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Infinitely many homoclinic solutions for a class of nonlinear difference equations

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Abstract: By using the Symmetric Mountain Pass Theorem, we establish some existence criteria to guarantee a class of nonlinear difference equation has infinitely many homoclinic orbits. Our conditions on the nonlinear term are rather relaxed and we generalize some existing results in the literature.

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1. Introduction

Difference equations have attracted the interest of many researchers in the past twenty years since they provided a natural description of several discrete models. Such discrete models are often investigated in various fields of science and technology such as computer science, economics, neural network, ecology, cybernetics, biological systems, optimal control, population dynamics, etc. These studies cover many of the branches of difference equation, such as stability, attractiveness, periodicity, oscillation and boundary value problem. Recently, there are some new results on periodic solutions and homoclinic solutions of nonlinear difference equations by using the critical point theory in the literature, see [1-3, 7-15, 20, 21, 30-33].

Consider the nonlinear difference equation of the form

$$\Delta\left[p(n)(\Delta u(n-1))^{\delta}\right]-q(n)(x(n))^{\delta}+f(n,u(n))=0,\quad n\in\mathbb{Z},\tag{1.1}$$

where Δ is the forward difference operator defined by $\Delta u(n)=u(n+1)-u(n)$, $\Delta^2u(n)=\Delta(\Delta u(n))$, $\delta>0$ is the ratio of odd positive integers, $\{p(n)\}$ and $\{q(n)\}$ are real sequences, $\{p(n)\}\neq 0$. $f:\mathbb{Z}\times\mathbb{R}\to\mathbb{R}$. As usual, we say that a solution $u(n)$ of (1.1) is homoclinic (to 0) if $u(n)\to 0$ as $n\to\pm\infty$. In addition, if $u(n)\not\equiv 0$ then $u(n)$ is called a nontrivial homoclinic solution.

In general, equation (1.1) may be regarded as a discrete analogue of the following second order differential equation

$$(p(t)\varphi(x'))'+q(t)x(t)+f(t,x)=0,\quad t\in\mathbb{R}.\tag{1.2}$$

Equation (1.2) can be regarded as the more general form of the Emden-Fowler equation, appearing in the study of astrophysics, gas dynamics, fluid mechanics, relativistic mechanics, nuclear physics and chemically

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reacting system in terms of various special forms of $f(t, x(t))$, for example, see [33] and the reference therein. In this survey paper, many well-known results concerning properties of solutions of (1.2) are collected. In the case of $\varphi(x) = |x|^{\delta-2}x$, Eq.(1.2) has been discussed extensively in the literature, we refer the reader to the monographs [4-6, 17-19, 22-29, 34].

It is well-known that the existence of homoclinic solutions for Hamiltonian systems and their importance in the study of the behavior of dynamical systems have been already recognized from Poincaré, homoclinic orbits play an important role in analyzing the chaos of dynamical system. In the past decade, this problem has been intensively studied using critical point theory and variational methods.

In some recent papers [7, 8, 10, 13-15, 20-21, 30], the authors studied the existence of periodic solutions, subharmonic solutions and homoclinic solutions of some special forms of (1.1) by using the critical point theory. These papers show that the critical point method is an effective approach to the study of periodic solutions for difference equations.

When $\delta = 1$, (1.1) reduces to the following equation:

$$\Delta\left[p(n)(\Delta u(n-1))\right] - q(n)x(n) + f(n, u(n)) = 0, \quad n \in \mathbb{Z}, \tag{1.3}$$

which has been studied in [21]. Ma and Guo applied the critical point theory to prove the existence of homoclinic solutions of (1.3) and obtained the following theorems.

Theorem A[21]**.** *Assume that p, q and f satisfy the following conditions:*

(p) $p(n) > 0$ *for all* $n \in \mathbb{Z}$*;*

(q) $q(n) > 0$ *for all* $n \in \mathbb{Z}$ *and* $\lim_{|n|\to+\infty} q(n) = +\infty$*;*

(f1) *There is a constant* $\mu > 2$ *such that*

$$0 < \mu \int_0^x f(n, s)ds \le xf(n, x), \qquad \forall\ (n, x) \in \mathbb{Z} \times (\mathbb{R} \setminus \{0\});$$

(f2) $\lim_{x\to 0} f(n, x)/x = 0$ *uniformly with respect to* $n \in \mathbb{Z}$.

(f3) $f(n, -x) = -f(n, x), \quad \forall\ (n, x) \in \mathbb{Z} \times \mathbb{R}$.

Then Eq. (1.3) possess an unbounded sequence of homoclinic solutions.

When $\delta \neq 1$, it seems that no similar results were obtained in the literature on the existence of homoclinic solutions. When $F(n, x)$ is an even function on x, however, generalize or improve Theorem A by using the Symmetric Mountain Pass Theorem, there has not been much work done up to now, because it is often very difficult to verify the last condition of the Symmetric Mountain Pass Theorem, different from the Mountain Pass Theorem.

Motivated by the above papers, we will obtain some new criteria for guaranteeing that (1.1) has infinitely many homoclinic orbits without any periodicity and generalize Theorem A. Especially, $F(n, x)$ satisfies a kind of new superquadratic condition which is different from the corresponding condition in the known literature.

In this paper, we always assume that $F(n, x) = \int_0^x f(n, s)ds, F_1(n, x) = \int_0^x f_1(n, s)ds, F_2(n, x) = \int_0^x f_2(n, s)ds$. Our main results are the following theorems.

Theorem 1.1. *Assume that p, q and F satisfy (p), (q), (f3) and the following assumptions:*

(F1) *$F(n,x)$ is continuously differentiable in x, and*

$$\frac{1}{q(n)}|f(n,x)| = o(|x|^{\delta}) \quad \text{as} \ \ x \to 0$$

uniformly in $n \in \mathbb{Z}$;

(F2) *For any $r > 0$, there exist $a = a(r)$, $b = b(r) > 0$ and $\nu < \delta + 1$ such that*

$$0 \leq \left(\delta + 1 + \frac{1}{a + b|x|^{\nu}}\right) F(n,x) \leq xf(n,x), \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}, \ \ |x| \geq r;$$

(F3) *For any $n \in \mathbb{Z}$*

$$\lim_{s \to +\infty} \left[s^{-(\delta+1)} \min_{|x|=1} F(n,sx)\right] = +\infty.$$

Then Eq.(1.1) possesses an unbounded sequence of homoclinic solutions.

Theorem 1.2. *Assume that p, q and F satisfy (p), (q), (f3) and the following conditions:*

(F1') *$F(n,x) = F_1(n,x) - F_2(n,x)$, for every $n \in \mathbb{Z}$, F_1 and F_2 are continuously differentiable in x and*

$$\frac{1}{q(n)}|f(n,x)| = o(|x|^{\delta}) \quad \text{as} \ \ x \to 0$$

uniformly in $n \in \mathbb{Z}$;

(F4) *There is a constant $\mu > \delta + 1$ such that*

$$0 < \mu F_1(n,x) \leq xf_1(n,x), \quad \forall\ (n,x) \in \mathbb{Z} \times (\mathbb{R} \setminus \{0\});$$

(F5) *$F_2(n,0) \equiv 0$ and there is a constant $\varrho \in (\delta + 1, \mu)$ such that*

$$xf_2(n,x) \leq \varrho F_2(n,x), \quad \forall\ (n,x) \in \mathbb{Z} \times \mathbb{R}.$$

Then Eq.(1.1) possesses an unbounded sequence of homoclinic solutions.

