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Multiple solutions to a quasilinear periodic boundary value problem with impulsive effects

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Abstract. The authors investigate the multiplicity of solutions to a quasilinear periodic boundary value problem with impulsive effects. They use variational methods and some critical points theorems for smooth functionals, due to Ricceri, that are defined on reflexive Banach spaces. They obtain the existence of at least three solutions to the problem. The applicability of the results is illustrated with an example.

Keywords: three solutions, quasilinear periodic, boundary value problem, impulsive effects, variational methods.

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1 Introduction

We study the existence of at least three distinct classical solutions to the quasilinear periodic boundary value problem with impulsive effects

$$\begin{cases} -p(x')x'' + \alpha(t)x = \lambda f(t,x) + \mu g(t,x), & t \neq t_j, \quad \text{a.e. } t \in [0,1], \\ \Delta(h'(u'(t_j))) = I_j(u(t_j)), & j = 1,2,\dots,m, \\ x(1) - x(0) = x'(1) - x'(0) = 0, \end{cases} \tag{$P^{f,g}_{\lambda,\mu}$}$$

where $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are L^1-Carathéodory functions, $\lambda > 0$ and $\mu \geq 0$ are parameters, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = 1$, $I_j : \mathbb{R} \to \mathbb{R}$, $j = 1,\dots,m$, are continuous functions, $\Delta(h'(u'(t_j))) = h'(u'(t_j^+)) - h'(u'(t_j^-))$ with $h'(u'(t_j^{\pm})) = \lim_{t \to t^{\pm}} h'(u'(t))$, and

$$h(y) = \int_0^y \left(\int_0^\tau p(\xi) d\xi \right) d\tau \quad \text{for every } y \in \mathbb{R}.$$

Recall that a function $h : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if it satisfies:

(a) $x \mapsto h(t,x)$ is measurable for every $x \in \mathbb{R}$;

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(b) $t \mapsto h(t,x)$ is continuous for a.e. $x \in [0,1]$;

(c) for every $\varepsilon > 0$ there exists a function $l_\varepsilon \in L^1([0,1])$ such that

$$\sup_{|x|\le\varepsilon} |h(t,x)| \le l_\varepsilon(t) \quad \text{for a.e. } t \in [0,1].$$

In this paper, and without further mention, we always assume that:

(Q_1) $p : \mathbb{R} \to (0,\infty)$ is continuous and nondecreasing on $[0,\infty)$, and there exist $M \ge m > 0$ such that

$$m \le p(x) \le M \quad \text{for all } x \in \mathbb{R}; \tag{1.1}$$

(Q_2) $\alpha \in C([0,1])$ and there exist $\alpha_1 \ge \alpha_0 > 0$ such that

$$\alpha_0 \le \alpha(t) \le \alpha_1 \quad \text{for all } t \in [0,1]; \tag{1.2}$$

(Q_3) $I_j : \mathbb{R} \to \mathbb{R}$ is nondecreasing and $I_j(0) = 0$ for $j = 1,\dots,m$.

In recent years, impulsive differential equations have played an important role in modern applied mathematical models of real processes arising in phenomena studied in physics, ecology, biological systems, biotechnology, and industrial robotics. Many authors have applied variational methods to study the existence of multiple solutions of impulsive systems of the form (1.1) or its variations, and we refer the reader to [2–4, 6, 7, 12, 17, 20] and references cited therein for some recent results. For example, Bonanno and Livrea [3] studied the existence and multiplicity of solutions to the periodic boundary value problem

$$\begin{cases} -x'' + A(t)x = \lambda b(t)\nabla G(x), & t \in [0,T], \\ x(T) - x(0) = x'(T) - x'(0) = 0, \end{cases}$$

where $A(t) = (a_{i,j}(t))_{n\times n}$ is a positive definite matrix for all $t \in [0,T]$, $a_{i,j} \in C([0,T],\mathbb{R})$, $G \in C^1(\mathbb{R}^n,\mathbb{R})$, and $b \in L^1([0,T])\backslash\{0\}$ that is nonnegative a.e. In [6], by using a three critical points theorem due to Bonanno and Marano, the existence of at least three solutions to a quasilinear second order differential equation was discussed. Using the symmetric mountain pass theorem and genus properties of critical point theory, Shen and Liu [17] investigated the existence of infinitely many solutions to the second-order quasilinear periodic boundary value problem with impulsive effects

$$\begin{cases} -u(t)'' + b(t)u(t) - (|u(t)|^2)''u(t) = f(t,u), & t \in J, \\ \Delta(u'(t_j)) = I_j(u(t_j)), & j = 1,2,\dots,m, \\ u(T) = u(0), \ u'(T) = u'(0), \end{cases}$$

where $b \in L^\infty(0,T;\mathbb{R})$ and $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is continuous. Using variational methods, Heidarkhani and Moradi [7] discussed the existence of at least one weak solution and infinitely many weak solutions to $(P^{f,g}_{\lambda,\mu})$ with $\mu = 0$ and $I_j \equiv 0$ for $j = 1,2,\dots,m$.

Motivated by the above studies, in this paper, we establish new criteria to guarantee that the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions for appropriate values of the parameters λ and μ. It is worth stressing that we only assume g to be a L^1-Carathéodory function which permits us to use variational methods. In addition, we obtain multiplicity results for two cases: (i) if the nonlinearity f is asymptotically quadratic, and (ii) if it is subquadratic as $|u| \to \infty$. Our approach is based on variational methods and a three critical points theorem due to Ricceri [14].

The remainder of this paper is organized as follows. Section 2 contains some preliminary lemmas, and Section 2.1 contains our main results and their proofs.

2 Preliminaries

Our main tool is a theorem of Ricceri [14, Theorem 2] which is recalled in Lemma 2.1 below. In what follows, we let X be a real Banach space, and as in [14], we let $\mathcal{W}_X$ denote the class of all functionals $\Phi : X \to \mathbb{R}$ having the property: If $\{u_n\}$ is a sequence in X converging weakly to $u \in X$ with $\liminf_{n\to\infty} \Phi(u_n) \leq \Phi(u)$, then $\{u_n\}$ has a subsequence converging strongly to u. For example, if X is uniformly convex and $g : [0, \infty) \to \mathbb{R}$ is a continuous and strictly increasing function, then the functional $u \to g(\|u\|)$ belongs to the class $\mathcal{W}_X$.

