lambda_{k+1} - \varepsilon_0\right]; \end{gathered} \tag{3.16}$$

$$\left|f_0(u) - f_\rho(\Phi(\rho, u))\right| < \varepsilon \qquad \forall u \in \mathcal{S}_{0,\delta} \text{ with } f_0(u) \leq \lambda_{k+1} - \varepsilon_0; \tag{3.17}$$

$$\left|f_\rho(u) - f_0(\Phi(0, (u))\right| < \varepsilon \qquad \forall u \in \mathcal{S}_{\rho,\delta} \text{ with } f_\rho(u) \leq \lambda_{k+1} - \varepsilon_0. \tag{3.18}$$

To prove (3.17) and (3.18) we use (3.8). If $\sigma \in [\varepsilon, 4\varepsilon]$, set $A^\sigma_\rho := \bar{f}_\rho^{\hat{\lambda}/2 - \sigma}$, $B^\sigma_\rho := \bar{f}_\rho^{\hat{\lambda}/2 + \sigma}$ i.e.:

$$A^\sigma_\rho = \left\{u \in S_{\rho,\delta} : f_\rho(u) \leq \hat{\lambda}/2 - \sigma\right\}, \qquad B^\sigma_\rho = \left\{u \in S_{\rho,\delta} : f_\rho(u) \leq \hat{\lambda}/2 + \sigma\right\}.$$

Moreover set $\tilde{A}^\sigma_\rho := \Phi(\rho, A^\sigma_0)$, $\tilde{B}^\sigma_\rho := \Phi(\rho, B^\sigma_0)$, $\hat{A}^\sigma_\rho := \Phi(0, A^\sigma_\rho)$, $\hat{B}^\sigma_\rho := \Phi(0, B^\sigma_\rho)$. From (3.17) and (3.18) (remind that $\Phi(\rho, \cdot)^{-1} = \Phi(0, \cdot)$) we get

$$\begin{aligned} &A^{4\varepsilon}_0 \subset \Phi(0, A^{3\varepsilon}_\rho) \subset A^{2\varepsilon}_0 \subset \Phi(0, A^\varepsilon_\rho), \qquad B^\varepsilon_0 \subset \Phi(0, B^{2\varepsilon}_\rho) \subset B^{3\varepsilon}_0 \Phi(0, B^{4\varepsilon}_\rho), \\ &A^{4\varepsilon}_\rho \subset \Phi(\rho, A^{3\varepsilon}_0) \subset A^{2\varepsilon}_\rho \subset \Phi(\rho, A^\varepsilon_0), \qquad B^\varepsilon_\rho \subset \Phi(\rho, B^{2\varepsilon}_0) \subset B^{3\varepsilon}_\rho \Phi(\rho, B^{4\varepsilon}_0). \end{aligned}$$

The above inclusions give rise to the following diagram in homology:

$$\begin{tikzcd} H_q(B^\varepsilon_\rho, A^{4\varepsilon}_\rho) \arrow[r, "i_1^*"] \arrow[d, "\phi_1^*"] & H_q(\tilde{B}^{2\varepsilon}_\rho, \tilde{A}^{3\varepsilon}_\rho) \arrow[r, "i_2^*"] & H_q(B^{3\varepsilon}_\rho, A^{2\varepsilon}_\rho) \arrow[r, "i_3^*"] \arrow[d, "\phi_3^*"] & H_q(\tilde{B}^{4\varepsilon}_\rho, \tilde{A}^\varepsilon_\rho) \\ H_q(\hat{B}^\varepsilon_\rho, \hat{A}^{4\varepsilon}_\rho) \arrow[r, "j_1^*"] & H_q(B^{2\varepsilon}_0, A^{3\varepsilon}_0) \arrow[r, "j_2^*"] \arrow[u, "\phi_2^*"] & H_q(\hat{B}^{3\varepsilon}_\rho, \hat{A}^{2\varepsilon}_\rho) \arrow[r, "j_3^*"] & H_q(B^{4\varepsilon}_0, A^\varepsilon_0) \arrow[u, "\phi_2^*"] \end{tikzcd}$$

where i_1, i_2, i_3, j_1, j_2, j_3 are embeddings and ϕ_1, ϕ_3 are restrictions of $\Phi(0, \cdot)$, while ϕ_2, ϕ_4 are restrictions of $\Phi(\rho, \cdot)$. It is clear that ϕ_i^* are isomorphisms. Notice that $i_2 \circ \phi_2 \circ j_1 \circ \phi_1$ is the embedding of $(B^\varepsilon_\rho, A^{4\varepsilon}_\rho)$ in $(B^{3\varepsilon}_\rho, A^{2\varepsilon}_\rho)$ and $j_3 \circ \phi_3 \circ i_2 \circ \phi_2$ is the embedding of $(B^{2\varepsilon}_0, A^{3\varepsilon}_0)$ in $(B^{4\varepsilon}_0, A^\varepsilon_0)$.

Since there are no critical values for $\bar{f}_0$ in $[\hat{\lambda}-4\varepsilon,\hat{\lambda}-\varepsilon]\cup[\hat{\lambda}+\varepsilon,\hat{\lambda}+4\varepsilon]$ (see (3.16)), then the pair $(B_\rho^{\varepsilon},A_\rho^{4\varepsilon})$ is a deformation retract of the pair $(B_\rho^{3\varepsilon},A_\rho^{2\varepsilon})$, so $i_2^*\circ\phi_2^*\circ j_1^*\circ\phi_1^*$ is an isomorphism. For analogous reasons $j_3^*\circ\phi_3^*\circ i_2^*\circ\phi_2^*$ is an isomorphism. It follows that $i_2^*\circ\phi_2^*: H_q(B_0^{2\varepsilon},A_0^{3\varepsilon})\to H_q(B_\rho^{3\varepsilon},A_\rho^{2\varepsilon})$ is an isomorphism.