Theorem 1.3. *Assume that p, q and F satisfy (p), (q), (f3), (F4), (F5) and the following assumption:*

(F1") *$F(n,x) = F_1(n,x) - F_2(n,x)$, for every $n \in \mathbb{Z}$, F_1 and F_2 are continuously differentiable in x and there is a bounded set $J \subset \mathbb{Z}$ such that*

$$F_2(n,x) \geq 0, \quad \forall\ (n,x) \in J \times \mathbb{R}, \ \ |x| \leq 1,$$

and

$$\frac{1}{q(n)}|f(n,x)| = o(|x|^{\delta}) \quad \text{as} \ \ x \to 0$$

uniformly in $n \in \mathbb{Z} \setminus J$.

Then Eq.(1.1) possesses an unbounded sequence of homoclinic solutions.

Remark 1.1. *If Ambrosetti-Rabinwitz (AR) condition: there exist some $\mu > 2$ such that*

$$0 < \mu F(n,x) \leq (\nabla F(n,x), x)$$

holds, then (F2) with $\delta = 1$ also holds by choosing $a > 1/(\mu - 2)$, $b > 0$ and $\nu \in (0, 2)$. In addition, by (AR), we have

$$F(n, sx) \geq s^{\mu} F(n, x) \quad \text{for} \ \ (n, x) \in \mathbb{Z} \times \mathbb{R}, \ \ s \geq 1.$$

It follows that for any $n \in \mathbb{Z}$

$$s^{-2} \min_{|x|=1} F(n, sx) \geq s^{\mu-2} \min_{|x|=1} F(n, x) \to +\infty, \qquad s \to +\infty.$$

This shows that (AR) implies (F3). Therefore, Theorem 1.1 also generalize Theorem A by relaxing conditions (f1) and (f2).

Remark 1.2. *Obviously, conditions (F1), (F1') and (F1") are weaker than (f1). Therefore, both Theorem 1.2 and Theorem 1.3 generalize Theorem A by relaxing conditions (f1) and (f2).*

2. Preliminaries

Let

$$S = \{\{u(n)\}_{n \in \mathbb{Z}} \ : \ u(n) \in \mathbb{R}, \ n \in \mathbb{Z}\},$$

$$E = \left\{ u \in S \ : \ \sum_{n \in \mathbb{Z}} \left[p(n)(\triangle u(n-1))^{\delta+1} + q(n)(u(n))^{\delta+1}) \right] < +\infty \right\},$$

and for $u \in E$, let

$$\|u\| = \left\{ \sum_{n \in \mathbb{Z}} \left[p(n)(\triangle u(n-1))^{\delta+1} + q(n)(u(n))^{\delta+1}) \right] < +\infty \right\}^{\frac{1}{\delta+1}}.$$

Then E is a uniform convex Banach space with this norm and is a reflexive Banach space, see details in ref.[36] or Lemma 2.4.

As usual, for $1 \leq p < +\infty$, let

$$l^p(\mathbb{Z}, \mathbb{R}) = \left\{ u \in S \ : \ \sum_{n \in \mathbb{Z}} |u(n)|^p < +\infty \right\},$$

and

$$l^{\infty}(\mathbb{Z}, \mathbb{R}) = \left\{ u \in S \ : \ \sup_{n \in \mathbb{Z}} |u(n)| < +\infty \right\},$$

and their norms are defined by

$$\|u\|_p = \left(\sum_{n \in \mathbb{Z}} |u(n)|^p \right)^{1/p}, \quad \forall \ u \in l^p(\mathbb{Z}, \mathbb{R}); \qquad \|u\|_{\infty} = \sup_{n \in \mathbb{Z}} |u(n)|, \quad \forall \ u \in l^{\infty}(\mathbb{Z}, \mathbb{R}),$$

respectively.

Let $I : E \to \mathbb{R}$ be defined by

$$I(u) = \frac{1}{\delta + 1} \|u\|^{\delta+1} - \sum_{n \in \mathbb{Z}} F(n, u(n)). \tag{2.1}$$

If $(p), (q)$ and (F1) or (F1') or (F1") hold, then $I \in C^1(E, \mathbb{R})$ and one can easily check that

$$\begin{aligned} \langle I'(u), v \rangle \ &= \ \sum_{n \in \mathbb{Z}} \left[(p(n)(\triangle u(n-1))^{\delta} \triangle v(n-1)) + q(n)(u(n))^{\delta} v(n) \right. \\ &\quad \left. - f(n, u(n)) v(n)) \right], \quad \forall \ u, v \in E. \end{aligned} \tag{2.2}$$

Furthermore, the critical points of I in E are classical solutions of (1.1) with $u(\pm\infty) = 0$.

We will obtain the critical points of I by using the Symmetric Mountain Pass Theorem. We recall it and a minimization theorem as:

Lemma 2.1[17, 25]**.** *Let E be a real Banach space and $I \in C^1(E, \mathbb{R})$ satisfy (PS)-condition. Suppose that I satisfies the following conditions:*

(i) $I(0) = 0$*;*

(ii) There exist constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho(0)} \geq \alpha$;

(iii) For each finite dimensional subspace $E' \subset E$, there is $r = r(E') > 0$ such that $I(u) \leq 0$ for $u \in E' \setminus B_r(0)$, where $B_r(0)$ is an open ball in E of radius r centered at 0.

Then I possesses an unbounded sequence of critical values.

Remark 2.1. *As shown in [6], a deformation lemma can be proved with condition (C) replacing the usual (PS)-condition, and it turns out that Lemma 2.1 hold true under condition (C). We say I satisfies condition (C), i.e., for every sequence $\{u_k\} \subset E$, $\{u_k\}$ has a convergent subsequence if $I(u_k)$ is bounded and $(1 + \|u_k\|)\|I'(u_k)\| \to 0$ as $k \to \infty$.*

Lemma 2.2. *For $u \in E$*

$$\|u\|_\infty \leq q^{-\frac{1}{\delta+1}} \|u\| = \lambda \|u\|, \tag{2.3}$$

where $q = \inf_{n\in\mathbb{Z}} q(n), \lambda = q^{-\frac{1}{\delta+1}}$.

Proof. Since $u \in E$, it follows that $\lim_{|t|\to\infty} |u(t)| = 0$. Hence, there exists $n^* \in \mathbb{Z}$ such that

$$\|u\|_\infty = |u(n^*)| = \max_{n\in\mathbb{Z}} |u(n)|.$$

By (q) and (2.2), we have

$$\|u\|^{\delta+1} \geq \sum_{n\in\mathbb{Z}} q(n)|u(n)|^{\delta+1} \geq q \sum_{n\in\mathbb{Z}} |u(n)|^{\delta+1} \geq q\|u\|_\infty^{\delta+1} = q|u(n^*)|^{\delta+1}. \tag{2.4}$$

It follows from (2.4) that (2.3) holds.

