On nonnegative radial entire solutions of second order quasilinear elliptic systems

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Abstract

In this article, we consider the second order quasilinear elliptic system of the form

$$\Delta_{p_i} u_i = H_i(|x|)u_{i+1}^{\alpha_i}, \quad x \in \mathbb{R}^N, i = 1, 2, \cdots, m$$

with nonnegative continuous functions H_i. Sufficient conditions are given to have nonnegative nontrivial radial entire solutions. When $H_i,\ i = 1, 2, \cdots, m$, behave like constant multiples of $|x|^\lambda,\ \lambda \in \mathbb{R}$, we can completely characterize the existence property of nonnegative nontrivial radial entire solutions.

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1 Introduction

This paper is concerned with existence and nonexistence of nonnegative radial entire solutions of second order quasilinear elliptic systems of the form

$$(1.1) \qquad \begin{cases} \Delta_{p_1} u_1 = H_1(|x|)u_2^{\alpha_1}, \\ \Delta_{p_2} u_2 = H_2(|x|)u_3^{\alpha_2}, \\ \quad \vdots \\ \Delta_{p_m} u_m = H_m(|x|)u_{m+1}^{\alpha_m},\ u_{m+1} = u_1, \end{cases} \qquad x \in \mathbb{R}^N,$$

where $\Delta_p u = \mathrm{div}(|Du|^{p-2}Du)$, $|x|$ denotes the Euclidean length of $x \in \mathbb{R}^N$, $m \geq 2,\ N \geq 1,\ p_i > 1$ and $\alpha_i > 0,\ i = 1, 2, \cdots, m$, are constants satisfying $\alpha_1\alpha_2 \cdots \alpha_m > (p_1 - 1)(p_2 - 1) \cdots (p_m - 1)$, and the functions $H_i,\ i = 1, 2, \cdots, m$, are nonnegative continuous functions on $[0, \infty)$. When $p = 2$, Δ_p reduces to the usual Laplacian.

An entire solution of (1.1) is defined to be a function $(u_1, u_2, \cdots, u_m) \in (C^1(\mathbb{R}^N))^m$ such that $|Du_i|^{p_i-2}Du_i \in C^1(\mathbb{R}^N)$ and satisfy (1.1) at every point of $\mathbb{R}^N$. Such a solution is said to be radial if it depends only on $|x|$.

The problem of existence and nonexistence of nonnegative radial entire solutions for the scalar equation

$$\Delta_p u = f(|x|, u), \quad x \in \mathbb{R}^N,$$

has been investigated by several authors, and numerous results have been obtained; see, e.g. [3, 6, 7, 10] and references therein. In particular, when f has the form $f(|x|, u) = \pm H(|x|)u^\alpha$ with $\alpha > 0$ and positive function H, critical decay rate of H to admit nonnegative radial entire solutions has been characterized. However, as far as the author knows, very little is known about this problem for the system (1.1) except for the case $p_i = 2,\ i = 1, 2, \cdots, m$. For $p_i = 2$, we refer to [2, 5, 11, 13, 14]. Recently, in [12], the author has considered the elliptic system (1.1) with $m = 2$ and has obtained existence and nonexistence criteria of nonnegative nontrivial radial entire solutions. The results in [12] are described roughly as follows :

Theorem 0.1 [12, Theorems 1 and 2] *Let $m = 2$. Suppose that $H_i,\ i = 1, 2$, satisfy*

$$\frac{C_1}{|x|^{\lambda_i}} \le H_i(|x|) \le \frac{C_2}{|x|^{\lambda_i}}, \quad |x| \ge r_0 > 0, \quad i = 1, 2, \tag{1.2}$$

where $C_i > 0,\ i = 1, 2$, are constants and $\lambda_i,\ i = 1, 2$, are parameters.
(i) *If $\lambda_i,\ i = 1, 2$, satisfy*

$$\begin{cases} \lambda_1 - p_1 + \dfrac{\alpha_1(\lambda_2 - p_2)}{p_2 - 1} > \dfrac{\alpha_1\alpha_2 - (p_1 - 1)(p_2 - 1)}{(p_1 - 1)(p_2 - 1)} \max\{0, p_1 - N\} \quad \text{and} \\[2ex] \lambda_2 - p_2 + \dfrac{\alpha_2(\lambda_1 - p_1)}{p_1 - 1} > \dfrac{\alpha_1\alpha_2 - (p_1 - 1)(p_2 - 1)}{(p_1 - 1)(p_2 - 1)} \max\{0, p_2 - N\}, \end{cases} \tag{1.3}$$

then the system (1.1) has infinitely many positive radial entire solutions.
(ii) *If $\lambda_i,\ i = 1, 2$, satisfy*

$$\begin{cases} \lambda_1 - p_1 + \dfrac{\alpha_1(\lambda_2 - p_2)}{p_2 - 1} \le \dfrac{\alpha_1\alpha_2 - (p_1 - 1)(p_2 - 1)}{(p_1 - 1)(p_2 - 1)} \max\{0, p_1 - N\} \quad \text{or} \\[2ex] \lambda_2 - p_2 + \dfrac{\alpha_2(\lambda_1 - p_1)}{p_1 - 1} \le \dfrac{\alpha_1\alpha_2 - (p_1 - 1)(p_2 - 1)}{(p_1 - 1)(p_2 - 1)} \max\{0, p_2 - N\}, \end{cases}$$

then the system (1.1) does not possess any nonnegative nontrivial radial entire solutions.

Theorem 0.2 [12, Theorems 3 and 4] *Let $m = 2$ and $p_i = N,\ i = 1, 2$. Suppose that $H_i,\ i = 1, 2$, satisfy*

$$\frac{C_1}{|x|^N (\log |x|)^{\lambda_i}} \le H_i(|x|) \le \frac{C_2}{|x|^N (\log |x|)^{\lambda_i}}, \quad |x| \ge r_0 > 1, \quad i = 1, 2,$$

where $C_i > 0,\ i = 1, 2$, are constants and $\lambda_i,\ i = 1, 2$, are parameters.
(i) *If $\lambda_i,\ i = 1, 2$, satisfy*

$$\begin{cases} \lambda_1 - N + \dfrac{\alpha_1(\lambda_2 - N)}{N-1} > \dfrac{\alpha_1\alpha_2 - (N-1)^2}{N-1} & \textit{and} \\[2ex] \lambda_2 - N + \dfrac{\alpha_2(\lambda_1 - N)}{N-1} > \dfrac{\alpha_1\alpha_2 - (N-1)^2}{N-1}, \end{cases}$$

then the system (1.1) has infinitely many positive radial entire solutions.
(ii) *If $\lambda_i,\ i = 1, 2$, satisfy*

$$\begin{cases} \lambda_1 - N + \dfrac{\alpha_1(\lambda_2 - N)}{N-1} < \dfrac{\alpha_1\alpha_2 - (N-1)^2}{N-1} & \textit{or} \\[2ex] \lambda_2 - N + \dfrac{\alpha_2(\lambda_1 - N)}{N-1} < \dfrac{\alpha_1\alpha_2 - (N-1)^2}{N-1}, \end{cases}$$

then the system (1.1) has no nonnegative nontrivial radial entire solutions.

Theorem 0.1 characterizes the decay rates of H_1 and H_2 for the system (1.1) to admit nonnegative nontrivial radial entire solutions. That is, under the assumption (1.2) the system (1.1) has a nonnegative nontrivial radial entire solution if and only if (1.3) holds.

Considering some results in [11], we conjecture that the conclusion (ii) of Theorem 0.2 is still true even if the condition for (λ_1, λ_2) is weakened to

$$\begin{cases} \lambda_1 - N + \dfrac{\alpha_1(\lambda_2 - N)}{N-1} \le \dfrac{\alpha_1\alpha_2 - (N-1)^2}{N-1} & \textit{or} \\[2ex] \lambda_2 - N + \dfrac{\alpha_2(\lambda_1 - N)}{N-1} \le \dfrac{\alpha_1\alpha_2 - (N-1)^2}{N-1}. \end{cases}$$

The aim of this paper is to extend Theorems 0.1 and 0.2 to the system (1.1) with $m \ge 3$ and to answer the conjecture mentioned above affirmatively.

For nonnegative functions $f_i,\ i = 1, 2$, there have been a great number of works on qualitative theory for solutions of the elliptic system

$$\begin{cases} -\Delta_{p_1} u_1 = f_1(x, u_1, u_2), \\ -\Delta_{p_2} u_2 = f_2(x, u_1, u_2), \end{cases} \quad x \in \mathbb{R}^N.$$

We can find in many works necessary and/or sufficient conditions for this system to have positive entire solutions with (or without) prescribed asymptotic forms near $+\infty$; see, e.g.[1, 8, 9] and references therein.

Let us introduce some notation used throughout this paper. Denote

$$A = \alpha_1\alpha_2\cdots\alpha_m$$

and

$$P = (p_1 - 1)(p_2 - 1)\cdots(p_m - 1).$$

It follows from these definitions that our assumption is written as $A > P$. For any sequence $\{s_1, s_2, \cdots, s_m\}$, we always make the agreement that $s_{m+j} = s_j,\ j = 1, 2, \cdots, m$, that is, the suffixes should be taken in the sense $\mathbb{Z}/m\mathbb{Z}$. For real constants $\lambda_1, \lambda_2, \cdots, \lambda_m$, we put

$$\begin{aligned}(1.4)\qquad \Lambda_i &= \lambda_i - p_i + \frac{(\lambda_{i+1} - p_{i+1})\alpha_i}{p_{i+1} - 1} + \frac{(\lambda_{i+2} - p_{i+2})\alpha_i\alpha_{i+1}}{(p_{i+1} - 1)(p_{i+2} - 1)} + \cdots \\ &\quad + \frac{(\lambda_{i+m-1} - p_{i+m-1})\alpha_i\alpha_{i+1}\cdots\alpha_{i+m-3}\alpha_{i+m-2}}{(p_{i+1} - 1)(p_{i+2} - 1)\cdots(p_{i+m-2} - 1)(p_{i+m-1} - 1)} \\ &= \lambda_i - p_i + \sum_{j=1}^{m-1}\left\{(\lambda_{i+j} - p_{i+j})\prod_{k=0}^{j-1}\frac{\alpha_{i+k}}{p_{i+1+k} - 1}\right\};\end{aligned}$$

and

$$(1.5)\qquad \beta_i = \frac{P\Lambda_i}{(A - P)(p_i - 1)},$$

$i = 1, 2, \cdots, m$. Since our assumptions imposed on $H_i,\ i = 1, 2, \cdots, m$, take the forms

$$\liminf_{|x|\to\infty} |x|^{\lambda_i} H_i(|x|) > 0$$

or

$$\limsup_{|x|\to\infty} |x|^{\lambda_i} H_i(|x|) < \infty,$$

all our results are formulated by means of the numbers $\lambda_i, \Lambda_i, \beta_i,\ i = 1, 2, \cdots, m$.