Lemma 2.1. *Let X be a separable and reflexive real Banach space, let $\Phi : X \to \mathbb{R}$ be a coercive, sequentially weakly lower semicontinuous C^1-functional belonging to $\mathcal{W}_X$ that is bounded on bounded subsets of X and whose derivative admits a continuous inverse on X^*. Let $J : X \to \mathbb{R}$ be a C^1-functional with a compact derivative and assume that Φ has a strict local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, set*

$$\rho = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\}, \quad \sigma = \sup_{u\in\Phi^{-1}((0,\infty))} \frac{J(u)}{\Phi(u)},$$

and assume that $\rho < \sigma$. Then for each compact interval $[c, d] \subset (1/\sigma, 1/\rho)$ (with the conventions that $1/0 = \infty$ and $1/\infty = 0$), there exists $R > 0$ with the property: for every $\lambda \in [c, d]$ and every C^1-functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\gamma > 0$ such that for each $\mu \in [0, \gamma]$, the equation

$$\Phi'(u) = \lambda J'(u) + \mu \Psi'(u)$$

has at least three solutions in X whose norms are less than R.

We refer the reader to the papers [5,8–10,18,19] in which Lemma 2.1 was successfully used to ensure the existence of at least three solutions to boundary value problems.

The following two results of Ricceri are taken from [15, Theorem 1] and [16, Proposition 3.1], respectively.

Lemma 2.2. *Let X be a reflexive real Banach space, $I \subseteq \mathbb{R}$ be an interval, and let $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous C^1 functional that is bounded on bounded subsets of X, and whose derivative admits a continuous inverse on X^*. Let $J : X \to \mathbb{R}$ be a functional with a compact derivative and assume that*

$$\lim_{\|x\|\to\infty} (\Phi(x) - \lambda J(x)) = \infty, \quad \textit{for all } \lambda \in I,$$

and that there exists $\rho \in \mathbb{R}$ such that

$$\sup_{\lambda\in I} \inf_{x\in X} (\Phi(x) - \lambda(\rho - J(x))) < \inf_{x\in X} \sup_{\lambda\in I} (\Phi(x) - \lambda(\rho - J(x))).$$

Then there exist a nonempty open set $A \subseteq I$ and a positive number R with the property: for every $\lambda \in A$ and every C^1 functional $\Psi : X \to \mathbb{R}$ with compact derivative, there exists $\delta > 0$ such that, for each $\mu \in [0, \delta]$, the equation

$$\Phi'(u) - \lambda J'(u) - \mu\Psi'(u) = 0$$

has at least three solutions in X whose norms are less than R.

Lemma 2.3. *Let X be a nonempty set and let Φ and Ψ be two real functions on X. Assume that there exist $r > 0$ and $x_0, x_1 \in X$ such that*

$$\Phi(x_0) = J(x_0) = 0, \quad \Phi(x_1) > r, \quad \textit{and} \quad \sup_{x\in\Phi^{-1}(-\infty,r]} J(x) < r\frac{J(x_1)}{\Phi(x_1)}.$$

Then for each ρ satisfying

$$\sup_{x\in\Phi^{-1}(-\infty,r]} J(x) < \rho < r\frac{J(x_1)}{\Phi(x_1)},$$

we have

$$\sup_{\lambda\geq 0}\inf_{x\in X} (\Phi(x) - \lambda(\rho - J(x))) < \inf_{x\in X}\sup_{\lambda\geq 0}(\Phi(x) - \lambda(\rho - J(x))).$$

We refer the reader to the paper of Sun *et al.* [18] in which Lemma 2.2 was successfully employed to ensure the existence of at least three solutions to boundary value problems.

To construct an appropriate function space and apply critical point theory, we introduce the following notations and results to be used in the proofs of our main results.

Let us define the Banach space E by

$$E = \left\{u : [0,1] \to \mathbb{R} \,|\, u \text{ is absolutely continuous, } u(1) = u(0),\ u' \in L^2([0,1]\right\},$$

equipped with the norm

$$\|u\| = \left(\int_0^1 \left(|u'|^2 + |u|^2\right) dt\right)^{\frac{1}{2}}.$$

Clearly, E is a Hilbert space with the dual space E^*.

For every $u \in E$, we define

$$\Phi(u) = \int_0^1 h(u'(t))dt + \frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2 dt + \sum_{j=1}^{m}\int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta, \tag{2.1}$$

$$J(u) = \int_0^1 F(t, u(t))dt, \tag{2.2}$$

and

$$\Psi(u) = \int_0^1 G(t, u(t))dt, \tag{2.3}$$

where

$$F(t,x) = \int_0^x f(t,s)ds \quad \text{and} \quad G(t,x) = \int_0^x g(t,s)ds \quad \text{for all } x \in \mathbb{R}.$$

Standard arguments show that $I_\lambda := \Phi - \mu\Psi - \lambda J$ is a Gâteaux differentiable functional whose Gâteaux derivative at the point $u \in E$ is given by

$$\begin{aligned}(\Phi' - \mu\Psi' - \lambda J')(u)(v) = &\int_0^1 h'(u'(t))v'(t)dt + \int_0^1 \alpha(t)u(t)v(t)dt \\ &+ \sum_{j=1}^{m} I_j(u(t_j))v(t_j) - \lambda\int_0^1 f(t,u(t))v(t)dt \\ &- \mu\int_0^1 g(t,u(t))v(t)dt, \quad \text{for all } v \in E.\end{aligned}$$

Furthermore, from the definition of Φ, we see that it is sequentially weakly lower semicontinuous.

Definition 2.4. By a weak solution of the problem ($P^{f,g}_{\lambda,\mu}$), we mean a function $u \in E$ such that

$$\int_0^1 h'(u'(t))v'(t)dt + \int_0^1 \alpha(t)u(t)v(t)dt + \sum_{j=1}^{m} I_j(u(t_j))v(t_j) \\ - \lambda \int_0^1 f(t,u(t))v(t)dt - \mu \int_0^1 g(t,u(t))v(t)dt = 0,$$

for every $v \in E$.