From the definitions (3.11) and (3.12) we have the inclusions:

$$(\mathcal{S}_{0,\delta}\cap(\mathbb{X}_1\oplus\mathbb{X}_2),\Sigma_{0,\delta}\cap(\mathbb{X}_1\oplus\mathbb{X}_2))\subset(B_\rho^{3\varepsilon},A_\rho^{2\varepsilon})\subset(\mathcal{S}_{0,\delta}\setminus\mathbb{X}_3,\mathcal{S}_{\rho,\delta}\setminus(\mathbb{X}_2\oplus\mathbb{X}_3))$$

which allow to repeat the arguments of [2] (see also the proof of Lemma 2.3 in [21]). To estimate the relative category:

$$\operatorname{cat}_{(B_\rho^{3\varepsilon},A_\rho^{2\varepsilon})}(B_\rho^{3\varepsilon})\geq 2\qquad\forall\rho\in\,]0,\rho(\varepsilon)].$$

This implies that $\bar{f}_\rho$ hat at least two critical points $\bar{u}_{1,\rho}$, $\bar{u}_{2,\rho}$ with $\hat{\lambda}-3\varepsilon\leq f_\rho(\bar{u}_{i,\rho})\leq\hat{\lambda}+2\varepsilon$. We have $\|\bar{u}_{i,\rho}\|_{\mathbb{L}}^2/2+\mathcal{H}_{1,\rho}(\bar{u}_{i,\rho})=1$ and

$$\langle A\bar{u}_{i,\rho}+\nabla\mathcal{H}_\rho(\bar{u}_{i,\rho}),v\rangle=\lambda_{i,\rho}\langle\bar{u}_{i,\rho}+\nabla_{\mathbb{L}}\mathcal{H}_{1,\rho}(\bar{u}_{i,\rho}),v\rangle_{\mathbb{L}}\qquad\forall v\in\mathbb{H}\tag{3.19}$$

for a suitable Lagrange multiplier $\lambda_{i,\rho}\in\mathbb{R}$ (there is no μ, due to (3.15)). Taking $v=\bar{u}_{i,\rho}$ in (3.19):

$$\begin{aligned}[\hat{\lambda}-2\varepsilon,\hat{\lambda}+3\varepsilon]\ni\bar{f}(\bar{u}_{i,\rho})&=\frac{1}{2}\langle A\bar{u}_{i,\rho},\bar{u}_{i,\rho}\rangle+\mathcal{H}_\rho(\bar{u}_{i,\rho})\\&=\mathcal{H}_\rho(\bar{u}_{i,\rho})-\frac{1}{2}\langle\nabla\mathcal{H}_\rho(\bar{u}_{i,\rho}),\bar{u}_{i,\rho}\rangle+\frac{\lambda_{i,\rho}}{2}\left(\|\bar{u}_{i,\rho}\|_{\mathbb{L}}^2+\langle\nabla_{\mathbb{L}}\mathcal{H}_{1,\rho}(\bar{u}_{i,\rho}),\bar{u}_{i,\rho}\rangle_{\mathbb{L}}\right)\\&=\underbrace{\mathcal{H}_\rho(\bar{u}_{i,\rho})-\frac{\langle\nabla\mathcal{H}_\rho(\bar{u}_{i,\rho}),\bar{u}_{i,\rho}\rangle}{2}}_{:=C_1(\rho)}+\lambda_{i,\rho}\left(1+\underbrace{\left(\frac{\langle\nabla_{\mathbb{L}}\mathcal{H}_{1,\rho}(\bar{u}_{i,\rho}),\bar{u}_{i,\rho}\rangle_{\mathbb{L}}}{2}-\mathcal{H}_{1,\rho}(\bar{u}_{i,\rho})\right)}_{:=C_2(\rho)}\right)\end{aligned}$$

By using (3.9) we obtain $C_1(\rho)\to 0$, $C_2(\rho)\to 0$, so for $\rho(\varepsilon)$ small enough we have $|\lambda_{i,\rho}-\hat{\lambda}|<4\varepsilon$. We have thus proven that $\lambda_{1,\rho}\to\hat{\lambda}$ as $\rho\to 0$. Let $u_{i,\rho}:=\rho\bar{u}_{1,\rho}$. Clearly $u_{i,\rho}\xrightarrow{\mathbb{L}}0$ as $\rho\to 0$. By multiplying (3.19) by ρ and using the definitions of $\mathcal{H}_\rho$ and $\mathcal{H}_{1,\rho}$ we get that $(u,\lambda)=(u_{i,\rho},\lambda_{i,\rho})$ verify (3.3). Taking the scalar product with $u_{i,\rho}$ in (3.3) gives $\langle Au_{i,\rho},u_{i,\rho}\rangle\to 0$. Then, by (3.1), we have $u_{i,\rho}\xrightarrow{\mathbb{H}}0$. □

4 A global bifurcation result for radial solutions

We consider the case $N=2$ and $\Omega=B(0,R)=\{x\in\mathbb{R}^2:\|X\|<R\}$. We look for radial solutions for Problem (2.2), i.e. $z(x,y)=w(\|(x,y)\|)$. Actually with similar arguments we could have considered the general case $N\geq 2$. Given $R>0$, it is therefore convenient to introduce the Hilbert space

$$E:=\left\{w:[0,R]\to\mathbb{R}:\int_0^R\rho\dot{w}^2\,d\rho<+\infty\right\}$$

endowed with $(v,w)_E:=\int_0^R\rho\dot{v}\dot{w}\,d\rho+\int_0^R\rho vw\,d\rho$ and for $\lambda>0$ the set

$$W_\lambda:=\left\{w\in E:1+\sqrt{\lambda}w(\rho)>0\right\},\qquad\mathcal{W}:=\{(w,\lambda)\in\mathbb{R}\times E:\lambda>0,w\in W_\lambda\}.$$

It is clear that $\|w\|_\infty \leq C\|w\|_E$, for a suitable constant C, so W is open in E and $\mathcal{W}$ is open in $\mathbb{R} \times E$. As well known the search for radial solutions leads to the equation

$$\begin{cases} \ddot{w} + \dfrac{\dot{w}}{\rho} = -\lambda w - \dfrac{\lambda w}{1 + \sqrt{\lambda w}} =: f_\lambda(w), \\ \dot{w}(0) = \dot{w}(R) = 0. \end{cases} \tag{4.1}$$

By the above we mean that

$$(w, \lambda) \in \mathcal{W}, \qquad \int_0^R \rho \dot{w} \dot{\delta} \, d\rho = \int_0^R \rho f_\lambda(w) \delta \, d\rho \qquad \forall v \in E. \tag{4.2}$$

It is standard to check that "weak solutions", i.e. solutions to (4.2) actually solve (4.1) in a classical sense.