This paper is organized as follows. In Section 2, we consider the existence of positive radial entire solutions. In Section 3, we give estimates for nonnegative entire solutions of (1.1). In Section 4, we give nonexistence criteria of nonnegative nontrivial radial entire solutions of (1.1) based on the results in Section 3.

2 Existence results

In this section we consider the existence of positive radial entire solutions of (1.1).

We first observe that $(u_1, u_2, \cdots, u_m)$ is a positive radial entire solution of (1.1) if and only if the function $(v_1(r), v_2(r), \cdots, v_m(r)) = (u_1(|x|), u_2(|x|), \cdots, u_m(|x|))$, $r = |x|$, satisfies the system of second order ordinary differential equations

$$(2.1) \qquad \begin{cases} r^{1-N}(r^{N-1}|v_i'|^{p_i-2}v_i')' = H_i(r)v_{i+1}^{\alpha_i}, \quad r > 0, \\ v_i'(0) = 0, \end{cases} \qquad i = 1, 2, \cdots, m,$$

where $' = d/dr$. Furthermore, integrating (2.1) on $[0, r]$ twice, we obtain the system of integral equations equivalent to (2.1) :

(2.2)

$$v_i(r) = a_i + \int_0^r \left(s^{1-N} \int_0^s t^{N-1} H_i(t) v_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} ds, \quad r \geq 0, \ i = 1, 2, \cdots, m,$$

where $a_i = v_i(0)$. Therefore a positive radial entire solution of (1.1) can be obtained, under suitable conditions on H_i, by solving the system of integral equations (2.2).

Theorem 2.1 *Suppose that H_i, $i = 1, 2, \cdots, m$, satisfy*

$$(2.3) \qquad H_i(|x|) \leq \frac{C_i}{|x|^{\lambda_i}}, \quad |x| \geq r_0 > 0,$$

where $C_i > 0$ and λ_i, $i = 1, 2, \cdots, m$, are constants. Moreover, for these λ_i, Λ_i defined by (1.4) satisfy

$$\Lambda_i > \frac{A - P}{P} \max\{0, p_i - N\}, \quad i = 1, 2, \cdots, m.$$

Then (1.1) has infinitely many positive radial entire solutions.

Theorem 2.2 *Let $p_i \leq N$, $i = 1, 2, \cdots, m$. Suppose that H_i, $i = 1, 2, \cdots, m$, satisfy*

$$(2.4) \qquad H_i(|x|) \leq \frac{C_i}{|x|^{p_i}(\log |x|)^{\lambda_i}}, \quad |x| \geq r_0 > 1,$$

where $C_i > 0$ and λ_i, $i = 1, 2, \cdots, m$, are constants. Moreover

$$\Lambda_i > \frac{(A - P)(p_i - 1)}{P}, \quad i = 1, 2, \cdots, m.$$

Then (1.1) has infinitely many positive radial entire solutions.

Remark 2.1 (i) When $m = 2$, Theorem 2.1 reduces to Theorem 1 of [12].
(ii) When $p_i = 2,\ i = 1, 2, \cdots, m$, and $N \neq 2$, Theorem 2.1 reduces to Theorems 3.1 and 3.3 of [13].
(iii) When $p_i = N = 2,\ i = 1, 2, \cdots, m$, Theorem 2.2 reduces to Theorem 3.2 of [13].

Proof of Theorem 2.1. Without loss of generality, we may assume that $r_0 = 1$ in (2.3). Choose constants $a_i > 0,\ i = 1, 2, \cdots, m$, so that

$$(2.5)\quad \begin{cases} \left((2a_{i+1})^{\alpha_i} \displaystyle\int_0^1 H_i(s)ds\right)^{\frac{1}{p_i-1}} \leq \dfrac{a_i}{2}, \\ M_i \left(2(2a_{i+1})^{\alpha_i} \max\left\{\displaystyle\int_0^1 s^{N-1} H_i(s)ds,\ \dfrac{C_i}{N - \lambda_i + \alpha_i\beta_{i+1}}\right\}\right)^{\frac{1}{p_i-1}} \leq \dfrac{a_i}{2}, \end{cases}$$

where

$$M_i = \begin{cases} \dfrac{p_i - 1}{p_i - \lambda_i + \alpha_i\beta_{i+1}}, & p_i \leq N, \\ \dfrac{p_i - 1}{p_i - N}, & p_i > N, \end{cases}$$

and $\beta_i,\ i = 1, 2, \cdots, m$, are defined by (1.5). It is possible to choose such constants by the assumption $A > P$. From the definitions of β_i and Λ_i we can see that

$$\begin{aligned} & p_i - \lambda_i + \alpha_i\beta_{i+1} \\ = \;& p_i - \lambda_i + \frac{P\alpha_i}{(A-P)(p_{i+1}-1)} \left\{\lambda_{i+1} - p_{i+1} + \sum_{j=1}^{m-1} \left\{(\lambda_{i+1+j} - p_{i+1+j}) \prod_{k=0}^{j-1} \frac{\alpha_{i+1+k}}{p_{i+2+k} - 1}\right\}\right\} \\ = \;& p_i - \lambda_i + \frac{P}{A-P} \left\{\frac{\alpha_i(\lambda_{i+1} - p_{i+1})}{p_{i+1} - 1} + \sum_{j=1}^{m-1} \left\{(\lambda_{i+1+j} - p_{i+1+j}) \prod_{k=-1}^{j-1} \frac{\alpha_{i+1+k}}{p_{i+2+k} - 1}\right\}\right\} \\ = \;& p_i - \lambda_i + \frac{P}{A-P} \sum_{j=0}^{m-1} \left\{(\lambda_{i+1+j} - p_{i+1+j}) \prod_{k=-1}^{j-1} \frac{\alpha_{i+1+k}}{p_{i+2+k} - 1}\right\} \\ = \;& p_i - \lambda_i + \frac{P}{A-P} \left[\sum_{j=0}^{m-2} \left\{(\lambda_{i+1+j} - p_{i+1+j}) \prod_{k=-1}^{j-1} \frac{\alpha_{i+1+k}}{p_{i+2+k} - 1}\right\} + \frac{A}{P}(\lambda_i - p_i)\right] \\ = \;& \frac{P(\lambda_i - p_i)}{A-P} + \frac{P}{A-P} \sum_{j=1}^{m-1} \left\{(\lambda_{i+j} - p_{i+j}) \prod_{k=0}^{j-1} \frac{\alpha_{i+k}}{p_{i+1+k} - 1}\right\} \\ = \;& \frac{P\Lambda_i}{A-P} > \max\{0, p_i - N\}. \end{aligned}$$

Define the functions $F_i,\ i = 1, 2, \cdots, m$, by

$$F_i(r) = \begin{cases} 1, & 0 \le r \le 1, \\ r^{\beta_i}, & r \ge 1. \end{cases}$$

We regard the space $(C[0,\infty))^m$ as Fréchet space equipped with the topology of uniform convergence of functions on each compact subinterval of $[0,\infty)$. Let $X \subset (C[0,\infty))^m$ denotes the subset defined by

$$X = \{(v_1, v_2, \cdots, v_m) \in (C[0,\infty))^m; a_i \le v_i(r) \le 2a_iF_i(r),\ r \ge 0,\ 1 \le i \le m\}.$$

Clearly, X is a non-empty closed convex subset of $(C[0,\infty))^m$. Consider the mapping $\mathcal{F} : X \to (C[0,\infty))^m$ defined by $\mathcal{F}(v_1, v_2, \cdots, v_m) = (\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_m)$, where

$$\tilde{v}_i(r) = a_i + \int_0^r \left(s^{1-N}\int_0^s t^{N-1}H_i(t)v_{i+1}(t)^{\alpha_i}dt\right)^{\frac{1}{p_i-1}} ds, \quad r \ge 0, \quad i = 1, 2, \cdots, m.$$

In order to apply the Schauder-Tychonoff fixed point theorem, we will show that $\mathcal{F}$ is a continuous mapping from X into itself such that $\mathcal{F}(X)$ is relatively compact.

(I) *$\mathcal{F}$ maps X into itself.* Let $(v_1, v_2, \cdots, v_m) \in X$. Clearly, $\tilde{v}_i(r) \ge a_i,\ r \ge 0$. For $0 \le r \le 1$, we have

$$\begin{aligned} \tilde{v}_i(r) \le\ & a_i + \int_0^r \left(\int_0^s H_i(t)v_{i+1}(t)^{\alpha_i}dt\right)^{\frac{1}{p_i-1}} ds \\ \le\ & a_i + \int_0^1 \left(\int_0^1 H_i(t)(2a_{i+1}F_{i+1}(t))^{\alpha_i}dt\right)^{\frac{1}{p_i-1}} ds \\ =\ & a_i + \left((2a_{i+1})^{\alpha_i}\int_0^1 H_i(t)dt\right)^{\frac{1}{p_i-1}} \\ \le\ & a_i + \frac{a_i}{2} < 2a_i, \quad i = 1, 2, \cdots, m. \end{aligned}$$

For $r \ge 1$, we then write

$$\begin{aligned} \tilde{v}_i(r) =\ & a_i + \left(\int_0^1 + \int_1^r\right)\left(s^{1-N}\int_0^s t^{N-1}H_i(t)v_{i+1}(t)^{\alpha_i}dt\right)^{\frac{1}{p_i-1}} ds \\ \equiv\ & a_i + I_1 + I_2. \end{aligned}$$

A similar computation shows that $I_1 \le a_i/2,\ i = 1, 2, \cdots, m$. When $p_i \le N$, we see that

$$
\begin{aligned}
I_2 \le\ & \int_1^r \left(s^{1-N} \int_0^s t^{N-1} H_i(t) (2a_{i+1} F_{i+1}(t))^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} ds \\
\le\ & \int_1^r s^{\frac{1-N}{p_i-1}} \left((2a_{i+1})^{\alpha_i} \int_0^1 t^{N-1} H_i(t) dt + (2a_{i+1})^{\alpha_i} C_i \int_1^s t^{N-1-\lambda_i + a_i\beta_{i+1}} dt \right)^{\frac{1}{p_i-1}} ds \\
\le\ & \left(2(2a_{i+1})^{\alpha_i} \max \left\{ \int_0^1 t^{N-1} H_i(t) dt,\ \frac{C_i}{N - \lambda_i + \alpha_i\beta_{i+1}} \right\} \right)^{\frac{1}{p_i-1}} \int_1^r s^{\frac{1-\lambda_i+\alpha_i\beta_{i+1}}{p_i-1}} ds \\
\le\ & M_i \left(2(2a_{i+1})^{\alpha_i} \max \left\{ \int_0^1 t^{N-1} H_i(t) dt,\ \frac{C_i}{N - \lambda_i + \alpha_i\beta_{i+1}} \right\} \right)^{\frac{1}{p_i-1}} r^{\frac{p_i-\lambda_i+\alpha_i\beta_{i+1}}{p_i-1}} \\
\le\ & \frac{a_i}{2} r^{\frac{p_i-\lambda_i+\alpha_i\beta_{i+1}}{p_i-1}} = \frac{a_i}{2} r^{\beta_i}.
\end{aligned}
$$