By a classical solution of the problem ($P^{f,g}_{\lambda,\mu}$), we mean a function $u \in E$ such that $u(t)$ satisfies the equation in ($P^{f,g}_{\lambda,\mu}$) for a.e. $t \in [0,1] \setminus \{t_1,\dots,t_m\}$ and both the impulse condition and the boundary condition in ($P^{f,g}_{\lambda,\mu}$) hold.

Clearly, a critical point $u \in E$ of the functional I_λ is a weak solution of the problem ($P^{f,g}_{\lambda,\mu}$). Next , we show that u is indeed a classical solution.

Lemma 2.5. *If $u \in E$ is a critical point of I_λ, then u is a classical solution of ($P^{f,g}_{\lambda,\mu}$).*

Proof. Let $u \in E$ be a critical point for I_λ. Then, for any $v \in E$, it follows that

$$\begin{aligned} 0 &= \int_0^1 h'(u'(t))v'(t)dt + \int_0^1 \alpha(t)u(t)v(t)dt + \sum_{j=1}^{m} I_j(u(t_j))v(t_j) \\ &\quad - \lambda \int_0^1 f(t,u(t))v(t)dt - \mu \int_0^1 g(t,u(t))v(t)dt \\ &= \sum_{j=0}^{m+1} h'(u'(t))v(t)\Big|_{t=t_j^+}^{t_{j+1}^-} + \sum_{j=1}^{m} I_j(u(t_j))v(t_j) \\ &\quad - \int_0^1 [(h'(u'(t)))' - \alpha(t)u(t) + \lambda f(t,u(t)) + \mu g(t,u(t))]v(t)dt \\ &= \sum_{j=1}^{m}[-\Delta(h'(u'(t_j))) + I_j(u(t_j))]v(t_j) + h'(u'(1))v(1) - h'(u'(0))v(0) \\ &\quad - \int_0^1 [(h'(u'(t)))' - \alpha(t)u(t) + \lambda f(t,u(t)) + \mu g(t,u(t))]v(t)dt. \end{aligned}$$

That is, we have

$$\sum_{j=1}^{m}[-\Delta(h'(u'(t_j))) + I_j(u(t_j))]v(t_j) + h'(u'(1))v(1) - h'(u'(0))v(0) \\ - \int_0^1 [(h'(u'(t)))' - \alpha(t)u(t) + \lambda f(t,u(t)) + \mu g(t,u(t))]v(t)dt = 0 \quad \text{for all } v \in E. \tag{2.4}$$

Without loss of generality, we assume that $v \in C_0^\infty(t_j, t_{j+1})$ and $v(t) = 0$ for $t \in [0, t_j] \cup [t_{j+1}, 1]$. Then, substituting into (2.4) gives

$$(h'(u'(t)))' - \alpha(t)u(t) + \lambda f(t,u(t)) + \mu g(t,u(t)) = 0 \quad \text{a.e. } t \in (t_j, t_{j+1}).$$

Thus, in view of the fact that $(h'(u'(t)))' = p(u'(t))u''(t)$, we see that u satisfies the equation in $(P^{f,g}_{\lambda,\mu})$. Now, by (2.4), we have

$$\sum_{j=1}^{m}[-\Delta(h'(u'(t_j))) + I_j(u(t_j))]v(t_j) + h'(u'(1))v(1) - h'(u'(0))v(0) = 0 \tag{2.5}$$

for all $v \in E$. Next we shall show that u satisfies the impulsive condition in $(P^{f,g}_{\lambda,\mu})$. If this is not the case, without loss of generality, we assume that there exists $j \in \{1, \ldots, m\}$ such that

$$-\Delta(h'(u'(t_j))) + I_j(u(t_j)) \neq 0.$$

Let

$$v(t) = \prod_{i=0,\, i\neq j}^{m+1}(t - t_i).$$

Then,

$$\begin{aligned}
&\sum_{k=1}^{m}[-\Delta(h'(u'(t_k))) + I_k(u(t_k))]v(t_k) + h'(u'(1))v(1) - h'(u'(0))v(0) \\
&\quad = \sum_{k=1}^{m}[-\Delta(h'(u'(t_k))) + I_k(u(t_k))]\prod_{i=0,\, i\neq j}^{m+1}(t_k - t_i) \\
&\qquad + h'(u'(1))\prod_{i=0,\, i\neq j}^{m+1}(t_{m+1} - t_i) - h'(u'(0))\prod_{i=0,\, i\neq j}^{m+1}(t_0 - t_i) \\
&\quad = [-\Delta(h'(u'(t_j))) + I_j(u(t_j))]\prod_{i=0,\, i\neq j}^{m+1}(t_k - t_i) \neq 0,
\end{aligned}$$

which contradicts (2.5). Thus, u satisfies the impulse condition in $(P^{f,g}_{\lambda,\mu})$. Similarly, we can show that u satisfies the boundary condition in $(P^{f,g}_{\lambda,\mu})$. Therefore, u is a solution of $(P^{f,g}_{\lambda,\mu})$. □

We will also need the following lemma in the proof of our main result.

Lemma 2.6. *Let $S : E \to E^*$ be the operator defined by*

$$S(u)(v) = \int_0^1 h'(u'(t))v'(t)dt + \int_0^1 \alpha(t)u(t)v(t)dt + \sum_{j=1}^{m} I_j(u(t_j))v(t_j)$$

for every $u, v \in E$. Then S admits a continuous inverse on E^*.