Lemma 2.3. *Assume that (F2) and (F3) hold. Then for every $(n, x) \in \mathbb{Z} \times \mathbb{R}$,*

(i) *$s^{-\mu} F_1(n, sx)$ is nondecreasing on $(0, +\infty)$;*

(ii) *$s^{-\varrho} F_2(n, sx)$ is nonincreasing on $(0, +\infty)$.*

The proof of Lemma 2.3 is routine and so we omit it.

Lemma 2.4[36] *Every uniformly convex Banach space is reflexive.*

Lemma 2.5[16] *Let E be a uniformly convex Banach space, $x_n \in E$, then $x_n \to x$ if and only if $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$.*

3. Proofs of theorems

Proof of Theorem 1.1. We first show that I satisfies condition (C). Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a (C) sequence of I, that is, $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $(1+\|u_k\|)\|I'(u_k)\|\to 0$ as $k\to+\infty$. Then it follows from (2.1) and (2.2) that

$$\begin{aligned} C_1 &\geq (\delta+1)I(u_k)-\langle I'(u_k),u_k\rangle \\ &= \sum_{n\in\mathbb{Z}}[u_k(n)f(n,u_k(n))-(\delta+1)F(n,u_k(n))]. \end{aligned} \tag{3.1}$$

By (F1), there exists $\eta\in(0,1)$ such that

$$|f(n,x)|\leq \frac{1}{2}q(n)|x|^{\delta} \quad \text{for } n\in\mathbb{Z}, \ |x|\leq\eta. \tag{3.2}$$

Since $F(n,0)=0$, it follows that

$$|F(n,x)|\leq \frac{1}{2(\delta+1)}q(n)|x|^{\delta+1} \quad \text{for } n\in\mathbb{Z}, \ |x|\leq\eta. \tag{3.3}$$

By (F2), we have

$$xf(n,x)\geq(\delta+1)F(n,x)\geq 0 \quad \text{for } (n,x)\in\mathbb{Z}\times\mathbb{R}, \ k\in\mathbb{N}, \tag{3.4}$$

and

$$F(n,x)\leq(a+b|x|^{\nu})[xf(n,x)-(\delta+1)F(n,x)] \quad \text{for } (n,x)\in\mathbb{Z}\times\mathbb{R}, \ |x|\geq\eta. \tag{3.5}$$

It follows from (F2), (2.1), (3.1), (3.2), (3.3), (3.4) and (3.5) that

$$\begin{aligned} \frac{1}{\delta+1}\|u_k\|^{\delta+1} &= I(u_k)+\sum_{n\in\mathbb{Z}}F(n,u_k(n)) \\ &= I(u_k)+\sum_{n\in\mathbb{Z}(|u_k(n)|\leq\eta)}F(n,u_k(n))+\sum_{n\in\mathbb{Z}(|u_k(n)|>\eta)}F(n,u_k(n)) \\ &\leq I(u_k)+\frac{1}{2(\delta+1)}\sum_{n\in\mathbb{Z}(|u_k(n)|\leq\eta)}q(n)|u_k(n)|^{\delta+1} \\ &\quad +\sum_{n\in\mathbb{Z}(|u_k(n)|>\eta)}(a+b|u_k(n)|^{\nu})[u_k(n)f(n,u_k(n))-(\delta+1)F(n,u_k(n))] \\ &\leq C_2+\frac{1}{2(\delta+1)}\|u_k\|^{\delta+1}+\sum_{n\in\mathbb{Z}}(a+b|u_k(n)|^{\nu})[u_k(n)f(n,u_k(n))-(\delta+1)F(n,u_k(n))] \\ &\leq C_2+\frac{1}{2(\delta+1)}\|u_k\|^{\delta+1}+(a+b\|u_k\|_{\infty}^{\nu})\sum_{n\in\mathbb{Z}}[u_k(n)f(n,u_k(n))-(\delta+1)F(n,u_k(n))] \\ &\leq C_2+\frac{1}{2(\delta+1)}\|u_k\|^{\delta+1}+C_1(a+b\|u_k\|_{\infty}^{\nu}) \\ &\leq C_2+\frac{1}{2(\delta+1)}\|u_k\|^{\delta+1}+C_1\{a+\lambda^{\nu}b\|u_k\|^{\nu}\}, \quad k\in\mathbb{N}. \end{aligned} \tag{3.6}$$

Since $\nu<\delta+1$, it follows that there exists a constant $A>0$ such that

$$\|u_k\|\leq A \quad \text{for } k\in\mathbb{N}. \tag{3.7}$$

So passing to a subsequence if necessary, it can be assumed that $u_k\rightharpoonup u_0$ in E (Since E is a reflexive Banach space). For any given number $\varepsilon>0$, by (F1), we can choose $\xi>0$ such that

$$|f(n,x)|\leq\varepsilon q(n)|x|^{\delta} \quad \text{for } n\in\mathbb{Z}, \text{ and } |x|\leq\xi. \tag{3.8}$$

Since $q(n)\to\infty$, we can also choose an integer $\Pi>0$ such that

$$q(n)\geq\frac{A^{\delta+1}}{\xi^{\delta+1}}, \quad |n|\geq\Pi. \tag{3.9}$$

By (2.1), (3.8) and (3.9), we have

$$\begin{aligned} |u_k(n)|^{\delta+1} &= \frac{1}{q(n)} q(n)|u_k(n)|^{\delta+1} \\ &\le \frac{\xi^{\delta+1}}{A^{\delta+1}} \sum_{n\in\mathbb{Z}} q(n)|u_k(n)|^{\delta+1} \\ &\le \frac{\xi^{\delta+1}}{A^{\delta+1}} \|u_k\|^{\delta+1} \\ &\le \xi^{\delta+1} \quad \text{for } |n| \ge \Pi, \quad k \in \mathbb{N}. \end{aligned} \tag{3.10}$$

Since $u_k \rightharpoonup u_0$ in E, it is easy to verify that $u_k(n)$ converges to $u_0(n)$ pointwise for all $n \in \mathbb{Z}$, that is

$$\lim_{k\to\infty} u_k(n) = u_0(n), \qquad \forall\, n \in \mathbb{Z}. \tag{3.11}$$

Hence, we have by (3.10) and (3.11)

$$|u_0(n)| \le \zeta \quad \text{for } |n| \ge \Pi. \tag{3.12}$$

It follows from (3.11) and the continuity of $f(n, x)$ on x that there exists $k_0 \in \mathbb{N}$ such that

$$\sum_{n=-\Pi}^{\Pi} |f(n, u_k(n)) - f(n, u_0(n))||u_k(n) - u_0(n)| < \varepsilon \quad \text{for } k \ge k_0. \tag{3.13}$$