When $p_i > N$, we see that

$$
\begin{aligned}
I_2 \le\ & \int_1^r s^{\frac{1-N}{p_i-1}} ds \left(\int_0^r t^{N-1} H_i(t) (2a_{i+1} F_{i+1}(t))^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} \\
\le\ & M_i r^{\frac{p_i-N}{p_i-1}} \left((2a_{i+1})^{\alpha_i} \int_0^1 t^{N-1} H_i(t) dt + (2a_{i+1})^{\alpha_i} C_i \int_1^r t^{N-1-\lambda_i+\alpha_i\beta_{i+1}} dt \right)^{\frac{1}{p_i-1}} \\
\le\ & M_i r^{\frac{p_i-N}{p_i-1}} \left(2(2a_{i+1})^{\alpha_i} \max \left\{ \int_0^1 t^{N-1} H_i(t) dt, \frac{C_i}{N - \lambda_i + \alpha_i\beta_{i+1}} \right\} r^{N-\lambda_i+\alpha_i\beta_{i+1}} \right)^{\frac{1}{p_i-1}} \\
\le\ & \frac{a_i}{2} r^{\frac{p_i-\lambda_i+\alpha_i\beta_{i+1}}{p_i-1}} = \frac{a_i}{2} r^{\beta_i}.
\end{aligned}
$$

Thus we obtain

$$
\tilde{v}_i(r) \le \frac{3}{2} a_i + \frac{a_i}{2} r^{\beta_i} \le 2a_i r^{\beta_i}, \quad r \ge 1, \quad i = 1, 2, \cdots, m.
$$

Therefore, $\mathcal{F}(X) \subset X$.

(II) $\mathcal{F}$ *is continuous.* Let $\{(v_{1,l}, v_{2,l}, \cdots, v_{m,l})\}_{l=1}^{\infty}$ be a sequence in X which converges to $(v_1, v_2, \cdots, v_m) \in X$ uniformly on each compact subinterval of $[0, \infty)$. We put

$$
\phi_{i,l}(r) = r^{1-N} \int_0^r s^{N-1} H_i(s) v_{i+1,l}(s)^{\alpha_i} ds
$$

and

$$
\phi_i(r) = r^{1-N} \int_0^r s^{N-1} H_i(s) v_{i+1}(s)^{\alpha_i} ds.
$$

Then we have

$$|\phi_{i,l}(r)-\phi_i(r)| \le \int_0^r H_i(s)|v_{i+1,l}(s)^{\alpha_i}-v_{i+1}(s)^{\alpha_i}|ds.$$

Let $R>0$ be an arbitrary constant. Since $\{v_{i,l}\}_{l=1}^{\infty}$, $i=1,2,\cdots,m$, converge to v_i uniformly on $[0,R]$, it follows that $\{\phi_{i,l}\}_{l=1}^{\infty}$, $i=1,2,\cdots,m$, converge to ϕ_i uniformly on $[0,R]$; and hence $\{\phi_{i,l}^{\frac{1}{p_i-1}}\}_{l=1}^{\infty}$, $i=1,2,\cdots,m$, converge to $\phi_i^{\frac{1}{p_i-1}}$ uniformly on $[0,R]$. From this fact and

$$|\tilde{v}_{i,l}(r)-\tilde{v}_i(r)| \le \int_0^r \left|\phi_{i,l}(s)^{\frac{1}{p_i-1}}-\phi_i(s)^{\frac{1}{p_i-1}}\right| ds,$$

we can see that $\{\tilde{v}_{i,l}\}_{l=1}^{\infty}$, $i=1,2,\cdots,m$, converge to $\tilde{v}_i$ uniformly on $[0,R]$. These imply that $\{\tilde{v}_{i,l}\}_{l=1}^{\infty}$, $i=1,2,\cdots,m$, converge to $\tilde{v}_i$ uniformly on each compact subinterval of $[0,\infty)$. Therefore $\mathcal{F}$ is continuous.

(Ⅲ) $\mathcal{F}(X)$ *is relatively compact.* It is sufficient to verify the local equicontinuity of $\mathcal{F}(X)$, since $\mathcal{F}(X)$ is locally uniformly bounded by the fact that $\mathcal{F}(X)\subset X$. Let $(v_1,v_2,\cdots,v_m)\in X$ and $R>0$. Then we have

$$\begin{aligned}\tilde{v}_i'(r) = &\left(\int_0^r \left(\frac{s}{r}\right)^{N-1} H_i(s)v_{i+1}(s)^{\alpha_i}ds\right)^{\frac{1}{p_i-1}}\\ \le &\left(\int_0^R H_i(s)(2a_{i+1}F_{i+1}(s))^{\alpha_i}ds\right)^{\frac{1}{p_i-1}} < \infty, \quad i=1,2,\cdots,m.\end{aligned}$$

Obviously, these imply the local boundedness of the set $\{(\tilde{v}_1',\tilde{v}_2',\cdots,\tilde{v}_m')|(v_1,v_2,\cdots,v_m)\in X\}$. Hence the relative compactness of $\mathcal{F}(X)$ is shown by the Ascoli-Arzelà theorem.

Therefore, there exists an element $(v_1,v_2,\cdots,v_m)\in X$ such that $(v_1,v_2,\cdots,v_m)=\mathcal{F}(v_1,v_2,\cdots,v_m)$ by the Schauder-Tychonoff fixed point theorem, that is, $(v_1,v_2,\cdots,v_m)$ satisfies the system of integral equations (2.2). The function $(u_1(x),u_2(x),\cdots,u_m(x))=(v_1(|x|),v_2(|x|),\cdots,v_m(|x|))$ then gives a solution of (1.1). Since infinitely many $(a_1,a_2,\cdots,a_m)$ satisfy (2.5), we can construct an infinitude of positive radial entire solutions of (1.1). This completes the proof.□

Proof of Theorem 2.2. Without loss of generality, we may assume that $r_0=e$ in (2.4). Take constants $a_i>0$, $i=1,2,\cdots,m$, so that

$$\begin{cases} e\left((2a_{i+1})^{\alpha_i}\displaystyle\int_0^e H_i(t)dt\right)^{\frac{1}{p_i-1}} \le \dfrac{a_i}{2},\\ \left(2(2a_{i+1})^{\alpha_i}\max\left\{\displaystyle\int_0^e t^{p_i-1}H_i(t)dt,\ \dfrac{C_i}{1-\lambda_i+\alpha_i\beta_{i+1}}\right\}\right)^{\frac{1}{p_i-1}} \le \dfrac{a_i}{2}.\end{cases}$$

It is possible to take such constants by the assumption $A > P$.

Define the functions $F_i,\ i = 1, 2, \cdots, m$, by

$$F_i(r) = \begin{cases} 1, & 0 \le r \le e, \\ (\log r)^{\beta_i}, & r \ge e. \end{cases}$$

Consider the set

$$Y = \{(v_1, v_2, \cdots, v_m) \in (C[0,\infty))^m; a_i \le v_i(r) \le 2a_iF_i(r),\ r \ge 0,\ 1 \le i \le m\}$$

and the mapping $\mathcal{F} : Y \to (C[0,\infty))^m$ defined by $\mathcal{F}(v_1, v_2, \cdots, v_m) = (\tilde{v}_1, \tilde{v}_2, \cdots, \tilde{v}_m)$, where

$$\tilde{v}_i(r) = a_i + \int_0^r \left(s^{1-N} \int_0^s t^{N-1} H_i(t) v_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i - 1}} ds.$$

Obviously, the set Y is closed convex subset of Fréchet space $(C[0,\infty))^m$. We first show that $\mathcal{F}(Y) \subset Y$. Let $(v_1, v_2, \cdots, v_m) \in Y$. Clearly, $\tilde{v}_i(r) \ge a_i,\ r \ge 0$. For $0 \le r \le e$ we have

$$\begin{aligned} \tilde{v}_i(r) \le &\ a_i + \int_0^r \left(\int_0^s H_i(t) v_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} ds \\ \le &\ a_i + \int_0^e \left(\int_0^e H_i(t) (2a_{i+1}F_{i+1}(t))^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} ds \\ = &\ a_i + e \left((2a_{i+1})^{\alpha_i} \int_0^e H_i(t) dt \right)^{\frac{1}{p_i-1}} \\ \le &\ a_i + \frac{a_i}{2} < 2a_i, \quad i = 1, 2, \cdots, m. \end{aligned}$$

For $r \ge e$, we then write

$$\begin{aligned} \tilde{v}_i(r) = &\ a_i + \left(\int_0^e + \int_e^r \right) \left(s^{1-N} \int_0^s t^{N-1} H_i(t) v_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i - l}} ds \\ \equiv &\ a_i + I_1 + I_2. \end{aligned}$$

A similar computation shows that $I_1 \le a_i/2,\ i = 1, 2, \cdots, m$. The integral I_2 is estimated as follows:

$$\begin{aligned}
I_2 \le & \int_e^r \left(s^{1-p_i} \int_0^s t^{p_i-1} H_i(t) v_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} ds \\
\le & \int_e^r s^{-1} ds \left(\int_0^r t^{p_i-1} H_i(t) v_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} \\
\le & \left((2a_{i+1})^{\alpha_i} \int_0^e t^{p_i-1} H_i(t) dt + (2a_{i+1})^{\alpha_i} C_i \int_e^r t^{-1} (\log t)^{-\lambda_i + \alpha_i \beta_{i+1}} dt \right)^{\frac{1}{p_i-1}} \log r \\
\le & \left(2(2a_{i+1})^{\alpha_i} \max \left\{ \int_0^e t^{p_i-1} H_i(t) dt, \frac{C_i}{1-\lambda_i + \alpha_i \beta_{i+1}} \right\} (\log r)^{1-\lambda_i+\alpha_i\beta_{i+1}} \right)^{\frac{1}{p_i-1}} \log r \\
\le & \frac{a_i}{2} (\log r)^{\frac{p_i - \lambda_i + \alpha_i \beta_{i+1}}{p_i - 1}} = \frac{a_i}{2} (\log r)^{\beta_i}.
\end{aligned}$$

Thus we obtain

$$\tilde{v}_i(r) \le \frac{3}{2} a_i + \frac{a_i}{2} (\log r)^{\beta_i} \le 2a_i (\log r)^{\beta_i}, \quad r \ge e, \quad i = 1, 2, \cdots, m.$$

Therefore, $\mathcal{F}(v_1, v_2, \cdots, v_m) \in Y$.

The continuity of $\mathcal{F}$ and the relative compactness of $\mathcal{F}(Y)$ can be verified without difficulty, and so by the Schauder-Tychonoff fixed point theorem there exists $(v_1, v_2 \cdots, v_m) \in Y$ such that $(v_1, v_2, \cdots, v_m) = \mathcal{F}(v_1, v_2, \cdots, v_m)$. It is clear that this fixed point $(v_1, v_2, \cdots, v_m)$ gives rise to a positive radial entire solution of (1.1). The proof is finished.□

3 Growth estimates for nonnegative entire solutions

In this section we consider estimates for nonnegative radial entire solutions of (1.1) which will play an important role to prove nonexistence theorems for nonnegative nontrivial radial entire solutions.