It is clear that $(0, \lambda)$ is a solution for (4.1) for any $\lambda \in \mathbb{R}$. We call "nontrivial " solution a pair (w, λ) with $w \neq 0$ such that (4.1) holds.

Remark 4.1. If (w, λ) is a nontrivial solution then $\lambda > 0$. To see this it suffices to multiply (4.1) by u and integrate over $[0, R]$. Actually this property is true in the general case (not just in the radial problem).

We shall use the following simple inequality.

Remark 4.2. Let $0 < a < b < +\infty$. We have

$$\frac{b-a}{b} \leq \ln\left(\frac{b}{a}\right) \leq \frac{b-a}{a}. \tag{4.3}$$

We have indeed

$$\ln\left(\frac{b}{a}\right) = \ln\left(1 + \frac{b-a}{a}\right) \leq \frac{b-a}{a}$$

and

$$\ln\left(\frac{b}{a}\right) = -\ln\left(\frac{a}{b}\right) = -\ln\left(1 + \frac{a-b}{b}\right) \geq -\frac{a-b}{b} = \frac{b-a}{b}.$$

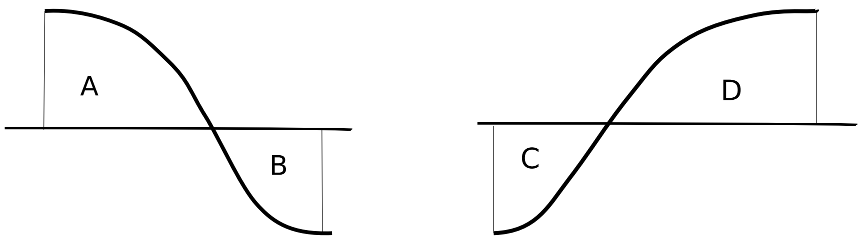

Figure 4.1: The different cases

Now let us suppose that a solution (w, λ) exists so we can find some properties and estimates on w. Arguing as in the proof of Lemma 2.2 in [8] we have that either $w = 0$ or $[0, R]$ can be split as the union of a finite number of subintervals $[r_{1,i}, r_{2,i}]$, $i = 1 \ldots, k$, where w has one of the following behaviors (see Figure 4.1, we are skipping the index i):

(A) $w(r_1) > 0$, $\dot{w}(r_1) = 0$, $\dot{w} < 0$ in $]r_1, r_2]$, and $w(r_2) = 0$;

(B) $w(r_1) = 0$, $\dot{w} < 0$ in $[r_1, r_2[$, $\dot{w}(r_2) = 0$, and $w(r_2) < 0$;

(C) $w(r_1) < 0$, $\dot w(r_1) = 0$, $\dot w > 0$ in $]r_1, r_2]$, and $w(r_2) = 0$;

(D) $w(r_1) = 0$, $\dot w > 0$ in $[r_1, r_2[$, $\dot w(r_2) = 0$, and $w(r_2) > 0$.

So let $w : [r_1, r_2] \to \mathbb{R}$ be as in one of the above cases. Multiplying (4.1) by $\dot w$ gives

$$\frac{1}{2}\ddot w \dot w' + \frac{\dot w^2}{\rho} = \frac{d}{d\rho}F_\lambda(w)$$

where

$$F_\lambda(s) = \ln(1 + \sqrt{\lambda}s) - \sqrt{\lambda}s - \frac{\lambda}{2}s^2.$$

Let $p := \dot w^2$, the previous equation can be written as

$$\frac{1}{2}\dot p + \frac{p}{\rho} = \frac{d}{d\rho}F_\lambda(w)$$

which is equivalent to

$$\frac{d}{d\rho}(\rho^2 p) = 2\rho^2 \frac{d}{d\rho}F_\lambda(w)\rho^2 = 2\rho^2 \frac{d}{d\rho}F_1\left(\sqrt{\lambda}w\right).$$

We integrate between ρ_1 and ρ_2, where $r_1 \le \rho_1 \le \rho_2 \le r_2$:

$$\rho_2^2 p(\rho_2) - \rho_1^2 p(\rho_1) = 2\rho_2^2 F_\lambda(w(\rho_2)) - 2\rho_1^2 F_\lambda(w(\rho_1)) - \int_{\rho_1}^{\rho_2} 4\sigma F_\lambda(w(\sigma))\, d\sigma.$$

Notice that F_λ is increasing on $\left]-\frac{1}{\sqrt{\lambda}}, 0\right[$ and decreasing on $]0, +\infty[$, so

$\sigma \mapsto F_\lambda(w(\sigma))$ is increasing (decreasing) in cases (A) and (C) (in cases (B) and (D)).

We hence get, in cases (A) and (C):

$$-2(\rho_2^2 - \rho_1^2)F_\lambda(w(\rho_2)) \le -\int_{\rho_1}^{\rho_2} 4\sigma F_\lambda(w(\sigma))\, d\sigma \le -2(\rho_2^2 - \rho_1^2)F_\lambda(w(\rho_1))$$

while in cases (B) and (D):

$$-2(\rho_2^2 - \rho_1^2)F_\lambda(w(\rho_1)) \le -\int_{\rho_1}^{\rho_2} 4\sigma F_\lambda(w(\sigma))\, d\sigma \le -2(\rho_2^2 - \rho_1^2)F_\lambda(w(\rho_2)).$$

So in cases (A) and (C) we have

$$2\rho_1^2(F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1)) \le \rho_2^2 p(\rho_2) - \rho_1^2 p(\rho_1) \le 2\rho_2^2(F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1)) \tag{4.4}$$

and in cases (B) and (D):

$$2\rho_2^2(F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1)) \le \rho_2^2 p(\rho_2) - \rho_1^2 p(\rho_1) \le 2\rho_1^2(F_\lambda(w(\rho_2)) - F_\lambda(w(\rho_1)). \tag{4.5}$$

Now we estimate $w(\rho)$ – we need to take into account all the four cases (A), (B), (C), (D).