On the other hand, it follows from (3.2), (3.9), (3.10), (3.11) and (3.12) that

$$\begin{aligned} &\sum_{|n|>\Pi} |f(n, u_k(n)) - f(n, u_0(n))||u_k(n) - u_0(n)| \\ \le\ & \sum_{|n|>\Pi} (|f(n, u_k(n))| + |f(n, u_0(n))|)(|u_k(n)| + |u_0(n)|) \\ \le\ & \varepsilon \sum_{|n|>\Pi} q(n)(|u_k(n)|^{\delta} + |u_0(n)|^{\delta})(|u_k(n)| + |u_0(n)|) \\ \le\ & 2\varepsilon \sum_{|n|>\Pi} q(n)(|u_k(n)|^{\delta+1} + |u_0(n)|^{\delta+1}) \\ \le\ & 2\varepsilon(\|u_k\|^{\delta+1} + \|u_0\|^{\delta+1}) \\ \le\ & 2\varepsilon(A^{\delta+1} + \|u_0\|^{\delta+1}), \quad k \in \mathbb{N}. \end{aligned} \tag{3.14}$$

Combining (3.13) with (3.14), we get

$$\sum_{n\in\mathbb{Z}} |f(n, u_k(n)) - f(n, u_0(n))|\, |u_k(n) - u_0(n)| \to 0 \quad \text{as } k \to \infty. \tag{3.15}$$

Using Hölder's inequality

$$ac + bd \le (a^p + b^p)^{1/p}(c^q + d^q)^{1/q},$$

where a, b, c, d are nonnegative numbers and $1/p + 1/q = 1, p > 1$, it follows from (2.2) that

$$\begin{aligned}
& \langle I'(u_k) - I'(u_0), u_k - u_0 \rangle \\
= & \sum_{n\in\mathbb{Z}} p(n)(\Delta u_k(n-1))^{\delta}(\Delta u_k(n-1) - \Delta u_0(n-1)) \\
& + \sum_{n\in\mathbb{Z}} q(n)(u_k(n))^{\delta}(u_k(n) - u_0(n)) \\
& - \sum_{n\in\mathbb{Z}} p(n)(\Delta u_0(n-1))^{\delta}(\Delta u_k(n-1) - \Delta u_0(n-1)) \\
& - \sum_{n\in\mathbb{Z}} q(n)(u_0(n))^{\delta}(u_k(n) - u_0(n)) \\
& - \sum_{n\in\mathbb{Z}} (f(n, u_k(n)) - f(n, u_0(n)), u_k(n) - u_0(n)) \\
= & \|u_k\|^{\delta+1} + \|u_0\|^{\delta+1} - \sum_{n\in\mathbb{Z}} p(n)(\Delta u_k(n-1))^{\delta}\Delta u_0(n-1) \\
& - \sum_{n\in\mathbb{Z}} q(n)(u_k(n))^{\delta} u_0(n) \\
& - \sum_{n\in\mathbb{Z}} p(n)(\Delta u_0(n-1))^{\delta}\Delta u_k(n-1) - \sum_{n\in\mathbb{Z}} q(n)(u_0(n))^{\delta} u_k(n) \\
& - \sum_{n\in\mathbb{Z}} (f(n, u_k(n)) - f(n, u_0(n)), u_k(n) - u_0(n)) \\
\geq & \|u_k\|^{\delta+1} + \|u_0\|^{\delta+1} - \left(\sum_{n\in\mathbb{Z}} p(n)(\Delta u_0(n-1))^{\delta+1}\right)^{\frac{1}{\delta+1}} \left(\sum_{n\in\mathbb{Z}} p(n)(\Delta u_k(n-1))^{\delta+1}\right)^{\frac{\delta}{\delta+1}} \\
& - \left(\sum_{n\in\mathbb{Z}} q(n)(u_0(n))^{\delta+1}\right)^{\frac{1}{\delta+1}} \left(\sum_{n\in\mathbb{Z}} q(n)(u_k(n))^{\delta+1}\right)^{\frac{\delta}{\delta+1}} \\
& - \left(\sum_{n\in\mathbb{Z}} p(n)(\Delta u_k(n-1))^{\delta+1}\right)^{\frac{1}{\delta+1}} \left(\sum_{n\in\mathbb{Z}} p(n)(\Delta u_0(n-1))^{\delta+1}\right)^{\frac{\delta}{\delta+1}} \\
& - \left(\sum_{n\in\mathbb{Z}} q(n)(u_k(n))^{\delta+1}\right)^{\frac{1}{\delta+1}} \left(\sum_{n\in\mathbb{Z}} q(n)(u_0(n))^{\delta+1}\right)^{\frac{\delta}{\delta+1}} \\
& - \sum_{n\in\mathbb{Z}} (f(n, u_k(n)) - f(n, u_0(n)), u_k(n) - u_0(n))
\end{aligned}$$

$$
\begin{aligned}
&\geq \|u_k\|^{\delta+1}+\|u_0\|^{\delta+1}\\
&\quad -\left(\sum_{n\in\mathbb{Z}}\left[p(n)(\Delta u_0(n-1))^{\delta+1}+q(n)(u_0(n))^{\delta+1}\right]\right)^{\frac{\delta}{\delta+1}}\\
&\quad \left(\sum_{n\in\mathbb{Z}}\left[p(n)(\Delta u_k(n-1))^{\delta+1}+q(n)(u_k(n))^{\delta+1}\right]\right)^{\frac{\delta}{\delta+1}}\\
&\quad -\left(\sum_{n\in\mathbb{Z}}\left[p(n)(\Delta u_k(n-1))^{\delta+1}+q(n)(u_k(n))^{\delta+1}\right]\right)^{\frac{\delta}{\delta+1}}\\
&\quad \left(\sum_{n\in\mathbb{Z}}\left[p(n)(\Delta u_0(n-1))^{\delta+1}+q(n)(u_0(n))^{\delta+1}\right]\right)^{\frac{\delta}{\delta+1}}\\
&\quad -\sum_{n\in\mathbb{Z}}(f(n,u_k(n))-f(n,u_0(n)),u_k(n)-u_0(n))\\
&= \|u_k\|^{\delta+1}+\|u_0\|^{\delta+1}-\|u_0\|\|u_k\|^{\delta}-\|u_k\|\|u_0\|^{\delta}\\
&\quad -\sum_{n\in\mathbb{Z}}(f(n,u_k(n))-f(n,u_0(n)),u_k(n)-u_0(n))\\
&= \left(\|u_k\|^{\delta}-\|u_0\|^{\delta}\right)(\|u_k\|-\|u_0\|)\\
&\quad -\sum_{n\in\mathbb{Z}}(f(n,u_k(n))-f(n,u_0(n)),u_k(n)-u_0(n)). \qquad (3.16)
\end{aligned}
$$

Since $I'(u_k)\to 0$ as $k\to+\infty$ and $u_k \rightharpoonup u_0$ in E, it follows from (3.16) that

$$\langle I'(u_k)-I'(u_0),u_k-u_0\rangle \to 0 \quad \text{as} \;\; k\to\infty,$$

which, together with (3.15) and (3.16), yields that $\|u_k\|\to\|u\|$ as $k\to+\infty$. By the uniform convexity of E and the fact that $u_k \rightharpoonup u_0$ in E, it follows from the Kadec-Klee property [16] or Lemma 2.5 that $u_k\to u_0$ in E. Hence, I satisfies (C)-condition.