Theorem 3.1 *Suppose that H_i, $i = 1, 2, \cdots, m$, satisfy*

(3.1) $$H_i(|x|) \ge \frac{C_i}{|x|^{\lambda_i}}, \quad |x| \ge r_0 > 0,$$

where $C_i > 0$ and λ_i are constants. Let $(u_1, u_2, \cdots, u_m)$ be a nonnegative radial entire solution of (1.1). Then u_i, $i = 1, 2, \cdots, m$, satisfy

(3.2) $$u_i(r) \le \tilde{C}_i r^{\beta_i} \quad \text{at} \quad \infty,$$

where $\tilde{C}_i > 0,\ i = 1, 2, \cdots, m,$ are constants and $\beta_i,\ i = 1, 2, \cdots, m,$ are defined by (1.5).

Theorem 3.2 *Let $p_i = N,\ i = 1, 2, \cdots, m$. Suppose that $H_i,\ i = 1, 2, \cdots, m,$ satisfy*

$$H_i(|x|) \geq \frac{C_i}{|x|^N(\log|x|)^{\lambda_i}}, \quad |x| \geq r_0 > 1, \tag{3.3}$$

where $C_i > 0$ and λ_i are constants. Let $(u_1, u_2, \cdots, u_m)$ be a nonnegative radial entire solution of (1.1). Then $u_i,\ i = 1, 2, \cdots, m,$ satisfy

$$u_i(r) \leq \tilde{C}_i(\log r)^{\beta_i} \quad \text{at} \quad \infty, \tag{3.4}$$

where $\tilde{C}_i > 0,\ i = 1, 2, \cdots, m,$ are constants, and $\beta_i,\ i = 1, 2, \cdots, m,$ are defined by (1.5).

Proof of Theorem 3.1. Let $(u_1, u_2, \cdots, u_m)$ be a nonnegative radial entire solution of (1.1). We may assume that $(u_1, u_2, \cdots, u_m) \not\equiv (0, 0, \cdots, 0)$. Then $(u_1, u_2, \cdots, u_m)$ satisfies the following system of ordinary differential equations

$$\begin{cases} (r^{N-1}|u_i'(r)|^{p_i-2}u_i'(r))' = r^{N-1}H_i(r)u_{i+1}(r)^{\alpha_i}, \quad r > 0, \\ u_i'(0) = 0, \end{cases} \qquad i = 1, 2, \cdots, m. \tag{3.5}$$

Integrating (3.5) over $[0, r]$, we have

$$r^{N-1}|u_i'(r)|^{p_i-2}u_i'(r) = \int_0^r s^{N-1}H_i(s)u_{i+1}(s)^{\alpha_i}ds, \quad i = 1, 2, \cdots, m.$$

Hence, we see that $u_i'(r) \geq 0$ for $r \geq 0$. Integrating (3.5) twice over $[R, r],\ R \geq 0$, we have

$$u_i(r) \geq u_i(R) + \int_R^r \left(s^{1-N}\int_R^s t^{N-1}H_i(t)u_{i+1}(t)^{\alpha_i}dt\right)^{\frac{1}{p_i-1}} ds, \quad i = 1, 2, \cdots, m. \tag{3.6}$$

Since $u_i,\ i = 1, 2, \cdots, m$, are nonnegative and nontrivial, there exists a point $x_* \in \mathbb{R}^N$ such that $u_{i_0}(r_*) > 0,\ r_* = |x_*|$ for some $i_0 \in \{1, 2, \cdots, m\}$. We may assume that $r_* \geq r_0$. Therefore we see from (3.6) with $R = r_*$ that $u_i(r) > 0$ for $r > r_*,\ i = 1, 2, \cdots, m$.

Let us fix $R > r_*$ arbitrarily. Using (3.1) and the inequality

$$\left(\frac{t}{s}\right)^{N-1} \geq \left(\frac{1}{3}\right)^{N-1}, \quad R \leq t \leq s \leq 3R$$

in (3.6), we have

$$\begin{aligned} u_i(r) &\geq u_i(R) + \int_R^r \left(\int_R^s \left(\frac{1}{3}\right)^{N-1} C_i t^{-\lambda_i} u_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} ds \\ &\geq \tilde{C}_i R^{-\frac{\lambda_i}{p_i-1}} \int_R^r \left(\int_R^s u_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} ds, \quad R \leq r \leq 3R, \end{aligned}$$

where $\tilde{C}_i > 0,\ i = 1, 2, \cdots, m$, are some constants independent of r and R. From now on, we use C to denote various positive constants independent of r and R as we will have no confusion. Put

$$f_i(r) = \tilde{C}_i R^{-\frac{\lambda_i}{p_i-1}} \int_R^r \left(\int_R^s u_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} ds, \quad R \leq r \leq 3R. \tag{3.7}$$

Clearly, $f_i,\ i = 1, 2, \cdots, m$, satisfy

$$u_i(r) \geq f_i(r), \quad R \leq r \leq 3R,$$

$$f_i(R) = f_i'(R) = 0,$$

$$f_i'(r) = \tilde{C}_i R^{-\frac{\lambda_i}{p_i-1}} \left(\int_R^r u_{i+1}(s)^{\alpha_i} ds \right)^{\frac{1}{p_i-1}} \geq 0, \quad R \leq r \leq 3R,$$

$$f_i''(r) > 0, \quad R < r \leq 3R,$$

and

$$\begin{aligned} (f_i'(r)^{p_i-1})' &= C R^{-\lambda_i} u_{i+1}(r)^{\alpha_i} \\ &\geq C R^{-\lambda_i} f_{i+1}(r)^{\alpha_i}, \quad R \leq r \leq 3R. \end{aligned} \tag{3.8}$$

From (3.7) and the monotonicity of u_i, we see that

$$f_i(r) \geq C R^{-\frac{\lambda_i}{p_i-1}} u_{i+1}(R)^{\frac{\alpha_i}{p_i-1}} (r-R)^{\frac{p_i}{p_i-1}}, \quad R \leq r \leq 3R. \tag{3.9}$$

Let us fix $i \in \{1, 2, \cdots, m\}$. Multiplying (3.8) by $f_{i+1}'(r) \geq 0$ and integrating by parts the resulting inequality on $[R+\varepsilon, r],\ \varepsilon > 0$, we have

$$f_{i+1}'(r) f_i'(r)^{p_i-1} \geq C R^{-\lambda_i} (f_{i+1}(r)^{\alpha_i+1} - f_{i+1}(R+\varepsilon)^{\alpha_i+1}), \quad R+\varepsilon \leq r \leq 3R.$$

Letting $\varepsilon \to 0$, we get

$$f_i'(r) f_{i+1}'(r)^{\frac{1}{p_i-1}} \geq C R^{-\frac{\lambda_i}{p_i-1}} f_{i+1}(r)^{\frac{\alpha_i+1}{p_i-1}}, \quad R \leq r \leq 3R.$$

Multiplying this inequality by f_{i+1}' and integrating by parts on $[R+\varepsilon, r]$ and letting $\varepsilon \to 0$, we obtain

$$f_i(r) f_{i+1}'(r)^{\frac{p_i}{p_i-1}} \geq C R^{-\frac{\lambda_i}{p_i-1}} f_{i+1}(r)^{\frac{\alpha_i+p_i}{p_i-1}}, \quad R \leq r \leq 3R.$$

From (3.8), we have

$$(f_{i-1}'(r)^{p_{i-1}-1})' f_{i+1}'(r)^{\frac{p_i \alpha_{i-1}}{p_i-1}} \geq C R^{-\frac{\lambda_i \alpha_{i-1}}{p_i-1} - \lambda_{i-1}} f_{i+1}(r)^{\frac{(\alpha_i+p_i)\alpha_{i-1}}{p_i-1}}, \quad R \leq r \leq 3R.$$

Again, multiplying this relation by f_{i+1}' and integrating by parts on $[R+\varepsilon, r]$ and letting $\varepsilon \to 0$ twice, we get

$$\begin{aligned} & f_{i-1}(r) f_{i+1}'(r)^{\frac{p_i \alpha_{i-1}}{(p_i-1)(p_{i-1}-1)} + \frac{p_{i-1}}{p_{i-1}-1}} \\ \geq \quad & C R^{-\frac{\lambda_i \alpha_{i-1}}{(p_i-1)(p_{i-1}-1)} - \frac{\lambda_{i-1}}{p_{i-1}-1}} f_{i+1}(r)^{\frac{(\alpha_i+p_i)\alpha_{i-1}}{(p_i-1)(p_{i-1}-1)} + \frac{p_{i-1}}{p_{i-1}-1}}, \quad R \leq r \leq 3R. \end{aligned}$$

From (3.8), we obtain

$$\begin{aligned} & (f_{i-2}'(r)^{p_{i-2}-1})' f_{i+1}'(r)^{\frac{p_i \alpha_{i-1} \alpha_{i-2}}{(p_i-1)(p_{i-1}-1)} + \frac{p_{i-1}\alpha_{i-2}}{p_{i-1}-1}} \\ \geq \quad & C R^{-\frac{\lambda_i \alpha_{i-1}\alpha_{i-2}}{(p_i-1)(p_{i-1}-1)} - \frac{\lambda_{i-1}\alpha_{i-2}}{p_{i-1}-1} - \lambda_{i-2}} f_{i+1}(r)^{\frac{(\alpha_i+p_i)\alpha_{i-1}\alpha_{i-2}}{(p_i-1)(p_{i-1}-1)} + \frac{p_{i-1}\alpha_{i-2}}{p_{i-1}-1}}, \quad R \leq r \leq 3R. \end{aligned}$$

By repeating this procedure we get

$$\begin{aligned} (3.10)\ & (f_{i-(m-1)}'(r)^{p_{i-(m-1)}-1})' f_{i+1}'(r)^{K_i} \\ & = (f_{i+1}'(r)^{p_{i+1}-1})' f_{i+1}'(r)^{K_i} \geq C R^{-L_i} f_{i+1}(r)^{M_i}, \quad R \leq r \leq 3R, \end{aligned}$$

where

$$K_i = \sum_{j=1}^{m-1} \left\{ p_{i-(j-1)} \prod_{k=j}^{m-1} \frac{\alpha_{i-k}}{p_{i+1-k}-1} \right\},$$

$$L_i = \sum_{j=1}^{m-1} \left\{ \lambda_{i-(j-1)} \prod_{k=j}^{m-1} \frac{\alpha_{i-k}}{p_{i+1-k}-1} \right\} + \lambda_{i+1},$$

and

$$\begin{aligned} M_i = & \frac{A}{\prod_{j=0}^{m-2}(p_{i-j}-1)} + \sum_{j=1}^{m-1}\left\{p_{i-(j-1)}\prod_{k=j}^{m-1}\frac{\alpha_{i-k}}{p_{i+1-k}-1}\right\} \\ = & \frac{A(p_{i+1}-1)}{P} + K_i. \end{aligned}$$