Case (A). We rename $\bar\rho := r_1$, $\rho_0 := r_2$ and let $h := w(\bar\rho) > 0$. We use (4.4) with $\rho_1 = \bar\rho$ and $\rho_2 = \sigma \in [\bar\rho, \rho_0]$:

$$2\bar\rho^2(F_\lambda(w(\sigma)) - F_\lambda(h)) \le \sigma^2 \dot w(\sigma)^2 \le 2\sigma^2(F_\lambda(w(\sigma)) - F_\lambda(h)).$$

Then we take the square root and divide:

$$\sqrt{2}\frac{\bar{\rho}}{\sigma} \leq \frac{-\dot{w}(\sigma)}{\sqrt{F_\lambda(w(\sigma)) - F_\lambda(h)}} \leq \sqrt{2}$$

and now we integrate between $\bar{\rho}$ and $\rho \in [\bar{\rho}, \rho_0]$ getting

$$\sqrt{2}\bar{\rho} \ln\left(\frac{\rho}{\bar{\rho}}\right) \leq -\Phi_{\lambda,h}(w(\rho)) + \Phi_{\lambda,h}(h) \leq \sqrt{2}(\rho - \bar{\rho})$$

where $\Phi_{\lambda,h} : [0, h] \to \mathbb{R}$ is defined by

$$\Phi_{\lambda,h}(s) := \int_0^s \frac{d\xi}{\sqrt{F_\lambda(\xi) - F_\lambda(h)}}$$

(it is simple to check the the integral converges at $\xi = h$). So we deduce

$$\Phi_{\lambda,h}^{-1}\left(\Phi_{\lambda,h}(h) - \sqrt{2}\,(\rho - \bar{\rho})\right) \leq w(\rho) \leq \Phi_{\lambda,h}^{-1}\left(\Phi_{\lambda,h}(h) - \sqrt{2}\bar{\rho}\ln\left(\frac{\rho}{\bar{\rho}}\right)\right)$$

which we prefer to write as

$$\Phi_{\lambda,h}^{-1}\left(\Phi_{\lambda,h}(h) + \sqrt{2}\,(\bar{\rho} - \rho)\right) \leq w(\rho) \leq \Phi_{\lambda,h}^{-1}\left(\Phi_{\lambda,h}(h) + \sqrt{2}\bar{\rho}\ln\left(\frac{\bar{\rho}}{\rho}\right)\right). \tag{4.6}$$

In particular, taking $\rho = \rho_0$, which gives $w(\rho_0) = 0$, (and using (4.3)) we have

$$\sqrt{2}\frac{\bar{\rho}}{\rho_0}(\rho_0 - \bar{\rho}) \leq \sqrt{2}\bar{\rho}\ln\left(\frac{\rho_0}{\bar{\rho}}\right) \leq \Phi_{\lambda,h}(h) \leq \sqrt{2}\,(\rho_0 - \bar{\rho}). \tag{4.7}$$

Moreover taking $\rho_1 = \bar{\rho}$ and $\rho_2 = \rho_0$ in (4.4) we have:

$$\sqrt{2}\frac{\bar{\rho}}{\rho_0}\sqrt{-F_\lambda(h))} \leq -\dot{w}(\rho_0) \leq \sqrt{2}\sqrt{-F_\lambda(h))} \tag{4.8}$$

Case (B). We rename $\rho_0 := r_1$, $\bar{\rho} := r_2$ and let $h := w(\bar{\rho}) < 0$. We use (4.5) with $\rho_1 = \sigma \in [\rho_0, \bar{\rho}]$ and $\rho_2 = \bar{\rho}$:

$$2\bar{\rho}^2(F_\lambda(h) - F_\lambda(w(\sigma))) \leq -\sigma^2\dot{w}(\sigma)^2 \leq 2\sigma^2(F_\lambda(h) - F_\lambda(w(\sigma))).$$

We change sign and proceed as in case (A):

$$2\sigma^2(F_\lambda(w(\sigma)) - F_\lambda(h)) \leq \sigma^2\dot{w}(\sigma)^2 \leq 2\bar{\rho}^2(F_\lambda(w(\sigma)) - F_\lambda(h))).$$

Take the square root and divide:

$$\sqrt{2} \leq \frac{-\dot{w}(\sigma)}{\sqrt{F_\lambda(w(\sigma)) - F_\lambda(h)}} \leq \sqrt{2}\frac{\bar{\rho}}{\sigma}.$$

Integrate on $[\rho, \bar{\rho}_0]$:

$$\sqrt{2}(\bar{\rho} - \rho) \leq -\Phi_{\lambda,h}(h) + \Phi_{\lambda,h}(w(\rho)) \leq \sqrt{2}\bar{\rho}\ln\left(\frac{\bar{\rho}}{\rho}\right)$$

defining $\Phi_{\lambda,h} : [h, 0] \to \mathbb{R}$ as in case (A). Applying $\Phi^{-1}_{\lambda,h}$ we get that (4.6) holds in case (B) too. In particular, taking $\rho = \rho_0$ (and using (4.3)):

$$\sqrt{2}(\bar{\rho} - \rho_0) \le -\Phi_{\lambda,h}(h) \le \sqrt{2}\bar{\rho} \ln\left(\frac{\bar{\rho}}{\rho_0}\right) \le \sqrt{2}\frac{\bar{\rho}}{\rho_0}(\bar{\rho} - \rho_0) \tag{4.9}$$

and taking $\rho_1 = \rho_0$ and $\rho_2 = \bar{\rho}$ in (4.5) we have

$$\sqrt{2}\sqrt{-F_\lambda(h))} \le -\dot{w}(\rho_0) \le \sqrt{2}\frac{\bar{\rho}}{\rho_0}\sqrt{-F_\lambda(h))} \tag{4.10}$$

Case (C). We rename $\bar{\rho} := r_1$, $\rho_0 := r_2$ end let $h := w(\bar{\rho}) < 0$. Using (4.4) with $\rho_1 = \bar{\rho}$ and $\rho_2 = \sigma \in [\bar{\rho}, \rho_0]$ we obtain the same inequality of case (A). After taking the square root and dividing:

$$\sqrt{2}\frac{\bar{\rho}}{\sigma} \le \frac{\dot{w}(\sigma)}{\sqrt{F_\lambda(w(\sigma)) - F_\lambda(h)}} \le \sqrt{2}.$$