Proof. For any $u \in E$, from conditions (Q_1)–(Q_3), it follows that

$$\begin{aligned}
S(u)(u) &= \int_0^1 h'(u'(t))u'(t)dt + \int_0^1 \alpha(t)|u(t)|^2dt + \sum_{j=1}^{m} I_j(u(t_j))u_i(t_j) \\
&\geq \min\{m, \alpha_0\}\|u\|^2,
\end{aligned}$$

which implies that S is coercive. Now, for any $u, v \in E$, we have

$$\begin{aligned}\langle S(u) - S(v), u - v\rangle &= \int_0^1 (h'(u'(t)) - h'(v'(t)))(u'(t) - v'(t))dt \\ &\quad + \int_0^1 \alpha(t)(u(t) - v(t))^2 dt \\ &\quad + \sum_{j=1}^m \left(I_j(u(t)) - I_j(v(t))\right)(u(t) - v(t)) \\ &\geq \min\{m, \alpha_0\}\|u - v\|^2.\end{aligned}$$

Thus, S is strongly monotone. Moreover, since E is reflexive, if $u_n \to u$ strongly in E as $n \to \infty$, it can be shown that $S(u_n) \to S(u)$ weakly in E^* as $n \to \infty$. Hence, S is demicontinuous. By [21, Theorem 26.A(d)], the inverse operator S^{-1} of S exists and is continuous. □

2.1 Main result

In this section, we state and prove our main results. Let

$$\lambda_1 = \inf_{u \in E\setminus\{0\}} \left\{ \frac{\int_0^1 h(u'(t))dt + \frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2 dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta}{\int_0^1 F(t, u(t))\mathrm{d}t} : \int_0^1 F(t, u(t))\mathrm{d}t > 0 \right\}$$

and

$$\lambda_2 = \frac{1}{\max\{0,\ \lambda_0,\ \lambda_\infty\}},$$

where

$$\lambda_0 = \limsup_{u \to 0} \frac{\int_0^1 F(t, u(t))\mathrm{d}t}{\int_0^1 h(u'(t))dt + \frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2 dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta}$$

and

$$\lambda_\infty = \limsup_{\|u\| \to \infty} \frac{\int_0^1 F(t, u(t))\mathrm{d}t}{\int_0^1 h(u'(t))dt + \frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2 dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta}.$$

Theorem 2.7. *Assume that*

$(\mathcal{A}_1)$ *there exists a constant $\varepsilon > 0$ such that*

$$\max\left\{ \limsup_{u \to 0} \frac{\max_{t\in[0,1]} F(t, u)}{|u|^2},\ \limsup_{|u| \to \infty} \frac{\max_{t\in[0,1]} F(t, u)}{|u|^2} \right\} < \varepsilon;$$

$(\mathcal{A}_2)$ *there exists a function $w \in E$ such that*

$$\int_0^1 h(w'(t))dt + \frac{1}{2}\int_0^1 \alpha(t)|w(t)|^2 dt + \sum_{j=1}^m \int_0^{w(t_j)} I_j(\zeta)\mathrm{d}\zeta \neq 0$$

and

$$\frac{8\varepsilon}{\min\{m, \alpha_0\}} < \frac{\int_0^1 F(t, w(t))\mathrm{d}t}{\int_0^1 h(w'(t))dt + \frac{1}{2}\int_0^1 \alpha(t)|w(t)|^2 dt + \sum_{j=1}^m \int_0^{w(t_j)} I_j(\zeta)\mathrm{d}\zeta}.$$

Then for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ such that for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for every $\mu \in [0,\gamma]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Remark 2.8. Under conditions $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$, it is true that $\lambda_1 < \lambda_2$ as is shown in the proof of Theorem 2.7 given below.

Proof of Theorem 2.7. Our aim is to apply Lemma 2.1 to the problem $(P^{f,g}_{\lambda,\mu})$. Take $X = E$; clearly, X is a separable and uniformly convex Banach space. From [13, Proposition 1.1] and its proof with $T = 1$ and $p = q = 2$, we have

$$\max_{t\in[0,1]} |u(t)| \le 2\|u\| \quad \text{for all } u \in X. \tag{2.6}$$

Let the functionals Φ, J, and Ψ be as given in (2.1)–(2.3). The functional Φ is C^1, and by Lemma 2.6, its derivative admits a continuous inverse on X^*. Moreover, Φ is sequentially weakly lower semicontinuous since Φ' is monotone (see the proof of Lemma 2.6). Since

$$\int_0^1 h(u'(t))dt = \int_0^1 \left(\int_0^{u'(t)} \left(\int_0^\tau p(\xi)d\xi \right) d\tau \right) dt,$$

from (1.1) and (1.2), it follows that

$$\begin{aligned}
\frac{1}{2}\min\{m,\alpha_0\}\|u\|^2 &\le \frac{m}{2}\int_0^1 |u'(t)|^2 dt + \frac{\alpha_0}{2}\int_0^1 |u(t)|^2 dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta \\
&\le \Phi(u) \\
&\le \frac{M}{2}\int_0^1 |u'(t)|^2 dt + \frac{\alpha_1}{2}\int_0^1 |u(t)|^2 dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta \\
&\le \frac{1}{2}\max\{M,\alpha_1\}\|u\|^2 + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta
\end{aligned} \tag{2.7}$$

for every $u \in X$. We then have

$$\lim_{\|u\|\to+\infty} \Phi(u) = \infty,$$

i.e., Φ is coercive. Now, let A be a bounded subset of X. Then there exist a constant $c > 0$ such that $\|u\| \le c$ for all $u \in A$. From (2.6), $\max_{t\in[0,1]} |u(t)| \le 2c$ for all $u \in A$. Thus, by the continuity of each I_j, we see that there exists $K > 0$ such that $\left|\sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta\right| < K$ for all $u \in A$. Then, by (2.7), we have

$$\Phi(u) \le \frac{1}{2}\max\{M,\alpha_1\}\|c\|^2 + K,$$

so Φ is bounded on each bounded subset of X.