Multiplying (3.10) by $f'_{i+1}(r) \geq 0$ and integrating by parts on $[R+\varepsilon, r]$ and letting $\varepsilon \to 0$, we have

$$f'_{i+1}(r)f_{i+1}(r)^{-\frac{M_i+1}{K_i+p_{i+1}}} \geq CR^{-\frac{L_i}{K_i+p_{i+1}}}, \quad R < r \leq 3R. \tag{3.11}$$

Since $(M_i+1)/(K_i+p_{i+1}) > 1$, we can set

$$\frac{M_i+1}{K_i+p_{i+1}} = \delta_i + 1, \ \delta_i = \frac{(A-P)(p_{i+1}-1)}{(K_i+p_{i+1})P}.$$

Integrating (3.11) on $[2R, 3R]$, we get

$$f_{i+1}(2R)^{-\delta_i} \geq CR^{-\frac{L_i}{K_i+p_{i+1}}+1}.$$

From (3.9) with $r = 2R$ and this inequality, we have

$$u_{i+2}(R) \leq CR^{\tau_i},$$

where

$$\tau_i = \frac{p_{i+1}-1}{\alpha_{i+1}\delta_i}\left\{\frac{L_i}{K_i+p_{i+1}} - 1 + \frac{(\lambda_{i+1}-p_{i+1})\delta_i}{p_{i+1}-1}\right\}.$$

From the definitions of K_i, L_i and δ_i, we see that

$$\begin{aligned} \tau_i = & \frac{p_{i+1}-1}{\alpha_{i+1}\delta_i(K_i+p_{i+1})}\left[L_i - K_i - p_{i+1} + \frac{(A-P)(\lambda_{i+1}-p_{i+1})}{P}\right] \\ = & \frac{P}{\alpha_{i+1}(A-P)}\left[\sum_{j=1}^{m-1}\left\{(\lambda_{i-j+1}-p_{i-j+1})\prod_{k=j}^{m-1}\frac{\alpha_{i-k}}{p_{i+1-k}-1}\right\} + \frac{A(\lambda_{i+1}-p_{i+1})}{P}\right] \\ = & \frac{P}{\alpha_{i+1}(A-P)}\left[\sum_{j=0}^{m-2}\left\{(\lambda_{i-j+1}-p_{i-j+1})\prod_{k=j}^{m-1}\frac{\alpha_{i-k}}{p_{i+1-k}-1}\right\} + \frac{(\lambda_{i+2}-p_{i+2})\alpha_{i+1}}{p_{i+2}-1}\right] \end{aligned}$$

$$
\begin{aligned}
&= \frac{P}{(A-P)(p_{i+2}-1)}\left[\sum_{j=0}^{m-2}\left\{(\lambda_{i-j+1}-p_{i-j+1})\prod_{k=j}^{m-2}\frac{\alpha_{i-k}}{p_{i+1-k}-1}\right\}+\lambda_{i+2}-p_{i+2}\right] \\
&= \frac{P}{(A-P)(p_{i+2}-1)}\left[\frac{(\lambda_{i+1}-p_{i+1})\alpha_i\alpha_{i-1}\cdots\alpha_{i-m+2}}{(p_{i+1}-1)(p_i-1)\cdots(p_{i-m+3}-1)}\right. \\
&\quad +\frac{(\lambda_i-p_i)\alpha_{i-1}\alpha_{i-2}\cdots\alpha_{i-m+2}}{(p_i-1)(p_{i-1}-1)\cdots(p_{i-m+3}-1)}+\cdots+\frac{(\lambda_{i-m+4}-p_{i-m+4})\alpha_{i-m+3}\alpha_{i-m+2}}{(p_{i-m+4}-1)(p_{i-m+3}-1)} \\
&\quad \left.+\frac{(\lambda_{i-m+3}-p_{i-m+3})\alpha_{i-m+2}}{p_{i-m+3}-1}+\lambda_{i+2}-p_{i+2}\right] \\
&= \frac{P}{(A-P)(p_{i+2}-1)}\left[\lambda_{i+2}-p_{i+2}+\sum_{j=1}^{m-1}\left\{(\lambda_{i+2+j}-p_{i+2+j})\prod_{k=0}^{j-1}\frac{\alpha_{i+2+k}}{p_{i+3+k}-1}\right\}\right] \\
&= \frac{P\Lambda_{i+2}}{(A-P)(p_{i+2}-1)}.
\end{aligned}
$$

Therefore we obtain (3.2) by the definition of β_i. Thus the proof is completed.□

The next lemma is needed in proving Theorem 3.2.

Lemma 3.3 *Let $p_i = N$, $i = 1, 2, \cdots, m$, and $(u_1, u_2, \cdots, u_m)$ be a nonnegative radial entire solution of (1.1). Then u_i, $i = 1, 2, \cdots, m$, satisfy*

$$u_i(r) \geq u_i(0) + \left(\int_0^r s^{N-1}H_i(s)\left(\log\left(\frac{r}{s}\right)\right)^{N-1} u_{i+1}(s)^{\alpha_i}ds\right)^{\frac{1}{N-1}}, \quad r \geq 0.$$

Proof. Let $(u_1, u_2, \cdots, u_m)$ be a nonnegative radial entire solution of (1.1). Then u_i, $i = 1, 2, \cdots, m$, satisfy the following system of ordinary differential equations

$$\begin{cases} (r^{N-1}|u_i'(r)|^{N-2}u_i'(r))' = r^{N-1}H_i(r)u_{i+1}(r)^{\alpha_i}, & r > 0, \\ u_i'(0) = 0, \end{cases} \quad i = 1, 2, \cdots, m.$$

Integrating these equations on $[0, r]$ twice, we have

$$
\begin{aligned}
u_i(r) = &\ u_i(0) + \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} H_i(t)u_{i+1}(t)^{\alpha_i}dt\right)^{\frac{1}{N-1}} ds \\
= &\ u_i(0) + \int_0^r \left(\int_0^r \Phi_i(s,t)dt\right)^{\frac{1}{N-1}} ds, \quad r \geq 0,
\end{aligned}
$$

where

$$\Phi_i(s,t) = \begin{cases} s^{1-N} t^{N-1} H_i(t) u_{i+1}(t)^{\alpha_i} & \text{for} \quad 0 \le t \le s, \\ 0 & \text{for} \quad t > s. \end{cases}$$

Using Minkowski's inequality (cf. [4, p.148]), we see that

$$\int_0^r \left(\int_0^r \Phi_i(s,t) dt \right)^{\frac{1}{N-1}} ds \ge \left(\int_0^r \left(\int_0^r \Phi_i(s,t)^{\frac{1}{N-1}} ds \right)^{N-1} dt \right)^{\frac{1}{N-1}}, \quad r \ge 0.$$

Then we have

$$\begin{aligned} u_i(r) \ge \;& u_i(0) + \left(\int_0^r \left(\int_0^r \Phi_i(s,t)^{\frac{1}{N-1}} ds \right)^{N-1} dt \right)^{\frac{1}{N-1}} \\ = \;& u_i(0) + \left(\int_0^r \left(\int_t^r s^{-1} t H_i(t)^{\frac{1}{N-1}} u_{i+1}(t)^{\frac{\alpha_i}{N-1}} ds \right)^{N-1} dt \right)^{\frac{1}{N-1}} \\ = \;& u_i(0) + \left(\int_0^r t^{N-1} H_i(t) \left(\log \frac{r}{t} \right)^{N-1} u_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{N-1}}. \end{aligned}$$

Thus the proof is finished.□

Proof of Theorem 3.2. Let $(u_1, u_2 \cdots, u_m)$ be a nonnegative radial entire solution of (1.1). We may assume that $(u_1, u_2 \cdots, u_m) \not\equiv (0, 0, \cdots, 0)$. As in the proof of Theorem 3.1 we see that $u_i(r) > 0, \ r \ge r_*, \ i = 1, 2, \cdots, m$, for some $r_* > r_0$.

Let us fix $R \ge r_*$ arbitrarily. From Lemma 3.3, we see that $u_i, \ i = 1, 2, \cdots, m$, satisfy

$$\text{(3.12)} \quad \begin{aligned} u_i(r) \ge\;& u_i(0) + \left(\int_0^r s^{N-1} H_i(s) \left(\log \frac{r}{s} \right)^{N-1} u_{i+1}(s)^{\alpha_i} ds \right)^{\frac{1}{N-1}} \\ \ge\;& \left(\int_{e^R}^r s^{N-1} H_i(s) (\log r - \log s)^{N-1} u_{i+1}(s)^{\alpha_i} ds \right)^{\frac{1}{N-1}}, \quad r \ge e^R. \end{aligned}$$

Let $\log s = t, \ \log r = \rho$. Then (3.12) becomes

$$u_i(e^\rho) \ge \left(\int_R^\rho e^{Nt} H_i(e^t) (\rho - t)^{N-1} u_{i+1}(e^t)^{\alpha_i} dt \right)^{\frac{1}{N-1}}, \quad \rho \ge R, \quad i = 1, 2, \cdots, m.$$

Now we discuss only on the interval $[R, 3R]$ for a moment. Let $R \leq \rho \leq 3R$. Then, from (3.3), we have

$$\begin{aligned} u_i(e^\rho) \geq & \left(C_i \int_R^\rho t^{-\lambda_i} (\rho - t)^{N-1} u_{i+1}(e^t)^{\alpha_i} dt \right)^{\frac{1}{N-1}} \\ \geq & \left(\tilde{C}_i R^{-\lambda_i} \int_R^\rho (\rho - t)^{N-1} u_{i+1}(e^t)^{\alpha_i} dt \right)^{\frac{1}{N-1}}, \quad R \leq \rho \leq 3R, \end{aligned}$$

where $\tilde{C}_i > 0$ are some constants independent of r and R. From now on we use the same letter C to denote various positive constants.