We integrate between $\bar{\rho}$ and $\rho \in [\bar{\rho}, \rho_0]$ getting

$$\sqrt{2}\bar{\rho} \ln\left(\frac{\rho}{\bar{\rho}}\right) \le \Phi_{\lambda,h}(w(\rho)) - \Phi_{\lambda,h}(h) \le \sqrt{2}(\rho - \bar{\rho})$$

with $\Phi_{\lambda,h} : [h, 0] \to \mathbb{R}$ defined as above. So we deduce

$$\Phi^{-1}_{\lambda,h}\left(\Phi_{\lambda,h}(h) + \sqrt{2}\bar{\rho} \ln\left(\frac{\rho}{\bar{\rho}}\right)\right) \le w(\rho) \le \Phi^{-1}_{\lambda,h}\left(\Phi_{\lambda,h}(h) + \sqrt{2}\,(\rho - \bar{\rho})\right). \tag{4.11}$$

In particular, taking $\rho = \rho_0$ (and using (4.3)):

$$\sqrt{2}\frac{\bar{\rho}}{\bar{\rho}_0}(\rho_0 - \bar{\rho}) \le \sqrt{2}\bar{\rho} \ln\left(\frac{\rho_0}{\bar{\rho}}\right) \le -\Phi_{\lambda,h}(h) \le \sqrt{2}\,(\rho_0 - \bar{\rho}). \tag{4.12}$$

Moreover taking $\rho_1 = \bar{\rho}$ and $\rho_2 = \rho_0$ in (4.4) we have

$$\sqrt{2}\frac{\bar{\rho}}{\rho_0}\sqrt{-F_\lambda(h))} \le \dot{w}(\rho_0) \le \sqrt{2}\sqrt{-F_\lambda(h))}. \tag{4.13}$$

Case (D). We rename $\rho_0 := r_1$, $\bar{\rho} := r_2$ and let $h := w(\bar{\rho}) > 0$. Using (4.5) with $\rho_1 = \sigma \in [\rho_0, \bar{\rho}]$ and $\rho_2 = \bar{\rho}$ we obtain the same inequalities of case (B). When we take the square root and divide:

$$\sqrt{2} \le \frac{\dot{w}(\sigma)}{\sqrt{F_\lambda(w(\sigma)) - F_\lambda(h)}} \le \sqrt{2}\frac{\bar{\rho}}{\sigma}.$$

Integrate on $[\rho, \bar{\rho}_0]$:

$$\sqrt{2}(\bar{\rho} - \rho) \le \Phi_{\lambda,h}(h) - \Phi_{\lambda,h}(w(\rho)) \le \sqrt{2}\bar{\rho} \ln\left(\frac{\bar{\rho}}{\rho}\right)$$

with the usual definition of $\Phi_{\lambda,h} : [h, 0] \to \mathbb{R}$. Applying $\Phi^{-1}_{\lambda,h}$ we obtain that (4.11) holds in case (D) too. In particular, taking $\rho = \rho_0$ (and using (4.3)):

$$\sqrt{2}\,(\bar{\rho} - \rho_0) \le \Phi_{\lambda,h}(h) \le \sqrt{2}\bar{\rho} \ln\left(\frac{\bar{\rho}}{\bar{\rho}_0}\right) \le \sqrt{2}\frac{\bar{\rho}}{\rho_0}(\bar{\rho} - \rho_0) \tag{4.14}$$

and taking $\rho_1 = \rho_0$ and $\rho_2 = \bar{\rho}_0$ in (4.5) we have

$$\sqrt{2}\sqrt{-F_\lambda(h))} \le \dot{w}(\rho_0) \le \sqrt{2}\frac{\bar{\rho}}{\rho_0}\sqrt{-F_\lambda(h))}. \tag{4.15}$$

Now we have

$$\sqrt{2}\Phi_{\lambda,h}(h) = \int_0^h \frac{d\xi}{\sqrt{F(\sqrt{\lambda}\xi) - F(\sqrt{\lambda}h)}} = \int_0^1 \frac{h\, d\sigma}{\sqrt{F(\sigma\sqrt{\lambda}h) - F(\sqrt{\lambda}h)}} = \frac{1}{\sqrt{\lambda}}\bar{\Phi}(\sqrt{\lambda}h)$$

where

$$\bar{\Phi}(s) := \int_0^1 \frac{s\, d\sigma}{\sqrt{F(\sigma s) - F(s)}} = \operatorname{sgn}(s)\int_0^1 \sqrt{\frac{s^2}{F(\sigma s) - F(s)}}\, d\sigma.$$

With simple computations:

$$\lim_{s\to 0} \frac{s^2}{F(\sigma s) - F(s)} = \frac{1}{1-\sigma^2}, \qquad \lim_{s\to +\infty} \frac{s^2}{F(\sigma s) - F(s)} = \frac{2}{1-\sigma^2},$$

and

$$\lim_{s\to -1^-} \frac{s^2}{F(\sigma s) - F(s)} = 0.$$

So we deduce that (see Figure 4.2)

$$\lim_{h\to 0^+} \Phi_{\lambda,h}(h) = \frac{\pi}{2\sqrt{2\lambda}}, \qquad \lim_{h\to +\infty} \Phi_{\lambda,h}(h) = \frac{\pi}{2\sqrt{\lambda}}, \tag{4.16}$$

$$\lim_{h\to 0^-} \Phi_{\lambda,h}(h) = -\frac{\pi}{2\sqrt{2\lambda}}, \qquad \lim_{h\to -1^+} \Phi_{\lambda,h}(h) = 0. \tag{4.17}$$