We now show that there exist constants $\rho,\alpha>0$ such that I satisfies assumption (ii) of Lemma 2.1 with these constants. Let $\vartheta\leq\eta$, if $\|u\|=\vartheta/\lambda:=\rho$, then by (2.3), $|u(n)|\leq\vartheta\leq\eta<1$ for $n\in\mathbb{Z}$.

Set

$$\alpha=\frac{\vartheta^{\delta+1}}{2(\delta+1)\lambda^{\delta+1}}.$$

Hence, from (2.1) and (3.3), we have

$$
\begin{aligned}
I(u) &= \frac{1}{\delta+1}\|u\|^{\delta+1}-\sum_{n\in\mathbb{Z}}F(n,u(n))\\
&\geq \frac{1}{\delta+1}\|u\|^{\delta+1}-\frac{1}{2(\delta+1)}\sum_{n\in\mathbb{Z}}q(n)|u(n)|^{\delta+1}\\
&\geq \frac{1}{2(\delta+1)}\|u\|^{\delta+1}\\
&= \alpha. \qquad (3.17)
\end{aligned}
$$

(3.17) shows that $\|u\|=\rho$ implies that $I(u)\geq\alpha$, i.e., I satisfies assumption (ii) of Lemma 2.1.

Finally, it remains to show that I satisfies assumption (iii) of Lemma 2.1. Let E' be a finite dimensional subspace of E. Since all the norms of a finite dimensional normed space are equivalent, so there exists a constant $d>0$ such that

$$\|u\|\leq d\|u\|_\infty \quad \text{for} \;\; u\in E'. \qquad (3.18)$$

Assume that $\dim E' = m$ and $u_1, u_2, \ldots, u_m$ is a base of E' such that

$$\|u_i\| = d, \qquad i = 1, 2, \ldots, m. \tag{3.19}$$

For any $u \in E'$, there exist $\lambda_i \in \mathbb{R},\ i = 1, 2, \ldots, m$ such that

$$u(n) = \sum_{i=1}^{m} \lambda_i u_i(n) \quad \text{ for } \ n \in \mathbb{Z}. \tag{3.20}$$

Let

$$\|u\|_* = \sum_{i=1}^{m} |\lambda_i| \|u_i\|. \tag{3.21}$$

It is easy to verify that $\|\cdot\|_*$ defined by (3.21) is a norm of E'. Hence, there exists $d' > 0$ such that $d'\|u\|_* \le \|u\|$.

Since $u_i \in E$, we can choose $\Pi_1 > \Pi$ such that

$$|u_i(n)| < \frac{d'\eta}{1+d'}, \qquad |n| > \Pi_1, \quad i = 1, 2, \ldots, m, \tag{3.22}$$

where η is given in (3.3). Set

$$\Theta = \left\{ \sum_{i=1}^{m} \lambda_i u_i(n) \ : \ \lambda_i \in \mathbb{Z},\ i = 1, 2, \ldots, m; \ \sum_{i=1}^{m} |\lambda_i| = 1 \right\} = \{ u \in E' \ : \ \|u\|_* = d \}. \tag{3.23}$$

Hence, for $u \in \Theta$, let $n_0 = n_0(u) \in \mathbb{Z}$ such that

$$|u(n_0)| = \|u\|_\infty. \tag{3.24}$$

Then by (3.19), (3.20), (3.21), (3.22), (3.23) and (3.24), we have

$$\begin{aligned} d'd &= d'd \sum_{i=1}^{m} |\lambda_i| = d' \sum_{i=1}^{m} |\lambda_i| \|u_i\| = d'\|u\|_* \\ &\le \|u\| \le d\|u\|_\infty = d|u(n_0)| \\ &\le d \sum_{i=1}^{m} |\lambda_i| |u_i(n_0)|, \qquad u \in \Theta. \end{aligned} \tag{3.25}$$

This shows that

$$|u(n_0)| \ge d' \tag{3.26}$$

and there exists $i_0 \in \{1, 2, \ldots, m\}$ such that $|u_{i_0}(n_0)| \ge d'$. By (F3), there exists $\sigma_0 = \sigma_0(d, \Pi_1) > 1$ such that

$$s^{-(\delta+1)} \min_{|x|=1} F(n, sx) \ge \left(\frac{2d}{d'} \right)^{\delta+1} \qquad \text{for } \ s \ge \frac{d'\sigma_0}{2}, \quad n \in \mathbb{Z}(-\Pi_1, \Pi_1). \tag{3.27}$$

It follows from (F3), (2.1) and (3.27) that

$$\begin{aligned} I(\sigma u) &= \frac{\sigma^{\delta+1}}{\delta+1} \|u\|^{\delta+1} - \sum_{n \in \mathbb{Z}} F(n, \sigma u(n)) \\ &\le \frac{\sigma^{\delta+1}}{\delta+1} \|u\|^{\delta+1} - F(n_0, \sigma u(n_0)) \\ &\le \frac{\sigma^{\delta+1}}{\delta+1} \|u\|^{\delta+1} - \min_{|x|=1} F(n_0, \sigma |u(n_0)| x) \\ &\le \frac{(d\sigma)^{\delta+1}}{\delta+1} - (d\sigma)^{\delta+1} |u(n_0)|^{\delta+1} \\ &\le \frac{(d\sigma)^{\delta+1}}{\delta+1} - (d\sigma)^{\delta+1} \\ &= -\frac{\delta (d\sigma)^{\delta+1}}{\delta+1}, \qquad u \in \Theta, \quad \sigma \ge \sigma_0. \end{aligned} \tag{3.28}$$

We deduce that there is $\sigma_0 = \sigma_0(d, \Pi_1) = \sigma_0(E') > 1$ such that

$$I(\sigma u) < 0 \quad \text{for} \;\; u \in \Theta \;\; \text{and} \;\; \sigma \geq \sigma_0.$$

That is

$$I(u) < 0 \quad \text{for} \;\; u \in E' \;\; \text{and} \;\; \|u\| \geq d\sigma_0.$$

This shows that condition (iii) of Lemma 2.1 holds. By Lemma 2.1, I possesses an unbounded sequence $\{d_k\}_{k\in\mathbb{N}}$ of critical values with $d_k = I(u_k)$, where u_k is such that $I'(u_k) = 0$ for $k = 1, 2, \ldots$. If $\{\|u_k\|\}$ is bounded, then there exists $B > 0$ such that

$$\|u_k\| \leq B \quad \text{for} \;\; k \in \mathbb{N}. \tag{3.29}$$

By a similar fashion for the proof of (3.3), for the given η in (3.3), there exists $\Pi_2 > 0$ such that

$$|u_k(n)| \leq \eta \quad \text{for} \;\; |n| \geq \Pi_2, \quad k \in \mathbb{N}. \tag{3.30}$$