Define the functions $f_i,\ i = 1, 2, \cdots, m$, by

$$f_i(\rho) = \tilde{C}_i R^{-\lambda_i} \int_R^\rho (\rho - t)^{N-1} u_{i+1}(e^t)^{\alpha_i} dt, \quad R \leq \rho \leq 3R. \tag{3.13}$$

Then we see that $f_i,\ i = 1, 2, \cdots, m$, are of class $C^N[R, 3R]$ and satisfy

$$u_i(e^\rho) \geq f_i(\rho)^{\frac{1}{N-1}}, \quad R \leq \rho \leq 3R,$$

$$f_i^{(k)}(r) \geq 0, \quad R \leq \rho \leq 3R, \quad f_i^{(k)}(R) = 0, \quad k = 0, 1, 2, \cdots, N-1,$$

and

$$\begin{aligned} f_i^{(N)}(\rho) &= CR^{-\lambda_i} u_{i+1}(e^\rho)^{\alpha_i} \\ &\geq CR^{-\lambda_i} f_{i+1}(\rho)^{\frac{\alpha_i}{N-1}}, \quad R \leq \rho \leq 3R. \end{aligned} \tag{3.14}$$

From (3.13) and the monotonicity of u_i we have

$$\begin{aligned} f_i(\rho) &\geq CR^{-\lambda_i} u_{i+1}(e^R)^{\alpha_i} \int_R^\rho (\rho - t)^{N-1} dt \\ &\geq CR^{-\lambda_i} (\rho - R)^N u_{i+1}(e^R)^{\alpha_i}, \quad R \leq \rho \leq 3R. \end{aligned} \tag{3.15}$$

Let us fix $i \in \{1, 2, \cdots, m\}$. Multiplying (3.14) by f'_{i+1} and integrating by parts the resulting inequality on $[R, \rho]$, we have

$$f_i^{(N-1)}(\rho) f'_{i+1}(\rho) \geq CR^{-\lambda_i} f_{i+1}(\rho)^{\frac{\alpha_i}{N-1}+1}, \quad R \leq \rho \leq 3R.$$

By repeating this process $(N-1)$ times, we get

$$f_i(\rho) f'_{i+1}(\rho)^N \geq CR^{-\lambda_i} f_{i+1}(\rho)^{\frac{\alpha_i}{N-1}+N}, \quad R \leq \rho \leq 3R.$$

From (3.14) we have

$$f_{i-1}^{(N)}(\rho)f_{i+1}'(\rho)^{\frac{N\alpha_{i-1}}{N-1}} \geq CR^{-\frac{\lambda_i\alpha_{i-1}}{N-1}-\lambda_{i-1}} f_{i+1}(\rho)^{\frac{\alpha_i\alpha_{i-1}}{(N-1)^2}+\frac{N\alpha_{i-1}}{N-1}}, \quad R \leq \rho \leq 3R.$$

Multiplying this inequality by f_{i+1}' and integrating by parts N times on $[R,\rho]$, we have

$$f_{i-1}(\rho)f_{i+1}'(\rho)^{\frac{N\alpha_{i-1}}{N-1}+N} \geq CR^{-\frac{\lambda_i\alpha_{i-1}}{N-1}-\lambda_{i-1}} f_{i+1}(\rho)^{\frac{\alpha_i\alpha_{i-1}}{(N-1)^2}+\frac{N\alpha_{i-1}}{N-1}+N}, \quad R \leq \rho \leq 3R.$$

From (3.14) we have

$$\begin{aligned} & f_{i-2}^{(N)}(\rho)f_{i+1}'(\rho)^{\frac{N\alpha_{i-1}\alpha_{i-2}}{(N-1)^2}+\frac{N\alpha_{i-2}}{N-1}} \\ \geq\ & CR^{-\frac{\lambda_i\alpha_{i-1}\alpha_{i-2}}{(N-1)^2}-\frac{\lambda_{i-1}\alpha_{i-2}}{N-1}-\lambda_{i-2}} f_{i+1}(\rho)^{\frac{\alpha_i\alpha_{i-1}\alpha_{i-2}}{(N-1)^3}+\frac{N\alpha_{i-1}\alpha_{i-2}}{(N-1)^2}+\frac{N\alpha_{i-2}}{N-1}}, \quad R \leq \rho \leq 3R. \end{aligned}$$

By repeating this procedure we get

$$\begin{aligned} \text{(3.16)} \qquad & f_{i-(m-1)}^{(N)}(\rho)f_{i+1}'(\rho)^{K_i} \\ & = f_{i+1}^{(N)}(\rho)f_{i+1}'(\rho)^{K_i} \quad \geq CR^{-L_i}f_{i+1}(\rho)^{M_i}, \quad R \leq \rho \leq 3R, \end{aligned}$$

where

$$K_i = \sum_{j=1}^{m-1}\left\{\frac{N}{(N-1)^{m-j}}\prod_{k=j}^{m-1}\alpha_{i-k}\right\},$$

$$L_i = \sum_{j=1}^{m-1}\left\{\frac{\lambda_{i-(j-1)}}{(N-1)^{m-j}}\prod_{k=j}^{m-1}\alpha_{i-k}\right\}+\lambda_{i+1},$$

and

$$\begin{aligned} M_i = & \frac{\prod_{j=0}^{m-1}\alpha_{i-j}}{(N-1)^m}+\sum_{j=1}^{m-1}\left\{\frac{N}{(N-1)^{m-j}}\prod_{k=j}^{m-1}\alpha_{i-k}\right\} \\ = & \frac{A}{(N-1)^m}+K_i. \end{aligned}$$

Multiplying (3.16) by f_{i+1}' and integrating by parts $(N-1)$ times on $[R,\rho]$, we get

$$f_{i+1}'(\rho)f_{i+1}(\rho)^{-\frac{A-(N-1)^m}{(K_i+N)(N-1)^m}-1} \geq CR^{-\frac{L_i}{K_i+N}}, \quad R < \rho \leq 3R.$$

Integrating this inequality on $[2R, 3R]$, we have

$$f_{i+1}(2R)^{-\frac{A-(N-1)^m}{(K_i+N)(N-1)^m}} \geq CR^{-\frac{L_i}{K_i+N}+1}.$$

From (3.15), we get

$$u_{i+2}(e^R)^{\alpha_{i+1}} \leq R^{\frac{(N-1)^m}{A-(N-1)^m}\left\{L_i-K_i+\frac{A(\lambda_{i+1}-N)}{(N-1)^m}-\lambda_{i+1}\right\}}.$$

From the definitions of K_i and L_i, we see that

$$\begin{aligned}
& L_i - K_i + \frac{A(\lambda_{i+1}-N)}{(N-1)^m} - \lambda_{i+1} \\
= \; & \sum_{j=1}^{m-1}\left\{\frac{\lambda_{i-j+1}}{(N-1)^{m-j}}\prod_{k=j}^{m-1}\alpha_{i-k}\right\} - \sum_{j=1}^{m-1}\left\{\frac{N}{(N-1)^{m-j}}\prod_{k=j}^{m-1}\alpha_{i-k}\right\} + \frac{A(\lambda_{i+1}-N)}{(N-1)^m} \\
= \; & \sum_{j=1}^{m-2}\left\{\frac{(\lambda_{i+1-j}-N)}{(N-1)^{m-j}}\prod_{k=j}^{m-1}\alpha_{i-k}\right\} + \frac{\alpha_{i+1}(\lambda_{i+2}-N)}{N-1} + \frac{A(\lambda_{i+1}-N)}{(N-1)^m} \\
= \; & \frac{\alpha_{i+1}}{N-1}\left\{\lambda_{i+2}-N+\sum_{j=0}^{m-2}\left\{\frac{(\lambda_{i+1-j}-N)}{(N-1)^{m-j-1}}\prod_{k=j}^{m-2}\alpha_{i-k}\right\}\right\} \\
= \; & \frac{\alpha_{i+1}}{N-1}\left\{\lambda_{i+2}-N+\frac{(\lambda_{i+1}-N)\alpha_i\alpha_{i-1}\cdots\alpha_{i-(m-2)}}{(N-1)^{m-1}} + \frac{(\lambda_i-N)\alpha_{i-1}\alpha_{i-2}\cdots\alpha_{i-(m-2)}}{(N-1)^{m-2}}\right. \\
& \left. + \cdots + \frac{(\lambda_{i-m+4}-N)\alpha_{i-m+3}\alpha_{i-m+2}}{(N-1)^2} + \frac{(\lambda_{i-m+3}-N)\alpha_{i-m+2}}{N-1}\right\} \\
= \; & \frac{\alpha_{i+1}}{N-1}\left\{\lambda_{i+2}-N+\sum_{j=1}^{m-1}\left\{\frac{(\lambda_{i+2+j}-N)}{(N-1)^j}\prod_{k=0}^{j-1}\alpha_{i+2+k}\right\}\right\} \\
= \; & \frac{\alpha_{i+1}\Lambda_{i+2}}{N-1}.
\end{aligned}$$

Therefore we see that

$$u_{i+2}(e^R)^{\alpha_i+1} \leq CR^{\frac{\alpha_{i+1}(N-1)^{m-1}\Lambda_{i+2}}{A-(N-1)^m}}.$$

Thus we obtain

$$u_{i+2}(e^\rho) \leq C\rho^{\frac{(N-1)^{m-1}\Lambda_{i+2}}{A-(N-1)^m}} \quad \text{at } \infty, \quad i = 1, 2, \cdots, m.$$

Hence we obtain (3.4) since $\rho = \log r$. The proof is completed.□

4 Nonexistence results

In this section we study the nonexistence of nonnegative nontrivial radial entire solutions of (1.1).

Theorem 4.1 *Suppose that* $H_i,\ i = 1, 2, \cdots, m$, *satisfy*

$$H_i(|x|) \geq \frac{C_i}{|x|^{\lambda_i}}, \quad |x| \geq r_0 > 0, \tag{4.1}$$

where $C_i > 0$ *and* $\lambda_i,\ i = 1, 2, \cdots, m$, *are constants. Moreover*

$$\Lambda_i \leq \frac{A - P}{P} \max\{0, p_i - N\} \quad \textit{for} \quad \textit{some} \quad i \in \{1, 2, \cdots, m\}.$$

If $(u_1, u_2, \cdots, u_m)$ *is a nonnegative radial entire solution of (1.1), then*

$$(u_1, u_2, \cdots, u_m) \equiv (0, 0, \cdots, 0).$$

Remark 4.1 (i) When $m = 2$, Theorem 4.1 reduces to Theorem 2 of [12]. However, the proof presented here is simpler than that of Theorem 2 of [12].
(ii) When $p_i = 2,\ i = 1, 2, \cdots, m$, and $N \neq 2$, Theorem 4.1 reduces to Theorems 2.3 and 2.5 of [13].

Theorem 4.2 *Let* $p_i = N,\ i = 1, 2, \cdots, m$. *Suppose that* $H_i,\ i = 1, 2, \cdots, m$, *satisfy*

$$H_i(|x|) \geq \frac{C_i}{|x|^N (\log |x|)^{\lambda_i}}, \quad |x| \geq r_0 > 1, \tag{4.2}$$

where $C_i > 0$ *and* $\lambda_i,\ i = 1, 2, \cdots, m$, *are constants. Moreover*

$$\Lambda_i \leq \frac{A - (N-1)^m}{(N-1)^{m-1}} \quad \textit{for} \quad \textit{some} \quad i \in \{1, 2, \cdots, m\}.$$

If $(u_1, u_2, \cdots, u_m)$ *is a nonnegative radial entire solution of (1.1), then*

$$(u_1, u_2, \cdots, u_m) \equiv (0, 0, \cdots, 0).$$

Remark 4.2 (i) Theorem 4.2 shows that the conjecture stated in the introduction is true.
(ii) When $p_i = 2,\ i = 1, 2, \cdots, m$, Theorem 4.2 reduces to Theorem 2.4 of [13].

We give an example to show the sharpness of our results.