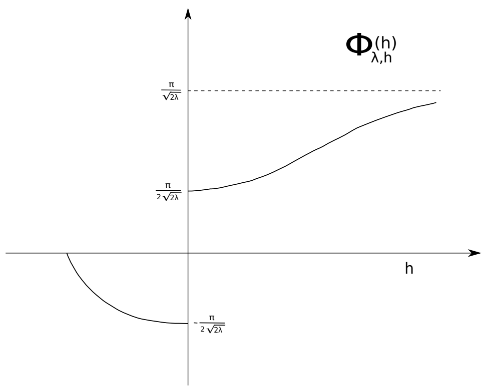

Figure 4.2: Graph of $\Phi_{\lambda,h}(h)$

To state the main result we need some notation, which we take from [8, 23]. For $k \in \mathbb{N}$, $k \ge 1$, we consider

$$\begin{aligned} \mathcal{S} &:= \{(w,\lambda) \in \mathcal{W} : (w,\lambda) \text{ is a solution to (4.1)}\} \\ \mathcal{S}_k^+ &:= \{(w,\lambda) \in \mathcal{S} :\ w \text{ has } k \text{ nodes in }]0,R[, w(0) > 0\}, \\ \mathcal{S}_k^- &:= \{(w,\lambda) \in \mathcal{S} :\ w \text{ has } k \text{ nodes in }]0,R[, w(0) < 0\}. \end{aligned}$$

We also consider the two eigenvalue problems:

$$\ddot{w} + \frac{\dot{w}}{\rho} = -\mu w, \qquad \dot{w}(0) = \dot{w}(R) = 0. \tag{4.18}$$

$$\ddot{v} + \frac{\dot{v}}{\rho} = -\nu v, \qquad \dot{v}(0) = 0, v(R) = 0. \tag{4.19}$$

It is clear that $w \neq 0$ and $\mu \neq 0$ solve (4.18) if and only if, for some integer $k \geq 1$,

$$\mu = \mu_k := \left(\frac{y_k}{R}\right)^2 \tag{4.20}$$

where y_k denotes the k-th nontrivial zero of J_0' and J_0 is the first Bessel function, and

$$w = \alpha w_k, \quad \alpha \in \mathbb{R}, \qquad w_k(\rho) := J_0\left(\frac{y_k}{R}\rho\right). \tag{4.21}$$

For the sake of completeness we can agree that $\mu_0 = 0$ and $w_0(\rho) = J_0(0)$. In the same way $v \neq 0$ and ν solve (4.19) if and only if, for some integer $k \geq 1$:

$$\nu = \nu_k := \left(\frac{z_k}{R}\right)^2 \tag{4.22}$$

where z_k is the k-th zero of J_0 and

$$v = \alpha v_k, \quad \alpha \in \mathbb{R}, \qquad v_k(\rho) := J_0\left(\frac{z_k}{R}\rho\right). \tag{4.23}$$

Notice that $\nu_k < \mu_k < \nu_{k+1}$ for all k.

Theorem 4.3. *Let $\mu_k > 0$ be an eigenvalue for (4.18). Then $\mathcal{S}_k^+$ is a connected set and*

- $(0, \mu_k/2) \in \overline{\mathcal{S}_k^+}$*;*
- $0 < \inf\left\{\lambda \in \mathbb{R} : \exists w \in E \text{ with } (w, \lambda) \in \mathcal{S}_k^+\right\}$*;*
- $\sup\left\{\lambda \in \mathbb{R} : \exists w \in E \text{ with } (w, \lambda) \in \mathcal{S}_k^+\right\} < +\infty$*;*
- $\mathcal{S}_k^+$ *is unbounded and contains a sequence (w_n, λ_n) such that $\|w_n\|_E \to \infty$ and*

$$\lim_{n\to\infty} \lambda_n = \begin{cases} \mu_{k/2} & \text{if } k \text{ is even,} \\ \nu_{(k+1)/2} & \text{if } k \text{ is odd.} \end{cases} \tag{4.24}$$

Figure 4.3 somehow illustrates Theorem (4.3).

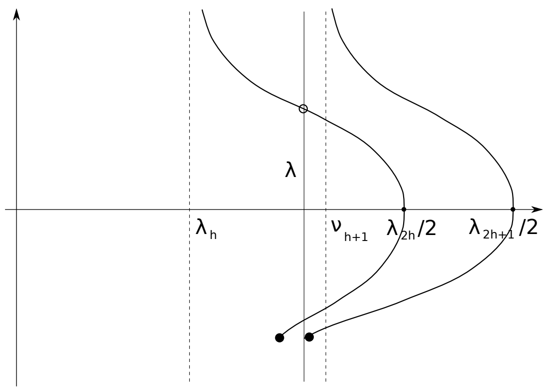

Figure 4.3: Bifurcation diagram

The proof of (4.3) will be obtained from some preliminary statements.

Remark 4.4. If $(w, \lambda) \in \mathcal{S}^+$ (resp. $(w, \lambda) \in \mathcal{S}^+$), and $0 = \rho_0 < \rho_1, \ldots, \rho_k < \rho_{k+1=R}$, $\rho_1, \ldots, \rho_k$ being the nodal points of w, then

$$\rho_{i+1} - \rho_i \geq \ (\leq) \ \frac{\pi}{4\sqrt{\lambda}} \quad \text{for } i \text{ even} \quad (\text{resp. for } i \text{ odd}). \tag{4.25}$$

This is easily seen using the right hand sides of the inequalities (4.7), (4.12), and (4.16).

Lemma 4.5. *For any integer k there esist two constants $\underline{\lambda}_k$ and $\overline{\lambda}_k$ such that*

$$(w, \lambda) \in \mathcal{S}_k^+ \cup \mathcal{S}_k^- \Rightarrow 0 < \underline{\lambda}_k \leq \lambda \leq \overline{\lambda}_k < +\infty. \tag{4.26}$$

Proof. Take any subinterval $[r_1, r_2]$ as in cases (A)–(D) and consider the first eigenvalue $\bar{\mu} = \bar{\mu}(r_1, r_2)$ for the mixed type boundary condition

$$\begin{cases} -(\rho \dot{w})' = \mu w \quad \text{on }]r_1, r_2[\\ \dot{w}(r_1) = 0, w(r_2) = 0 \qquad (\text{resp. } w(r_1) = 0, \dot{w}(r_2) = 0) \end{cases}$$

in cases (A), (C) (resp. cases (C), (D)). We can choose an eigenfunction $\bar{e}$ corresponding to $\bar{\mu}$ so that $z\bar{e} > 0$ in $]r_1, r_2[$. Multiplying (4.1) by $\bar{e}$ and integrating over $[r_1, r_2]$ yields

$$\bar{\mu} \int_{r_1}^{r_2} \rho z \bar{e} \, d\rho = \lambda \int_{r_1}^{r_2} \rho z \bar{e} \left(1 + \frac{1}{1 + \sqrt{\lambda z}}\right) d\rho.$$