To prove that $\Phi \in \mathcal{W}_X$, define

$$\Phi_1(u) = \int_0^1 h(u'(t))dt \quad \text{and} \quad \Phi_2(u) = \frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2 dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)\mathrm{d}\zeta$$

for all $u \in X$. Then,

$$\Phi(u) = \Phi_1(u) + \Phi_2(u) \quad \text{for all } u \in X.$$

As in (2.7), we have

$$\Phi_1(u) \geq d(u) := \frac{m}{2}\int_0^1 |u'(t)|^2 dt \quad \text{for all } u \in X. \tag{2.8}$$

Let $\{u_k\}$ be a sequence in X and let $u \in X$ be such that $u_k \rightharpoonup u$ and $\liminf_{k\to\infty} \Phi(u_k) \leq \Phi(u)$. To show that $\{u_k\}$ has a subsequence strongly converging to u, assume, to the contrary, that $\{u_k\}$ does not have such a subsequence. Then, there exist $\epsilon > 0$ and a subsequence $\{u_{k_n}\}$ of $\{u_k\}$ such that

$$\left\|\frac{u_{k_n} - u}{2}\right\| \geq \epsilon \quad \text{for all } n \in \mathbb{N}.$$

Note that $\{u_{k_n}\}$ converges uniformly to u by [13, Proposition 1.2]. Then, in view of the definition of $\|\cdot\|$, there exists $\epsilon_1 > 0$ such that

$$d\left(\frac{u_{k_n} - u}{2}\right) \geq \epsilon_1 \quad \text{for all } n \in \mathbb{N}.$$

Thus, from (2.8)

$$\Phi_1\left(\frac{u_{k_n} - u}{2}\right) \geq \epsilon_1 \quad \text{for all } n \in \mathbb{N}. \tag{2.9}$$

Now, the sequentially weakly lower semicontinuity of Φ implies that $\liminf_{n\to\infty} \Phi(u_{k_n}) = \Phi(u)$. Hence, there exists a subsequence $\{w_\ell\} = \{u_{k_{n_\ell}}\}$ of $\{u_{k_n}\}$ such that

$$\lim_{\ell\to\infty} \Phi(w_\ell) = \Phi(u).$$

Since $\{w_\ell\}$ converges uniformly to u and I_j, $j = 1\ldots, m$, are continuous, we see that

$$\lim_{\ell\to\infty} \Phi_2(w_\ell) = \Phi_2(u),$$

and so

$$\lim_{\ell\to\infty} \Phi_1(w_\ell) = \Phi_1(u). \tag{2.10}$$

It is clear that Φ_1 is sequentially weakly lower semicontinuous and that $(w_\ell + u)/2 \rightharpoonup u$ as $\ell \to \infty$. Then,

$$\Phi_1(u) \leq \liminf_{\ell\to\infty} \Phi_1\left(\frac{w_\ell + u}{2}\right). \tag{2.11}$$

By simple calculations and the nondecreasing nature of p, we have that for $y > 0$,

$$\begin{aligned} h''(\sqrt{y}) &= \frac{1}{4}y^{-1}p(\sqrt{y}) - \frac{1}{4}y^{-3/2}\int_0^{\sqrt{y}} p(\xi)d\xi \\ &\geq \frac{1}{4}y^{-1}p(\sqrt{y}) - \frac{1}{4}y^{-3/2}\sqrt{y}p(\sqrt{y}) = 0. \end{aligned}$$

Hence, $h(\sqrt{y})$ is convex. Moreover, $h(y)$ is continuous, strictly increasing for $y \geq 0$, and $h(0) = 0$. Thus, from [11, Theorem 2.1], we have

$$\frac{1}{2}\Phi_1(w_\ell) + \frac{1}{2}\Phi_1(u) \geq \Phi_1\left(\frac{w_\ell + u}{2}\right) + \Phi_1\left(\frac{w_\ell - u}{2}\right) \quad \text{for all } \ell \in \mathbb{N}.$$

Taking limit superior as $\ell \to \infty$ and using (2.8), (2.9), and (2.10) in the above inequality, we obtain

$$\Phi_1(u) - \epsilon_1 \geq \limsup_{\ell\to\infty} \Phi_1\left(\frac{w_\ell + u}{2}\right),$$

which contradicts (2.11). This shows that $\{u_k\}$ has a subsequence converging strongly to u. Therefore, $\Phi \in \mathcal{W}_X$.

The functionals J and Ψ are C^1-functionals with compact derivatives. Moreover, Φ has a strict local minimum 0 with $\Phi(0) = J(0) = 0$. Therefore, the regularity assumptions on Φ, J, and Ψ, as required in Lemma 2.1, are satisfied. In view of $(\mathcal{A}_1)$, there exist τ_1, τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(t,u) \leq \varepsilon |u|^2 \tag{2.12}$$

for every $t \in [0,1]$ and every u with $|u| \in [0,\tau_1) \cup (\tau_2, \infty)$. Since $F(t,u)$ is continuous on $[0,1] \times \mathbb{R}$, $F(t,u)$ is bounded on $[0,1] \times [\tau_1, \tau_2]$. Thus, we can choose $\eta > 0$ and $v > 2$ such that

$$F(t,u) \leq \varepsilon |u|^2 + \eta |u|^v$$

for all $(t,u) \in [0,1] \times \mathbb{R}$. Then, from (2.6), we have

$$J(u) \leq 4\varepsilon \|u\|^2 + \eta 2^v \|u\|^v \tag{2.13}$$

for all $u \in X$. Hence, from (2.7) and (2.13), we have

$$\limsup_{|u| \to 0} \frac{J(u)}{\Phi(u)} \leq \frac{8\varepsilon}{\min\{m, \alpha_0\}}. \tag{2.14}$$

Moreover, by (2.12), for each $u \in X \setminus \{0\}$,

$$\begin{aligned}
\frac{J(u)}{\Phi(u)} &= \frac{\int_{|u| \leq \tau_2} F(t,u(t))\mathrm{d}t}{\Phi(u)} + \frac{\int_{|u| > \tau_2} F(t,u(t))dt}{\Phi(u)} \\
&\leq \frac{\sup_{t \in [0,1], |u| \in [0,\tau_2]} F(t,u)}{\Phi(u)} + \frac{4\varepsilon \|u\|^2}{\Phi(u)} \\
&\leq \frac{\sup_{t \in [0,1], |u| \in [0,\tau_2]} F(t,u)}{\frac{1}{2}\min\{m, \alpha_0\} \|u\|^2} + \frac{8\varepsilon}{\min\{m, \alpha_0\}}.
\end{aligned}$$

Therefore,

$$\limsup_{\|u\| \to \infty} \frac{J(u)}{\Phi(u)} \leq \frac{8\varepsilon}{\min\{m, \alpha_0\}}. \tag{2.15}$$

In view of (2.14) and (2.15), we have

$$\rho = \max\left\{0, \limsup_{\|u\| \to \infty} \frac{J(u)}{\Phi(u)}, \limsup_{u \to 0} \frac{J(u)}{\Phi(u)}\right\} \leq \frac{8\varepsilon}{\min\{m, \alpha_0\}}. \tag{2.16}$$