Example. Let us consider the elliptic system

$$
(4.3)\qquad \begin{cases} \Delta_{p_1} u_1 = \dfrac{1}{(1+|x|)^{\lambda_1}} u_2^{\alpha_1}, \\ \Delta_{p_2} u_2 = \dfrac{1}{(1+|x|)^{\lambda_2}} u_3^{\alpha_2}, \\ \qquad \vdots & x \in \mathbb{R}^N, \\ \Delta_{p_m} u_m = \dfrac{1}{(1+|x|)^{\lambda_m}} u_1^{\alpha_m}, \end{cases}
$$

where $N \geq 1,\ p_i > 1,\ \alpha_i > 0,\ i = 1, 2, \cdots, m,$ are constants satisfying $A > P$. Since

$$
\frac{C_i}{|x|^{\lambda_i}} \leq \frac{1}{(1+|x|)^{\lambda_i}} \leq \frac{\tilde{C}_i}{|x|^{\lambda_i}}, \quad |x| \geq 1, \quad i = 1, 2, \cdots, m
$$

hold for some positive constants C_i and $\tilde{C}_i,\ i = 1, 2, \cdots, m$, we can see from Theorems 2.1 and 4.1 that a necessary and sufficient condition for (4.3) to have a positive radial entire solution is

$$
\Lambda_i > \frac{A-P}{P} \max\{0, p_i - N\}, \quad i = 1, 2, \cdots, m.
$$

Proof of Theorem 4.1. Let $(u_1, u_2, \cdots, u_m)$ be a nonnegative nontrivial radial entire solution of (1.1). From Theorem 3.1 and its proof, we see that $u_i(r) > 0,\ r \geq r_*,\ i = 1, 2, \cdots, m$, for some $r_* > r_0$ and $u_i,\ i = 1, 2, \cdots, m$, satisfy

$$
(4.4)\qquad u_i(r) \leq C_i r^{\beta_i} \quad \text{at} \ \ \infty, \quad i = 1, 2, \cdots, m
$$

for some constants $C_i > 0,\ i = 1, 2, \cdots, m$.

If there exists an $i_0 \in \{1, 2, \cdots, m\}$ such that

$$
\Lambda_{i_0} < \frac{A-P}{P} \max\{0, p_{i_0} - N\},
$$

then we can see from the definition of β_{i_0} that

$$
\begin{cases} \beta_{i_0} < 0 & \text{if} \ \ p_{i_0} \leq N, \\ \beta_{i_0} < \dfrac{p_{i_0} - N}{p_{i_0} - 1} & \text{if} \ \ p_{i_0} > N. \end{cases}
$$

If $p_{i_0} \le N$, then it is found that $\lim_{r\to\infty} u_{i_0}(r) = 0$. On the other hand, since u_{i_0} is nondecreasing and $u_{i_0}(r_*) > 0$, we have

$$u_{i_0}(r) \ge u_{i_0}(r_*) > 0, \quad r \ge r_*.$$

This is a contradiction. If $p_{i_0} > N$, then integrating (3.5) on $[0, r]$ twice we have

$$\begin{aligned}
u_{i_0}(r) = \;& u_{i_0}(0) + \int_0^r s^{\frac{1-N}{p_{i_0}-1}} \left(\int_0^s t^{N-1} H_{i_0}(t) u_{i_0+1}(t)^{\alpha_{i_0}} dt \right)^{\frac{1}{p_{i_0}-1}} ds \\
\ge \;& \int_{r_*}^r s^{\frac{1-N}{p_{i_0}-1}} ds \left(\int_0^{r_*} t^{N-1} H_{i_0}(t) u_{i_0+1}(t)^{\alpha_{i_0}} dt \right)^{\frac{1}{p_{i_0}-1}} \\
= \;& \left(\int_0^{r_*} t^{N-1} H_{i_0}(t) u_{i_0+1}(t)^{\alpha_{i_0}} dt \right)^{\frac{1}{p_{i_0}-1}} \frac{p_{i_0}-1}{p_{i_0}-N} \left\{ r^{\frac{p_{i_0}-N}{p_{i_0}-1}} - r_*^{\frac{p_{i_0}-N}{p_{i_0}-1}} \right\} \\
\ge \;& C r^{\frac{p_{i_0}-N}{p_{i_0}-1}}, \quad r \ge \tilde{r}_* > r_*
\end{aligned}$$

for some constant $C > 0$. This contradicts to (4.4) with $\beta_{i_0} < (p_{i_0} - N)/(p_{i_0} - 1)$. It remains to discuss the case that

$$\Lambda_i \ge \frac{A-P}{P} \max\{0, p_i - N\}, \quad i = 1, 2, \cdots, m.$$

From the assumption of Λ_i, there exists an $i_0 \in \{1, 2, \cdots, m\}$ such that

$$\Lambda_{i_0} = \frac{A-P}{P} \max\{0, p_{i_0} - N\}.$$

Without loss of generality, we may assume that $i_0 = m$, that is,

$$\Lambda_i \ge \frac{A-P}{P} \max\{0, p_i - N\}, \quad i = 1, 2, \cdots, m-1$$

and

$$\Lambda_m = \frac{A-P}{P} \max\{0, p_m - N\}.$$

We first observe that

$$\begin{aligned}
\text{(4.5)} \qquad \lambda_i \;\le\; & \sum_{j=1}^{m-i-1} \left\{ (p_{i+j} - \lambda_{i+j}) \prod_{k=0}^{j-1} \frac{\alpha_{i+k}}{p_{i+1+k} - 1} \right\} + \min\{p_i, N\} \\
& + \max\{0, p_m - N\} \prod_{k=0}^{m-i-1} \frac{\alpha_{i+k}}{p_{i+1+k} - 1}, \quad i = 1, 2, \cdots, m-2,
\end{aligned}$$

and

$$\lambda_{m-1} \leq \frac{\alpha_{m-1}\max\{0, p_m - N\}}{p_m - 1} + \min\{p_{m-1}, N\}. \tag{4.6}$$

In fact, from the definition of Λ_i, we obtain

$$\begin{aligned}
\lambda_i \geq \ & -\sum_{j=1}^{m-1}\left\{(\lambda_{i+j} - p_{i+j})\prod_{k=0}^{j-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\right\} + p_i + \frac{A-P}{P}\max\{0, p_i - N\} \\
= \ & -\left(\sum_{j=1}^{m-i-1} + \sum_{j=m-i+1}^{m-1}\right)\left\{(\lambda_{i+j} - p_{i+j})\prod_{k=0}^{j-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\right\} \\
& -(\lambda_m - p_m)\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1} + p_i + \frac{A-P}{P}\max\{0, p_i - N\} \\
\equiv \ & -S_1 - S_2 - S_3 + p_i + \frac{A-P}{P}\max\{0, p_i - N\}.
\end{aligned}$$

From the assumption of Λ_m we have

$$\lambda_m - p_m = -\sum_{j=1}^{m-1}\left\{(\lambda_{m+j} - p_{m+j})\prod_{k=0}^{j-1}\frac{\alpha_{m+k}}{p_{m+1+k}-1}\right\} + \frac{A-P}{P}\max\{0, p_m - N\}.$$

Substituting this relation to S_3 we have

$$\begin{aligned}
S_3 = \ & -\sum_{j=1}^{m-1}\left\{(\lambda_{m+j} - p_{m+j})\prod_{k=0}^{j-1}\frac{\alpha_{m+k}}{p_{m+1+k}-1}\right\}\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1} \\
& +\frac{A-P}{P}\max\{0, p_m - N\}\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1} \\
= \ & -\sum_{j=1}^{m-1}\left\{(\lambda_{m+j} - p_{m+j})\prod_{k=0}^{m-i+j-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\right\} \\
& +\frac{A-P}{P}\max\{0, p_m - N\}\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1} \\
= \ & -\sum_{j=m-i+1}^{2m-i-1}\left\{(\lambda_{i+j} - p_{i+j})\prod_{k=0}^{j-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\right\} + \frac{A-P}{P}\max\{0, p_m - N\}\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{j=m-i+1}^{m-1}\left\{(\lambda_{i+j}-p_{i+j})\prod_{k=0}^{j-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\right\}-\sum_{j=m}^{2m-i-1}\left\{(\lambda_{i+j}-p_{i+j})\prod_{k=0}^{j-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\right\}\\
&\quad +\frac{A-P}{P}\max\{0,p_m-N\}\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\\
&= -S_2-\sum_{j=0}^{m-i-1}\left\{(\lambda_{i+j}-p_{i+j})\prod_{k=0}^{j+m-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\right\}\\
&\quad +\frac{A-P}{P}\max\{0,p_m-N\}\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\\
&= -S_2-\frac{A}{P}S_1-\frac{A}{P}(\lambda_i-p_i)+\frac{A-P}{P}\max\{0,p_m-N\}\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
\lambda_i \geq\ & \left(\frac{A}{P}-1\right)S_1+\frac{A}{P}(\lambda_i-p_i)+p_i+\frac{A-P}{P}\max\{0,p_i-N\}\\
&-\frac{A-P}{P}\max\{0,p_m-N\}\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1},
\end{aligned}
$$

namely,

$$
\begin{aligned}
0 \geq\ & S_1+\lambda_i-p_i+\max\{0,p_i-N\}-\max\{0,p_m-N\}\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\\
=\ & \sum_{j=1}^{m-i-1}\left\{(\lambda_{i+j}-p_{i+j})\prod_{k=0}^{j-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}\right\}+\lambda_i-\min\{p_i,N\}\\
& -\max\{0,p_m-N\}\prod_{k=0}^{m-i-1}\frac{\alpha_{i+k}}{p_{i+1+k}-1}.
\end{aligned}
$$

Therefore we obtain (4.5). Similarly we obtain (4.6). From the above computation we see that if

$$\Lambda_i > \frac{A-P}{P}\max\{0,p_i-N\},$$

then "$<$" holds in (4.5) and (4.6), and if

$$\Lambda_i = \frac{A-P}{P}\max\{0,p_i-N\},$$

then "$=$" holds in (4.5) and (4.6).

From now on, the letter C denotes various positive constants independent of r and R. Integrating (3.5) twice over $[r_*, r]$, from (4.1), we have

$$(4.7)\; u_i(r) \geq u_i(r_*) + \int_{r_*}^{r} \left(s^{1-N} \int_{r_*}^{s} t^{N-1} H_i(t) u_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i - 1}} ds$$

$$\geq C \int_{r_*}^{r} \left(s^{1-N} \int_{r_*}^{s} t^{N-1-\lambda_i} u_{i+1}(t)^{\alpha_i} dt \right)^{\frac{1}{p_i-1}} ds, \quad i = 1, 2, \cdots, m.$$

In what follows of the proof the argument is divided into two cases according to p_m.