This implies:

$$\lambda \int_{r_1}^{r_2} \rho z \bar{e} \, d\rho \leq \bar{\mu} \int_{r_1}^{r_2} \rho z \bar{e} \, d\rho \leq 2\lambda \int_{r_1}^{r_2} \rho z \bar{e} \, d\rho$$

which gives $\frac{\bar{\mu}}{2} \leq \lambda \leq \bar{\mu}$. Now since $]r_1, r_2[\subset]0, R[$ we have $\bar{\mu} \geq \bar{\mu}[0, R]$. On the other side since w has k nodal points we can choose r_1, r_2 such that $r_2 - r_1 \geq R/k$, which implies $\bar{\mu} \leq \sup_{b-a=R/k} \bar{\mu}(a, b) < +\infty$. This proves (4.26). □

Lemma 4.6. *Let (w_n, λ_n) be a sequence in $\mathcal{S}_k^+$. Then we can consider $0 < \rho_{1,n} < \cdots < \rho_{k,n} < R$ to be the nodes of w_n and set $\rho_{0,n} := 0$, $\rho_{k+1,n} := R$; in ths way $wn(\rho) > 0$ on $]\rho_1, \rho_{i+1}[$ if i is even and $wn(\rho) < 0$ on $]\rho_1, \rho_{i+1}[$ if i is odd. The following facts are equivalent:*

(a) $$\lim_{n \to \infty} \sup_{\rho \in [0,R]} w_n(\rho) = +\infty;$$

(b) $$\lim_{n \to \infty} \inf_{\rho \in [0,R]} (1 + \lambda_n w_n(\rho)) = 0;$$

(c) $$\lim_{n \to \infty} \sup_{\rho \in [\rho_{i,n}, \rho_{i+1,n}]} w_n(\rho) = +\infty \quad \text{if } i \text{ is even;}$$

(d) $$\lim_{n \to \infty} \inf_{\rho \in [\rho_{i,n}, \rho_{i+1,n}]} (1 + \lambda_n w_n(\rho)) = 0 \quad \text{if } i \text{ is odd;}$$

(e) $$\lim_{n \to \infty} \rho_{1+1,n} - \rho_{i,n} = 0 \quad \text{if } i \text{ is odd;}$$

Moreover, if any of the above holds, then (4.24) holds.

Proof. We can assume, passing to a subsequence that $\lambda_n \to \hat{\lambda} \in [\underline{\lambda}_k, \overline{\overline{\lambda}}_k]$. First notice that for all i even (corresponding to $w > 0$) we have

$$\rho_{i+1,n} - \rho_{i,n} \geq \frac{\pi}{4\sqrt{\underline{\lambda}_k}}$$

as we can infer from (4.7) or (4.14) and the behaviour of $\Phi_{\lambda,h}(h)$ in (4.16).

Let

$$h_{i,n} := \max_{\rho_{i,n} \le \rho_{i+1,n}} w(\rho) \text{ for } i \text{ even}, \qquad h_{i,n} := \min_{\rho_{i,n} \le \rho_{i+1,n}} w(\rho) \text{ for } i \text{ odd}.$$

Then for any i even:

$$h_{i,n} \to +\infty \Leftrightarrow \Phi_{\lambda_n,h_{i,n}}(h_{i,n}) \to \frac{\pi}{2\sqrt{\hat{\lambda}}} \Leftrightarrow \dot{w}(\rho_{i,n}) \to +\infty \Leftrightarrow \dot{w}(\rho_{i+1,n}) \to -\infty.$$

This can be deduced from (4.16), (4.8), and (4.15). In the same way, using (4.17), (4.10), and (4.13) we get that, for i odd:

$$1 + \sqrt{\lambda_n} h_{i,n} \to 0 \Leftrightarrow \Phi_{\lambda_n,h_{i,n}}(h_{i,n}) \to 0 \Leftrightarrow \dot{w}(\rho_{i,n}) \to -\infty \Leftrightarrow \dot{w}(\rho_{i+1,n}) \to +\infty.$$

Now we prove our claims. Let $\bar{i} \in \{0,\dots,k\}$ with $\bar{i}$ even (resp. odd) and suppose that $h_{\bar{i},n} \to +\infty$ (resp. $1 + \sqrt{\lambda_n} h_{\bar{i},n} \to 0$). Then $F_{\lambda_n}(h_{\bar{i},n}) \to +\infty$ (resp. $F_{\lambda_n}(h_{\bar{i},n}) \to -\infty$) and by (4.8), (4.15) ((4.10), (4.13)) we get that

$$\dot{w}_n(\rho_{\bar{i},n}) \to +\infty,\ \dot{w}_n(\rho_{\bar{i}+1,n}) \to -\infty \quad (\dot{w}_n(\rho_{\bar{i},n}) \to -\infty,\ \dot{w}_n(\rho_{\bar{i}+1,n}) \to +\infty)$$

which in turn implies

$$F_{\lambda_n}(h_{\bar{i}-1,n}) \to -\infty \text{ (resp. } +\infty),\ F_{\lambda_n}(h_{\bar{i}+1,n}) \to -\infty \text{ (resp. } +\infty)$$

(with the obvious exceptions when $\bar{i} - 1 < 0$ or $\bar{i} + 1 > k$). So we get

$$1 + \sqrt{\lambda_n} h_{\bar{i}-1,n} \to 0\ (h_{\bar{i}-1,n} \to +\infty),\ 1 + \sqrt{\lambda_n} h_{\bar{i}+1,n} \to 0\ (h_{\bar{i}+1,n} \to +\infty).$$

This shows that the property $|F_\lambda(h_{i,n})| \to +\infty$ "propagates" from the i-th interval to the previous and to the next one. From this it is easy to deduce that (a)–(d) are all equivalent. To prove that they are equivalent to (e) just use (4.7), (4.9), (4.12), (4.14), depending on the case, noticing that $\rho_{1,n} \ge \frac{\pi}{4\pi \underline{\lambda}_k}$, as from (4.25) (this would not be possible if we were considering $\mathcal{S}_k^-$).