Condition $(\mathcal{A}_2)$ together with (2.16) yield

$$\begin{aligned}
\sigma &= \sup_{u \in \Phi^{-1}((0,\infty))} \frac{J(u)}{\Phi(u)} = \sup_{X \setminus \{0\}} \frac{J(u)}{\Phi(u)} \\
&\geq \frac{\int_0^1 F(t,w(t))dt}{\Phi(w(t))} = \frac{\int_0^1 F(t,w(t))dt}{\frac{1}{2}\max\{M, \alpha_1\}\|u\|^2 + \sum_{j=1}^m \int_0^{w(t_j)} I_j(\zeta)d\zeta} \\
&> \frac{8\varepsilon}{\min\{m, \alpha_0\}} \geq \rho.
\end{aligned}$$

Thus, all the conditions of Lemma 2.1 are satisfied. With $\lambda_1 = 1/\sigma$ and $\lambda_2 = 1/\rho$, by Lemmas 2.1 and 2.5, for each compact interval $[c,d] \subset (\lambda_1, \lambda_2)$, there exists $R > 0$ such that for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions whose norms in X are less than R. $\square$

The following result is another application of Lemma 2.1.

Theorem 2.9. *Assume that*

$$\max_{u\in E}\left\{\limsup_{u\to 0}\frac{\max_{t\in[0,1]}F(t,u)}{|u|^2},\ \limsup_{|u|\to\infty}\frac{\max_{t\in[0,1]}F(t,u)}{|u|^2}\right\} \le 0 \tag{2.17}$$

and

$$\sup_{u\in E}\frac{\int_0^1 F(t,u(t))\mathrm{d}t}{\int_0^1 h(u'(t))dt + \frac{1}{2}\int_0^1 \alpha(t)|u(t)|^2 dt + \sum_{j=1}^m \int_0^{u(t_j)} I_j(\zeta)d\zeta} > 0. \tag{2.18}$$

Then for each compact interval $[c,d] \subset (\lambda_1, \infty)$, there exists $R > 0$ such that for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : [0,1] \times \mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Proof. For any $\varepsilon > 0$, (2.17) implies that there exist τ_1 and τ_2 with $0 < \tau_1 < \tau_2$ such that

$$F(t,u) \le \varepsilon|u|^2$$

for every $t \in [0,1]$ and every u with $|u| \in [0,\tau_1) \cup (\tau_2,\infty)$. Since $F(t,u)$ is continuous on $[0,1] \times \mathbb{R}$, $F(t,u)$ is bounded on $[0,1] \times [\tau_1,\tau_2]$. Thus, as before, we can choose $\eta > 0$ and $v > 2$ so that

$$F(t,u) \le \varepsilon|u|^2 + \eta|u|^v$$

for all $(t,u) \in [0,1] \times \mathbb{R}$. Then, by the same process as in the proof of Theorem 2.7, we obtain (2.14) and (2.15). Since ϵ is arbitrary, (2.14) and (2.15) give

$$\max\left\{0, \limsup_{\|u\|\to+\infty}\frac{J(u)}{\Phi(u)}, \limsup_{u\to 0}\frac{J(u)}{\Phi(u)}\right\} \le 0.$$

Then, with ρ and σ defined as in Lemma 2.1, we have $\rho = 0$, and by (2.18), we have $\sigma > 0$. In this case, $\lambda_1 = 1/\sigma$ and $\lambda_2 = \infty$. Thus, by Lemma 2.1 the theorem is proved. $\square$

Remark 2.10. In condition $(\mathcal{A}_2)$ of Theorem 2.7, if we choose

$$w_0(t) = \begin{cases} \sigma, & t \in [0,1/4], \\ 2\sigma t + \sigma/2, & t \in [1/4,1/2], \\ -2\sigma t + 5\sigma/2, & t \in [1/2,3/4], \\ \sigma, & t \in [3/4,1], \end{cases} \tag{2.19}$$

where $\sigma > 0$, then $w_0 \in E$, and $(\mathcal{A}_2)$ now takes the form

$(\widehat{\mathcal{A}}_2)$ there exists a positive constant σ such that

$$\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta \neq 0$$

and

$$\frac{8\varepsilon}{\min\{m,\alpha_0\}}<\frac{\int_0^1 F(t,w_0(t))dt}{\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta}.$$

Next, we point out some results in which the function f is separable. To be precise, we consider the problem

$$\begin{cases}-p(x')x''+\alpha(t)x=\lambda\theta(t)f(x)+\mu g(t,x), \quad t\neq t_j, \quad \text{a.e. } t\in[0,1],\\ \Delta(h'(u'(t_j)))=I_j(u(t_j)), \quad j=1,2,\dots,m,\\ x(1)-x(0)=x'(1)-x'(0)=0,\end{cases} \qquad (\phi^{\theta}_{\lambda,\mu})$$

where $\theta:[0,1]\to\mathbb{R}$ is a nonzero function with $\theta\in L^1([0,1])$, $f:\mathbb{R}\to\mathbb{R}$ is a continuous function, and $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ is an L^1-Carathéodory function. Let $F(t,x)=\theta(t)F(x)$ for every $(t,x)\in[0,1]\times\mathbb{R}$, where

$$F(x)=\int_0^x f(\xi)d\xi \quad \text{for all } x\in\mathbb{R}.$$

The following existence results are then consequences of Theorem 2.7.

Theorem 2.11. *Assume that*

$(\mathcal{A}_3)$ *there exists a constant $\varepsilon>0$ such that*

$$\sup_{t\in[0,1]}\theta(t)\cdot\max\left\{\limsup_{u\to 0}\frac{F(u)}{|u|^2},\ \limsup_{|u|\to\infty}\frac{F(u)}{|u|^2}\right\}<\varepsilon;$$

$(\mathcal{A}_4)$ *there exists a positive constant σ such that*

$$\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta \neq 0$$

and

$$\frac{8\varepsilon}{\min\{m,\alpha_0\}}<\frac{f(w_0(t))\int_0^1 \theta(t)dt}{\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta},$$

where w_0 is defined by (2.19).