Thus, from (2.1), (2.3) and (3.3), we have

$$\begin{aligned} \frac{1}{\delta+1}\|u_k\|^{\delta+1} &= d_k + \sum_{n\in\mathbb{Z}} F(n, u_k(n)) \\ &= d_k + \sum_{|n|>\Pi_2} F(n, u_k(n)) + \sum_{|n|\leq\Pi_2} F(n, u_k(n)) \\ &\geq d_k - \frac{1}{2(\delta+1)} \sum_{|n|>\Pi_2} q(n)|u_k(n)|^{\delta+1} - \sum_{|n|\leq\Pi_2} |F(n, u_k(n))| \\ &\geq d_k - \frac{1}{2(\delta+1)}\|u_k\|^{\delta+1} - \sum_{|n|\leq\Pi_2} \max_{|x|\leq\lambda B} |F(n, x)|. \end{aligned} \tag{3.31}$$

It follows that

$$d_k \leq \frac{3}{2(\delta+1)}\|u_k\|^{\delta+1} + \sum_{|n|\leq\Pi_2} \max_{|x|\leq\lambda B} |F(n, x)| < +\infty.$$

This contradicts to the fact that $\{d_k\}_{k=1}^{\infty}$ is unbounded, and so $\{\|u_k\|\}$ is unbounded. The proof is complete.

Proof of Theorem 1.2. It is clear that $I(0) = 0$. We first show that I satisfies the (PS)-condition. Assume that $\{u_k\}_{k\in\mathbb{N}} \subset E$ is a sequence such that $\{I(u_k)\}_{k\in\mathbb{N}}$ is bounded and $I'(u_k) \to 0$ as $k \to +\infty$. Then there exists a constant $c > 0$ such that

$$|I(u_k)| \leq c, \quad \|I'(u_k)\|_{E^*} \leq \mu c \quad \text{for} \;\; k \in \mathbb{N}. \tag{3.32}$$

From (2.1), (2.2), (3.32), (F4) and (F5), we obtain

$$\begin{aligned} & (\delta+1)c + (\delta+1)c\|u_k\| \\ \geq\; & (\delta+1)I(u_k) - \frac{\delta+1}{\mu}\langle I'(u_k), u_k\rangle \\ =\; & \frac{\mu-(\delta+1)}{\mu}\|u_k\|^{\delta+1} + (\delta+1)\sum_{n\in\mathbb{Z}}\left[F_2(n, u_k(n)) - \frac{1}{\mu}u_k(n)f_2(n, u_k(n))\right] \\ & -(\delta+1)\sum_{n\in\mathbb{Z}}\left[F_1(n, u_k(n)) - \frac{1}{\mu}u_k(n)f_1(n, u_k(n))\right] \\ \geq\; & \frac{\mu-(\delta+1)}{\mu}\|u_k\|^{\delta+1}, \quad k \in \mathbb{N}. \end{aligned}$$

It follows that there exists a constant $A > 0$ such that

$$\|u_k\| \le A \quad \text{for} \;\; k \in \mathbb{N}. \tag{3.33}$$

Similar to the proof of Theorem 1.1, we can prove that $\{u_k\}$ has a convergent subsequence in E. Hence, I satisfies condition (PS)-condition. By a similar fashion for the proof in Theorem, we can verify that I satisfies assumption (ii) of Lemma 2.1.

Finally, it remains to show that I satisfies assumption (iii) of Lemma 2.1. Let E' be a finite dimensional subspace of E. Since all norms of a finite dimensional normed space are equivalent, so there is a constant $d' > 0$ such that (3.22) holds. Let η, Π_1 and Θ be the same as in the proof of Theorem 1.1, then (3.26) holds.

Set

$$\tau = \min\{F_1(n,x) : |n| \le \Pi_1, |x| \le d'\}, \tag{3.34}$$

where d' is given in (3.22).

Since $F_1(n,x) > 0$ for all $n \in \mathbb{Z}$ and $x \in \mathbb{R} \setminus \{0\}$, and $F_1(n,x)$ is continuous in x, so $\tau > 0$. It follows from (3.26), (3.34) and Lemma 2.3 (i) that

$$\begin{aligned}
\sum_{n=-\Pi_1}^{\Pi_1} F_1(n,u(n)) &\ge F_1(n_0,u(n_0)) \\
&\ge F_1\left(n_0, \frac{u(n_0)d'}{|u(n_0)|}\right)\left(\frac{|u(n_0)|}{d'}\right)^{\mu} \\
&\ge \left[\min_{|x|\le d'} F_1(n_0,x)\right]\left(\frac{|u(n_0)|}{d'}\right)^{\mu} \\
&\ge \tau \quad \text{for} \;\; u \in \Theta.
\end{aligned} \tag{3.35}$$

For any $u \in E$, it follows from (2.3) and Lemma 2.3 (ii) that

$$\begin{aligned}
&\sum_{n=-\Pi_1}^{\Pi_1} F_2(n,u(n)) \\
= &\sum_{n\in\mathbb{Z}(-\Pi_1,\Pi_1),\ |u(n)|>1} F_2(n,u(n)) + \sum_{n\in\mathbb{Z}(-\Pi_1,\Pi_1),\ |u(n)|\le 1} F_2(n,u(n)) \\
\le &\sum_{n\in\mathbb{Z}(-\Pi_1,\Pi_1),\ |u(n)|>1} F_2\left(n, \frac{u(n)}{|u(n)|}\right)|u(n)|^{\varrho} \\
&+ \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|\le 1} |F_2(n,x)| \\
\le &\|u\|_\infty^{\varrho} \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|=1} |F_2(n,x)| + \sum_{n=\Pi_1}^{\Pi_1} \max_{|x|\le 1} |F_2(n,x)| \\
\le &\lambda^{\varrho}\|u\|^{\varrho} \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|=1} |F_2(n,x)| + \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|\le 1} |F_2(n,x)| \\
= &M_1\|u\|^{\varrho} + M_2,
\end{aligned} \tag{3.36}$$

where

$$M_1 = \lambda^{\varrho} \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|=1} |F_2(n,x)|, \qquad M_2 = \sum_{n=-\Pi_1}^{\Pi_1} \max_{|x|\le 1} |F_2(n,x)|.$$