Finally suppose that (w_n, λ_n) verifies any of (a)–(e). Then $\|w_n\|_\infty \to +\infty$. Let $\hat{w}_n := \frac{w_n}{\|w_n\|_\infty}$. We can suppose that $\hat{w}_n \rightharpoonup \hat{w}$ in E and that

$$\rho_{1,n} \to \rho_1,\ \rho_{2j-1,n} \to \rho_j,\ \rho_{2j,n} \to \rho_j\ 1 \le j \le k/2,\ \rho_{k,n} \to R \text{ if } k \text{ is odd},$$

where $0 = \rho_0 < \rho_1 < \cdots < \rho_h < \rho_h + 1 = R$ and $h = \lfloor k/2 \rfloor$ (so $\rho_1 = R$ when $k = 1$). It is not difficult to prove that $\hat{w}(\rho) > 0$ in $]\rho_i, \rho_{i+1}[$ if $i = 0,\dots,h$, $\hat{w}(\rho_1) = \cdots = \hat{w}(\rho_h) = 0$, $\hat{w}'(0) = 0$ and $\hat{w}'(R) = 0$ is k is even while $\hat{w}(R) = 0$ is k is odd. Moreover for any $i = 0,\dots,h$:

$$-(\rho \hat{w}')' = \hat{\lambda} \hat{w} \quad \text{on }]\rho_i, \rho_{i+1}[$$

Now we can rearrange $\hat{w}$ defining $\tilde{w} := \sum_{j=0}^h (-1)^j \alpha_j \hat{w} \mathbb{1}_{[\rho_j,\rho_{j+1}]}$, where $\alpha_1 = 1$ and $\alpha_j \hat{w}'_-(\rho_j) = \alpha_{j+1} \hat{w}'_+(\rho_j)$, $j = 1,\dots,h$. In this way $(\hat{\lambda}, \tilde{w})$ is an eigenvalue – eigenfunction pair relative for problem (4.21) if k is even and of (4.23) if k is odd. Since $\tilde{w}$ has $h = k/2$ nodal points for k even and $h + 1 = (k+1)/2$ if k is odd, then (4.24) holds. □

Proof of Theorem 4.3. If $\varepsilon \in]0,1[$ we set

$$\mathcal{O}_\varepsilon := \left\{ (w,\lambda) \in E : \varepsilon < \lambda < \varepsilon^{-1}, 1 + \sqrt{\lambda} w(\rho) > \varepsilon, w(\rho) < \varepsilon^{-1}\ \forall \rho \in [0,R] \right\}.$$

Clearly $\mathcal{O}_\varepsilon$ is an open set with $\mathcal{O}_\varepsilon \subset \mathcal{W}$. Moreover, $(\mu_{k/2}, 0) \in \mathcal{O}_\varepsilon$ if ε is sufficiently small. Define $\tilde{h}_{\lambda,\varepsilon}$ as in (2.4) with $s_0 = \varepsilon$ and let $\tilde{h}_\lambda(s) := \tilde{h}_1(\sqrt{\lambda}s)$. Using [23] we get there that there exists a pair $(w_\varepsilon, \lambda_\varepsilon)$ in $\partial\mathcal{O}_\varepsilon$, with w_ε having k nodal points, which solves Problem (4.1) with $\tilde{h}_{\varepsilon,\lambda} := \tilde{h}_\varepsilon(\lambda, \cdot)$ instead of h_λ. Since $(w, \lambda) \in \partial\mathcal{O}_\varepsilon \Rightarrow \tilde{h}_\varepsilon(w, \lambda) = h_\lambda(w)$, we get that $(w_\varepsilon, \lambda_\varepsilon) \in \mathcal{S}_k^+$. For ε small we have $\varepsilon < \underline{\lambda}_k \leq \overline{\lambda}_k < \varepsilon^{-1}$ so we get $w_\varepsilon \in \partial\left\{1 + \sqrt{\lambda_\varepsilon}w > \varepsilon, w < \varepsilon^{-1}\right\}$ i.e. there exists a point $\rho_\varepsilon \in [0, R]$ such that

$$\text{either} \quad 1 + \sqrt{\lambda_\varepsilon} w_\varepsilon(\rho_\varepsilon) = \varepsilon \quad \text{or} \quad w_\varepsilon(\rho_\varepsilon) = \varepsilon^{-1}.$$

We can find a sequence $\varepsilon_n \to 0$ such that the corresponding $(w_n, \lambda_n) := (w_{\varepsilon_n}, \lambda_{\varepsilon_n})$ verify one of the above properties for all $n \in \mathbb{N}$. If the first one holds for all n, then (w_n, λ_n) verifies (b) of Lemma (4.6); in the second case (w_n, λ_n) verifies (a) of Lemma (4.6). Then by Lemma (4.6) $\|w_n\|_\infty \to \infty$ and (4.24) holds. This proves the theorem. □

Remark 4.7. As a consequence of Theorem (4.3) we get that for any $h \geq 1$ integer and any λ strictly between λ_h and $\lambda_{2h}/2$ there exists u such that (u, λ) solves Problem (1.2). The same is true for all λ strictly between ν_h and $\lambda_{2h-1}/2$.

Remark 4.8. The above proof fails if we follow the bifurcation branch (w_ρ, λ_ρ) with $w_\rho(0) < 0$. In this case it seems possible that the branch tends to a point $(\tilde{\lambda}, \tilde{w})$ where $\sqrt{\tilde{\lambda}}\tilde{w}(0) = -1$ (but $\sqrt{\tilde{\lambda}}\tilde{w}(0) > -1$ for $\rho > 0$). This phenomenon, if true, would be worth studying.

Remark 4.9. The computations of this section show that, if Ω is the ball, then there are no solutions for the Dirichlet problem. It is indeed impossible to construct a (nontrivial) solution (w, λ) for (4.1) with $w(R) = 0$.

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