Then for each compact interval $[c,d]\subset(\lambda_3,\lambda_4)$, where λ_3 and λ_4 are the same as λ_1 and λ_2, but with $\int_0^1 F(t,u(t))dt$ replaced by $\int_0^1\theta(t)F(u(t))dt$, there exists $R>0$ such that for every $\lambda\in[c,d]$ and every L^1-Carathéodory function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the problem $(\phi^{\theta}_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Theorem 2.12. *Assume that there exists a positive constant σ such that*

$$\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta>0$$

and

$$\int_0^1 \theta(t)F(w_0(t))dt>0, \tag{2.20}$$

where w_0 is given by (2.19). In addition, assume that

$$\limsup_{u\to 0}\frac{F(u)}{|u|^2}=\limsup_{|u|\to\infty}\frac{F(u)}{|u|^2}=0. \tag{2.21}$$

Then for each compact interval $[c,d]\subset(\lambda_3,\infty)$, where λ_3 is the same as λ_1 but with $\int_0^1 F(t,u(t))\mathrm{d}t$ replaced by $\int_0^1 \theta(t)F(u(t))\mathrm{d}t$, there exists $R>0$ such that for every $\lambda\in[c,d]$ and every L^1-Carathéodory function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$, there exists $\gamma>0$ such that for each $\mu\in[0,\gamma]$, the problem $(\phi^{\theta}_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Proof. From (2.21), we easily see that $(\mathcal{A}_3)$ is satisfied for every $\varepsilon>0$. Moreover, using (2.20), by choosing $\varepsilon>0$ small enough, $(\mathcal{A}_4)$ will hold. Hence, the conclusion of this theorem follows from Theorem 2.11. □

As an example in which the hypotheses of Theorem 2.12 are satisfied, we have the following.

Example 2.13. Let $p(x)=4-\cot(x)$ for each $x\in\mathbb{R}$, $\alpha(t)=\theta(t)=1$ for every $t\in[0,1]$, $m=1$, $t_1=1/5$, $I_1(x)=x^3$ for each $x\in\mathbb{R}$, and

$$f(x)=\begin{cases}4x^3, & |x|\le 1,\\ 4x, & 1<|x|\le 2,\\ 8, & |x|\ge 2.\end{cases}$$

Then, it is easy to check that

$$F(x)=\begin{cases}x^4, & |x|\le 1,\\ 2x^2-1, & 1<|x|\le 2,\\ 8x-9, & x>2,\\ 8x+23, & x<-2.\end{cases}$$

By choosing $\sigma=1$, $w_0(t)$ becomes

$$w_0(t)=\begin{cases}1, & t\in[0,1/4],\\ 2t+1/2, & t\in[1/4,1/2],\\ -2t+5/2, & t\in[1/2,3/4],\\ 1, & t\in[3/4,1].\end{cases}$$

It is trivial to verify that

$$\frac{1}{4}h(2\sigma)+\frac{1}{4}h(-2\sigma)+\frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta>0,$$

$$\int_0^1 \theta(t)F(w_0(t))dt > 0,$$

and

$$\lim_{u\to 0}\frac{F(u)}{|u|^2} = \lim_{|u|\to\infty}\frac{F(u)}{|u|^2} = 0.$$

Hence, by Theorem 2.12, for each compact interval $[c,d] \subset (0,\infty)$, there exists $R > 0$ such that for every $\lambda \in [c,d]$ and every L^1-Carathéodory function $g : [0,1]\times\mathbb{R} \to \mathbb{R}$, there exists $\gamma > 0$ such that for each $\mu \in [0,\gamma]$, the problem

$$\begin{cases} -p(x')x'' + x = \lambda f(x) + \mu g(t,x), \quad t \neq \frac{1}{5}, \quad \text{a.e. } t \in [0,1], \\ \Delta(h'(u'(\frac{1}{5}))) = I_1(u(\frac{1}{5})), \\ x(1) - x(0) = x'(1) - x'(0) = 0, \end{cases}$$

has at least three classical solutions whose norms in E are less than R.

The following theorem is a consequences of Lemma 2.3.

Theorem 2.14. *Assume that there exist three positive constants $1 \leq \zeta < 2$, θ, and σ, with*

$$\theta < \sqrt{\frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{31\alpha_0\sigma^2}{48}}, \tag{2.22}$$

such that

$(\mathcal{B}_1)$ *$f(t,x) \geq 0$ for every $(t,x) \in \big([0,1/4]\times[0,\sigma]\big) \cup \big([3/4,1]\times[0,\sigma]\big) \cup \big([1/4,3/4]\times[\sigma,3\sigma/2]\big)$;*

$(\mathcal{B}_2)$ $$\frac{\int_0^1 \sup_{|u|\leq\theta} F(t,u)dt}{\theta^2} < \frac{\min\{m,\alpha_0\}}{8}\,\frac{\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\sigma)dt}{\frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{31\alpha_1\sigma^2}{48} + \sum_{j=1}^m \int_0^{w_0(t_j)} I_j(\zeta)\mathrm{d}\zeta};$$

$(\mathcal{B}_3)$ *there exists $p > 0$ and a positive constant q such that*

$$|F(t,u)| \leq p|u|^\zeta + q \quad \textit{for all } (t,u) \in [0,1]\times\mathbb{R};$$

$(\mathcal{B}_4)$ *there exists $l > 0$ and a positive constant $\varrho \in \mathbb{R}$ such that*

$$G(t,u) \leq lu^\zeta + \varrho \quad \textit{for all } (t,u) \in [0,1]\times\mathbb{R}.$$

Then there exist a nonempty open set $A \subset [0,\infty)$ and a positive number $R > 0$ such that for every $\lambda \in A$ and every L^1-Carathéodory function $g : [0,1]\times\mathbb{R} \to \mathbb{R}$, there exists $\delta > 0$ such that for each $\mu \in [0,\delta]$, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions whose norms in E are less than R.