From (3.3), (3.35), (3.36) and Lemma 2.3, we have for $u \in \Theta$ and $\sigma > 1$

$$\begin{aligned}
I(\sigma u) &= \frac{\sigma^{\delta+1}}{\delta+1}\|u\|^{\delta+1} - \sum_{n\in\mathbb{Z}} F(n, \sigma u(n)) \\
&= \frac{\sigma^{\delta+1}}{\delta+1}\|u\|^{\delta+1} + \sum_{n\in\mathbb{Z}} F_2(n, \sigma u(n)) - \sum_{n\in\mathbb{Z}} F_1(n, \sigma u(n)) \\
&\le \frac{\sigma^{\delta+1}}{\delta+1}\|u\|^{\delta+1} + \sigma^{\varrho}\sum_{n\in\mathbb{Z}} F_2(n, u(n)) - \sigma^{\mu}\sum_{n\in\mathbb{Z}} F_1(n, u(n)) \\
&= \frac{\sigma^{\delta+1}}{\delta+1}\|u\|^{\delta+1} + \sigma^{\varrho}\sum_{|n|>\Pi_1} F_2(n, u(n)) - \sigma^{\mu}\sum_{|n|>\Pi_1} F_1(n, u(n)) \\
&\quad +\sigma^{\varrho}\sum_{n=-\Pi_1}^{\Pi_1} F_2(n, u(n)) - \sigma^{\mu}\sum_{n=-\Pi_1}^{\Pi_1} F_1(n, u(n)) \\
&\le \frac{\sigma^{\delta+1}}{\delta+1}\|u\|^{\delta+1} - \sigma^{\varrho}\sum_{|n|>\Pi_1} F(n, u(n)) \\
&\quad +\sigma^{\varrho}\sum_{n=-\Pi_1}^{\Pi_1} F_2(n, u(n)) - \sigma^{\mu}\sum_{n=-\Pi_1}^{\Pi_1} F_1(n, u(n)) \\
&\le \frac{\sigma^{\delta+1}}{\delta+1}\|u\|^{\delta+1} + \frac{\sigma^{\varrho}}{2(\delta+1)}\sum_{|n|>\Pi_1} q(n)|u(n)|^{\delta+1} + \sigma^{\varrho}(M_1\|u\|^{\varrho} + M_2) - \tau\sigma^{\mu} \\
&\le \frac{\sigma^{\delta+1}}{\delta+1}\|u\|^{\delta+1} + \frac{\sigma^{\varrho}}{2(\delta+1)}\|u\|^{\delta+1} + \sigma^{\varrho}(M_1\|u\|^{\varrho} + M_2) - \tau\sigma^{\mu} \\
&= \frac{(d\sigma)^{\delta+1}}{\delta+1} + \frac{d^{\delta+1}\sigma^{\varrho}}{2(\delta+1)} + M_1(d\sigma)^{\varrho} + M_2\sigma^{\varrho} - \tau\sigma^{\mu}. \qquad (3.37)
\end{aligned}$$

Since $\mu > \varrho > \delta+1$, we deduce that there is $\sigma_0 = \sigma_0(d, M_1, M_2, \tau) = \sigma_0(E') > 1$ such that

$$I(\sigma u) < 0 \quad \text{for } u \in \Theta \text{ and } \sigma \ge \sigma_0.$$

That is

$$I(u) < 0 \quad \text{for } u \in E' \text{ and } \|u\| \ge d\sigma_0.$$

This shows that (iii) of Lemma 2.1 holds. By Lemma 2.1, I possesses an unbounded sequence $\{d_k\}_{k\in\mathbb{N}}$ of critical values with $d_k = I(u_k)$, where u_k is such that $I'(u_k) = 0$ for $k = 1, 2, \ldots$. If $\{\|u_k\|\}_{k\in\mathbb{N}}$ is bounded, then there exists $B > 0$ such that

$$\|u_k\| \le B \quad \text{for } k \in \mathbb{N}. \qquad (3.38)$$

By a similar fashion for the proof of (3.2) and (3.3), for the given η in (3.2), there exists $\Pi_2 > 0$ such that

$$|u_k(n)| \le \eta \quad \text{for } |n| \ge \Pi_2, \quad k \in \mathbb{N}. \qquad (3.39)$$

Thus, from (2.1), (2.3), (3.3), (3.36) and (3.37), we have

$$\begin{aligned}
\frac{1}{\delta+1}\|u_k\|^{\delta+1} &= d_k + \sum_{n\in\mathbb{Z}} F(n, u_k(n)) \\
&= d_k + \sum_{|n|>\Pi_2} F(n, u_k(n)) + \sum_{n=-\Pi_2}^{\Pi_2} F(n, u_k(n)) \\
&\geq d_k - \frac{1}{2(\delta+1)} \sum_{|n|>\Pi_2} q(n)|u_k(n)|^{\delta+1} - \sum_{n=-\Pi_2}^{\Pi_2} F_2(n, u_k(n)) \\
&\geq d_k - \frac{1}{2(\delta+1)}\|u_k\|^{\delta+1} - \sum_{n=-\Pi_2}^{\Pi_2} \max_{|x|\leq\lambda B} |F_2(n,x)|. \qquad (3.40)
\end{aligned}$$

It follows that

$$d_k \leq \frac{3}{2(\delta+1)}\|u_k\|^p + \sum_{n=-\Pi_2}^{\Pi_2} \max_{|x|\leq\lambda B} |F_2(n,x)| < +\infty.$$

This contradicts to the fact that $\{d_k\}_{k\in\mathbb{N}}$ is unbounded, and so $\{\|u_k\|\}_{k\in\mathbb{N}}$ is unbounded.

Proof of Theorem 1.3. In the proof of Theorem 1.1, the condition that $F_2(n,x) \geq 0$ for $(n,x) \in \mathbb{Z}\times\mathbb{R}$, $|x| \leq 1$ in (F1') is only used in the the proofs of assumption (ii) of Lemma 2.1. Therefore, we only prove assumption (ii) of Lemma 2.1 still hold use (F1") instead of (F1'). By (F1"), there exists $\eta \in (0,1)$ such that

$$|f(n,x)| \leq \frac{1}{2}q(n)|x|^{\delta} \quad \text{for } \; n \in \mathbb{Z}\setminus J, \;\; |x| \leq \eta. \qquad (3.41)$$

Since $F(n,0) = 0$, it follows that

$$|F(n,x)| \leq \frac{1}{2(\delta+1)}q(n)|x|^{\delta+1} \quad \text{for } \; n \in \mathbb{Z}\setminus J, \;\; |x| \leq \eta. \qquad (3.42)$$

Set

$$M = \sup\left\{ \frac{F_1(n,x)}{q(n)} \;\middle|\; n \in J, \; x \in \mathbb{R}, \; |x| = 1 \right\}. \qquad (3.43)$$

Set $\delta = \min\{1/(2(\delta+1)M+1)^{1/(\mu-(\delta+1))}, \eta\}$. if $\|u\| = \vartheta/\lambda := \rho$, then by (2.3), $|u(n)| \leq \vartheta \leq \eta < 1$ for $n \in \mathbb{Z}$. By (3.43) and Lemma 2.3 (i), we have

$$\begin{aligned}
\sum_{n\in J} F_1(n, u(n)) &\leq \sum_{\{n\in J,\; u(n)\neq 0\}} F_1\left(n, \frac{u(n)}{|u(n)|}\right)|u(n)|^{\mu} \\
&\leq M\sum_{n\in J} q(n)|u(n)|^{\mu} \\
&\leq M\delta^{\mu-(\delta+1)} \sum_{n\in J} q(n)|u(n)|^{\delta+1} \\
&\leq \frac{1}{2(\delta+1)} \sum_{n\in J} q(n)|u(n)|^{\delta+1}. \qquad (3.44)
\end{aligned}$$

Set

$$\alpha = \frac{\vartheta^{\delta+1}}{2(\delta+1)\lambda^{\delta+1}}.$$

Hence, from (2.1), (3.42), (3.44) and (F1”), we have

$$\begin{aligned}
I(u) &= \frac{1}{\delta+1}\|u\|^{\delta+1} - \sum_{n\in\mathbb{Z}} F(n,u(n)) \\
&= \frac{1}{\delta+1}\|u\|^{\delta+1} - \sum_{n\in\mathbb{Z}\setminus J} F(n,u(n)) - \sum_{n\in J} F(n,u(n)) \\
&\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{2(\delta+1)}\sum_{n\in\mathbb{Z}\setminus J} q(n)|u(n)|^{\delta+1} - \sum_{n\in J} F_1(n,u(n)) \\
&\geq \frac{1}{\delta+1}\|u\|^{\delta+1} - \frac{1}{2(\delta+1)}\sum_{n\in\mathbb{Z}\setminus J} q(n)|u(n)|^{\delta+1} - \frac{1}{2(\delta+1)}\sum_{n\in J} q(n)|u(n)|^{\delta+1} \\
&= \frac{1}{\delta+1}\sum_{n\in\mathbb{Z}} |\Delta u(n-1)|^{\delta+1} - \frac{1}{2(\delta+1)}\sum_{n\in\mathbb{Z}} q(n)|u(n)|^{\delta+1} \\
&\geq \frac{1}{2(\delta+1)}\sum_{n\in\mathbb{Z}} \left[|\Delta u(n-1)|^{\delta+1} + q(n)|u(n)|^{\delta+1}\right] \\
&= \frac{1}{2(\delta+1)}\|u\|^{\delta+1} \\
&= \alpha. \qquad (3.45)
\end{aligned}$$

(3.45) shows that $\|u\| = \rho$ implies that $I(u) \geq \alpha$, i.e., I satisfies assumption (ii) of Lemma 2.1. It is obvious that I is even and $I(0) = 0$ and so assumption (ii) of Lemma 2.1 holds. The proof of assumption (iii) of Lemma 2.1 is the same as in the proof of Theorem 1.2, we omit its details.

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. Consider the second-order difference equation

$$\Delta\left[p(n)(\Delta x(n-1))^{\frac{1}{3}}\right] - q(n)(x(n))^{\frac{1}{3}} + f(n,x(n)) = 0, \qquad (4.1)$$

where $\delta = \frac{1}{3}, q : \mathbb{Z} \to (0,\infty)$ such that $q(n) \to +\infty$ as $|n| \to +\infty$, and

$$F(n,x) = q(n)(2-\cos n)|x|^{\frac{4}{3}}\ln(1+|x|).$$

Since

$$\begin{aligned}
xf(n,x) &= q(n)(2-\cos n)\left[\frac{4}{3}|x|^{\frac{4}{3}}\ln(1+|x|) + \frac{|x|^{\frac{7}{3}}}{1+|x|}\right] \\
&\geq \left(\frac{4}{3} + \frac{1}{1+|x|}\right)F(n,x) \geq 0, \quad \forall\ (n,x) \in \mathbb{Z}\times\mathbb{R}.
\end{aligned}$$

This shows that (F1) holds with $a = b = \nu = 1$. In addition, for any $n \in \mathbb{Z}$

$$\begin{aligned}
s^{-\frac{4}{3}}\min_{|x|=1} F(n,sx) &= s^{-\frac{4}{3}}\min_{|x|=1}\left[q(n)(2-\cos n)|sx|^{\frac{4}{3}}\ln(1+|sx|)\right] \\
&= q(n)(2-\cos n)\ln(1+s) \\
&\to +\infty, \quad s \to +\infty.
\end{aligned}$$

This shows that (F3) also holds. It is easy to verify that assumptions (q) and (F1) of Theorem 1.1 are satisfied. By Theorem 1.1, Eq. (1.1) has an unbounded sequence of homoclinic solutions.

Example 4.2. Consider the second-order difference equation

$$\Delta\left[p(n)(\Delta x(n-1))^3\right] - q(n)(x(n))^3 + f(n,x(n)) = 0, \tag{4.2}$$

where $\delta = 3$, $n \in \mathbb{Z}, u \in \mathbb{R}$, $q : \mathbb{Z} \to (0,\infty)$ such that $q(n) \to +\infty$ as $|n| \to \infty$. Let

$$F(n,x) = q(n)\left(\sum_{i=1}^{m} a_i|x|^{\mu_i} - \sum_{j=1}^{n} b_j|x|^{\varrho_j}\right),$$

where $\mu_1 > \mu_2 > \cdots > \mu_m > \varrho_1 > \varrho_2 > \cdots > \varrho_n > 4$, $a_i, b_j > 0$, $i = 1,2,\ldots,m; j = 1,2,\ldots,n$. Let $\mu = \mu_m$, $\varrho = \varrho_1$, and

$$F_1(n,x) = q(n)\sum_{i=1}^{m} a_i|x|^{\mu_i}, \qquad F_2(n,x) = q(n)\sum_{j=1}^{n} b_j|x|^{\varrho_j}.$$

Then it is easy to verify that all conditions of Theorem 1.2 are satisfied. By Theorem 1.2, Eq. (1.1) has an unbounded sequence of homoclinic solutions..

Example 4.3. Consider the second-order difference equation

$$\Delta\left[p(n)(\Delta x(n-1))^{\frac{11}{5}}\right] - q(n)(x(n))^{\frac{11}{5}} + f(n,x(n)) = 0, \tag{4.3}$$

where $\delta = \frac{11}{5}$, $n \in \mathbb{Z}, u \in \mathbb{R}$, $q : \mathbb{Z} \to (0,\infty)$ such that $q(n) \to +\infty$ as $|n| \to \infty$. Let

$$F(n,x) = q(n)\left[a_1|x|^{\mu_1} + a_2|x|^{\mu_2} - (2-|n|)|x|^{\varrho_1} - (2-|n|)|x|^{\varrho_2}\right],$$

where $q : \mathbb{Z} \to (0,\infty)$ such that $q(n) \to +\infty$ as $|n| \to +\infty$, $\mu_1 > \mu_2 > \varrho_1 > \varrho_2 > \frac{16}{5}$, $a_1, a_2 > 0$. Let $\mu = \mu_2$, $\varrho = \varrho_1$, $J = \{-2,-1,0,1,2\}$ and

$$F_1(n,x) = q(n)\left(a_1|x|^{\mu_1} + a_2|x|^{\mu_2}\right), \qquad F_2(n,x) = q(n)\left[(2-|n|)|x|^{\varrho_1} + (2-|n|)|x|^{\varrho_2}\right].$$

Then it is easy to verify that all conditions of Theorem 1.3 are satisfied. By Theorem 1.3, Eq.(1.1) has an unbounded sequence of homoclinic solutions.

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