Proof. Since the embeddings $E \hookrightarrow L^q$ $(q \geq 1)$ and $E \hookrightarrow L^\infty$ are compact (see Adams and Fournier [1]), there exists a positive constant C such that

$$|u|_{L^q([0,1])} \leq C\|u\|.$$

For any $\lambda \geq 0$ and $u \in E$, from (Q_3), $(\mathcal{B}_3)$, and $(\mathcal{B}_4)$, we have

$$\begin{aligned}\Phi(u) - \lambda\Psi(u) &\geq \frac{1}{2}\min\{m,\alpha_0\}\|u\|^2 - \lambda\int_0^1 [F(t,u(t)) + \frac{\mu}{\lambda}G(t,u(t))]dt \\ &\geq \frac{1}{2}\min\{m,\alpha_0\}\|u\|^2 - \lambda\left(\int_0^1 p|u|^\zeta dt + q\right) - \mu\left(l\int_0^1 |u(t)|^\zeta dt + \varrho\right) \\ &\geq \frac{1}{2}\min\{m,\alpha_0\}\|u\|^2 - \lambda pC_0\|u\|^\zeta - \mu lC_1\|u\|^\zeta - \lambda q - \mu\varrho.\end{aligned}$$

Since $\zeta < 2$,

$$\lim_{\|u\|\to+\infty} \Phi(u) - \lambda\Psi(u) = \infty \quad \text{for all } \lambda > 0.$$

Let w_0 be defined by (2.19) with σ given in the theorem. Now, $\min_{t\in[\frac{1}{4},\frac{3}{4}]}\{w_0(t)\} = \sigma$ and $\max_{t\in[\frac{1}{4},\frac{3}{4}]}\{w_0(t)\} = \frac{3\sigma}{2}$, so

$$\begin{aligned} J(w_0) &= \int_0^{\frac{1}{4}}\int_0^{\sigma} f(t,\xi)d\xi dt + \int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{w_0(t)} f(t,\xi)d\xi dt + \int_{\frac{3}{4}}^{1}\int_0^{\sigma} f(t,\xi)d\xi dt \\ &\geq \int_{\frac{1}{4}}^{\frac{3}{4}}\int_0^{\sigma} f(t,\xi)d\xi dt = \int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\sigma)dt. \end{aligned}$$

Moreover, simple calculations show that

$$\begin{aligned} \Phi(w_0) &= \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt + \sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta \\ &\leq \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{\alpha_1}{2}\int_0^1 |w_0(t)|^2dt + \sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)\mathrm{d}\zeta \\ &= \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{31\alpha_1\sigma^2}{48} + \sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)\mathrm{d}\zeta \end{aligned} \tag{2.23}$$

and

$$\begin{aligned} \Phi(w_0) &= \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{1}{2}\int_0^1 \alpha(t)|w_0(t)|^2dt + \sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)d\zeta \\ &\geq \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{\alpha_0}{2}\int_0^1 |w_0(t)|^2dt \\ &= \frac{1}{4}h(2\sigma) + \frac{1}{4}h(-2\sigma) + \frac{31\alpha_0\sigma^2}{48}. \end{aligned} \tag{2.24}$$

Let $r = \frac{\min\{m,\alpha_0\}}{8}\theta^2$. Then, from (2.22) and (2.24), we have $\Phi(w_0) > r$. From the definition of Φ, (2.6), and (2.7), it follows that

$$\begin{aligned} \Phi^{-1}(-\infty,r] &= \{x \in E : \ \Phi(x) \leq r\} \\ &\subseteq \left\{x \in E : \ \max_{t\in[0,1]} |x(t)| \leq \sqrt{\frac{8r}{\min\{m,\alpha_0\}}}\right\} \\ &\subseteq \left\{x \in E : \ \max_{t\in[0,1]} |x(t)| \leq \theta\right\}. \end{aligned}$$

Therefore,

$$\sup_{u\in\Phi^{-1}((-\infty,r])} J(u) \leq \int_0^1 \sup_{|u|\leq\theta} F(t,u)dt.$$

Thus, from $(\mathcal{B}_2)$ and (2.23), we have

$$\begin{aligned} r\frac{J(w_0)}{\Phi(w_0)} &= \frac{r}{\Phi(w_0)}\left(\int_0^1 F(t,w_0(t))dt\right) \\ &\geq \frac{\dfrac{\min\{m,\alpha_0\}}{8}\theta^2\left(\int_{\frac{1}{4}}^{\frac{3}{4}} F(t,\sigma)dt\right)}{\dfrac{1}{4}h(2\sigma)+\dfrac{1}{4}h(-2\sigma)+\dfrac{31\alpha_1\sigma^2}{48}+\sum_{j=1}^{m}\int_0^{w_0(t_j)} I_j(\zeta)\mathrm{d}\zeta} \\ &> \int_0^1 \sup_{|u|\leq\theta} F(t,u)dt \geq \sup_{u\in\Phi^{-1}((-\infty,r])} J(u). \end{aligned}$$

We can then fix ρ so that

$$\sup_{u\in\Phi^{-1}((-\infty,r])} J(u) < \rho < r\frac{J(w_0)}{\Phi(w_0)}.$$

From Lemma 2.3, we obtain

$$\sup_{\lambda\geq 0}\inf_{u\in E}(\Phi(u)-\lambda(\rho-J(u)) < \inf_{u\in E}\sup_{\lambda\geq 0}(\Phi(u)-\lambda(\rho-J(u)).$$

Hence, by Lemma 2.2, for each compact interval $[c,d] \subset (\lambda_1,\lambda_2)$, there exists $R>0$ such that for every $\lambda\in[c,d]$, and every L^1-Carathéodory function $g:[0,1]\times\mathbb{R}\to\mathbb{R}$ there exists $\delta>0$ such that, for each $\mu\in[0,\delta]$, $\Phi'(u)-\lambda J'(u)-\mu\Psi'(u)=0$ has at least three solutions in E. Hence, the problem $(P^{f,g}_{\lambda,\mu})$ has at least three classical solutions whose norms are less than R. □

2.2 Results and discussion

In this paper we investigate the existence of multiple solutions to a quasilinear periodic boundary value problem with impulsive effects. The main technique of proof involves variational methods and critical points theorems for smooth functionals. We obtain the existence of at least three solutions to the problem. The applicability of the results are illustrated by an example.

Use of AI tools declaration

The authors declare they have not used Artificial Intelligence (AI) tools in the creation of this article.

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Conflict of interest

All authors declare that there are no conflicts of interest in this paper.

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