(i) Let $p_m \leq N$. We first consider the case that

$$\Lambda_{m-1} = \frac{A-P}{P} \max\{0, p_{m-1} - N\}.$$

Then from (4.6) we see that $\lambda_{m-1} = \min\{p_{m-1}, N\}$. From (4.7) with $i = m-1$ we have

$$u_{m-1}(r) \geq C u_m(r_*)^{\frac{\alpha_{m-1}}{p_{m-1}-1}} \int_{r_*+1}^{r} \left(s^{1-N} \int_{r_*}^{s} t^{N-1-\min\{p_{m-1}, N\}} dt \right)^{\frac{1}{p_{m-1}-1}} ds.$$

Therefore we see that, for $p_{m-1} < N$

$$u_{m-1}(r) \geq C \int_{r_*+1}^{r} \left(s^{1-N} \int_{r_*}^{s} t^{N-1-p_{m-1}} dt \right)^{\frac{1}{p_{m-1}-1}} ds$$

$$\geq C \int_{r_*+1}^{r} s^{-1} ds$$

$$\geq C \log r, \quad r \geq r_1 > r_* + 1,$$

for $p_{m-1} = N$

$$u_{m-1}(r) \geq C \int_{r_*+1}^{r} \left(s^{1-N} \int_{r_*}^{s} t^{-1} dt \right)^{\frac{1}{p_{m-1}-1}} ds$$

$$\geq C \int_{r_*+1}^{r} s^{-1} (\log s)^{\frac{1}{p_{m-1}-1}} ds$$

$$\geq C (\log r)^{\frac{p_{m-1}}{p_{m-1}-1}}, \quad r \geq r_1 > r_* + 1,$$

and for $p_{m-1} > N$

$$\begin{aligned} u_{m-1}(r) \geq & \; C\int_{r_*+1}^{r} \left(s^{1-N} \int_{r_*}^{s} t^{-1} dt \right)^{\frac{1}{p_{m-1}-1}} ds \\ \geq & \; C\int_{r_*+1}^{r} s^{\frac{1-N}{p_{m-1}-1}} (\log s)^{\frac{1}{p_{m-1}-1}} ds \\ \geq & \; Cr^{\frac{p_{m-1}-N}{p_{m-1}-1}} (\log r)^{\frac{1}{p_{m-1}-1}}, \quad r \geq r_1 > r_* + 1. \end{aligned}$$

Here, the last inequality is given by integration by parts. On the other hand, from (4.4) with $i = m-1$ and the definition of β_{m-1} we see that

$$u_{m-1}(r) \leq \begin{cases} C & \text{if } \; p_{m-1} \leq N, \\ Cr^{\frac{p_{m-1}-N}{p_{m-1}-1}} & \text{if } \; p_{m-1} > N \end{cases}$$

for large $r > r_*$. This is a contradiction.

Next we consider the case that

$$\Lambda_{m-2} = \frac{A-P}{P} \max\{0, p_{m-2} - N\}.$$

Then we see from (4.5) with $i = m-2$ and (4.6) that

$$\lambda_{m-1} < \min\{p_{m-1}, N\} \quad \text{and} \quad \lambda_{m-2} = \frac{(p_{m-1} - \lambda_{m-1})\alpha_{m-2}}{p_{m-1}-1} + \min\{p_{m-2}, N\}.$$

From (4.7) with $i = m-1$ we have

$$\begin{aligned} u_{m-1}(r) \geq & \; Cu_m(r_*)^{\frac{\alpha_{m-1}}{p_{m-1}-1}} \int_{r_*+1}^{r} \left(s^{1-N} \int_{r_*}^{s} t^{N-1-\lambda_{m-1}} dt \right)^{\frac{1}{p_{m-1}-1}} ds \\ \geq & \; C\int_{r_*+1}^{r} s^{\frac{1-\lambda_{m-1}}{p_{m-1}-1}} ds \\ \geq & \; Cr^{\frac{p_{m-1}-\lambda_{m-1}}{p_{m-1}-1}}, \quad r \geq r_1 > r_* + 1. \end{aligned}$$

From this estimate and (4.7) with $i = m-2$ we obtain

$$\begin{aligned} u_{m-2}(r) \geq & \; C\int_{r_1+1}^{r} \left(s^{1-N} \int_{r_1}^{s} t^{N-1-\lambda_{m-2}+\frac{\alpha_{m-2}(p_{m-1}-\lambda_{m-1})}{p_{m-1}-1}} dt \right)^{\frac{1}{p_{m-2}-1}} ds \\ = & \; C\int_{r_1+1}^{r} \left(s^{1-N} \int_{r_1}^{s} t^{N-1-\min\{p_{m-2},N\}} dt \right)^{\frac{1}{p_{m-2}-1}} ds. \end{aligned}$$

Therefore we see that for $r \geq r_2 > r_1 + 1$

$$u_{m-2}(r) \geq \begin{cases} C \log r & \text{if } \ p_{m-2} < N, \\ C(\log r)^{\frac{p_{m-2}}{p_{m-2}-1}} & \text{if } \ p_{m-2} = N, \\ Cr^{\frac{p_{m-2}-N}{p_{m-2}-1}}(\log r)^{\frac{1}{p_{m-2}-1}} & \text{if } \ p_{m-2} > N. \end{cases}$$

On the other hand, from (4.4) with $i = m-2$ and the definition of β_{m-2} we see that

$$u_{m-2}(r) \leq \begin{cases} C & \text{if } \ p_{m-2} \leq N, \\ Cr^{\frac{p_{m-2}-N}{p_{m-2}-1}} & \text{if } \ p_{m-2} > N, \end{cases}$$

for large $r \geq r_*$. This is a contradiction.

Similarly, suppose that there exists an $i_0 \in \{1, 2, \cdots, m\}$ such that

$$\Lambda_{i_0} = \frac{A-P}{P} \max\{0, p_{i_0} - N\}$$

and

$$\Lambda_i > \frac{A-P}{P} \max\{0, p_i - N\}, \quad i = i_0 + 1, \cdots, m-1.$$

Then we see from (4.6) and (4.7) with $i = m-1$ that

$$u_{m-1}(r) \geq Cr^{\frac{p_{m-1}-\lambda_{m-1}}{p_{m-1}-1}}, \quad r \geq r_1 > r_* + 1.$$

From this estimate, (4.5) with $i = m-2$, (4.7) with $i = m-2$ we have

$$\begin{aligned} u_{m-2}(r) \geq \;& C \int_{r_1+1}^{r} \left(s^{1-N} \int_{r_1}^{s} t^{N-1-\lambda_{m-2}+\frac{\alpha_{m-2}(p_{m-1}-\lambda_{m-1})}{p_{m-1}-1}} dt \right)^{\frac{1}{p_{m-2}-1}} ds \\ \geq \;& C \int_{r_1+1}^{r} s^{\frac{1-\lambda_{m-2}}{p_{m-2}-1}+\frac{\alpha_{m-2}(p_{m-1}-\lambda_{m-1})}{(p_{m-1}-1)(p_{m-2}-1)}} ds \\ \geq \;& Cr^{\frac{p_{m-2}-\lambda_{m-2}}{p_{m-2}-1}+\frac{\alpha_{m-2}(p_{m-1}-\lambda_{m-1})}{(p_{m-1}-1)(p_{m-2}-1)}}, \quad r \geq r_2 > r_1 + 1. \end{aligned}$$

By repeating this procedure, we get a sequence $\{r_j\}_{j=2}^{m-i_0-1}$ such that

$$u_i(r) \geq Cr^{\tau_i}, \quad r \geq r_j > r_{j-1} + 1, \quad i = m-2, m-3, \cdots, i_0 + 1,$$

where

$$\begin{aligned} \tau_i = \;& \frac{1}{p_i - 1} \left\{ p_i - \lambda_i + \sum_{j=1}^{m-i-1} \left\{ (p_{i+j} - \lambda_{i+j}) \prod_{k=0}^{j-1} \frac{\alpha_{i+k}}{p_{i+1+k} - 1} \right\} \right\} \\ = \;& \frac{p_i - \lambda_i + \alpha_i \tau_{i+1}}{p_i - 1}. \end{aligned}$$

From this estimate, (4.5) with $i = i_0$, (4.7) with $i = i_0$ we have

$$\begin{aligned} u_{i_0}(r) \geq\ & C \int_{r_{m-i_0-1}+1}^{r} \left(s^{1-N} \int_{r_{m-i_0-1}}^{s} t^{N-1-\lambda_{i_0}+\alpha_{i_0}\tau_{i_0+1}} dt \right)^{\frac{1}{p_{i_0}-1}} ds \\ =\ & C \int_{r_{m-i_0-1}+1}^{r} \left(s^{1-N} \int_{r_{m-i_0-1}}^{s} t^{N-1-\min\{p_{i_0},N\}} dt \right)^{\frac{1}{p_{i_0}-1}} ds. \end{aligned}$$

Therefore we see that for $r \geq r_{m-i_0} > r_{m-i_0-1} + 1$

$$u_{i_0}(r) \geq \begin{cases} C \log r & \text{if } \ p_{i_0} < N, \\ C(\log r)^{\frac{p_{i_0}}{p_{i_0}-1}} & \text{if } \ p_{i_0} = N, \\ C r^{\frac{p_{i_0}-N}{p_{i_0}-1}} (\log r)^{\frac{1}{p_{i_0}-1}} & \text{if } \ p_{i_0} > N. \end{cases}$$

On the other hand, from (4.4) with $i = i_0$ and the definition of β_{i_0} we see that

$$u_{i_0}(r) \leq \begin{cases} C & \text{if } \ p_{i_0} \leq N, \\ C r^{\frac{p_{i_0}-N}{p_{i_0}-1}} & \text{if } \ p_{i_0} > N, \end{cases}$$

for large $r \geq r_*$. This is a contradiction. Thus the proof is completed for the case $p_m \leq N$.

(ii) Let $p_m > N$. Then, integrating (3.5) on $[0, r]$ twice, we have

$$\begin{aligned} \text{(4.8)} \qquad u_m(r) \ &= \ u_m(0) + \int_0^r s^{\frac{1-N}{p_m-1}} \left(\int_0^s t^{N-1} H_m(t) u_1(t)^{\alpha_m} dt \right)^{\frac{1}{p_m-1}} ds \\ &\geq \ \int_{r_*}^r s^{\frac{1-N}{p_m-1}} ds \left(\int_0^{r_*} t^{N-1} H_m(t) u_1(t)^{\alpha_m} dt \right)^{\frac{1}{p_m-1}} \\ &\geq \ C r^{\frac{p_m-N}{p_m-1}}, \quad r \geq r_1 > r_*. \end{aligned}$$

Let us consider the case that

$$\Lambda_{m-1} = \frac{A-P}{P} \max\{0, p_{m-1} - N\}.$$

Then from (4.6) we see that

$$\lambda_{m-1} = \frac{\alpha_{m-1}(p_m - N)}{p_m - 1} + \min\{p_{m-1}, N\}.$$

From (4.7) with $i = m-1$ and (4.8) we have

$$\begin{aligned} u_{m-1}(r) &\geq \int_{r_1+1}^{r} \left(s^{1-N} \int_{r_1}^{s} t^{N-1-\lambda_{m-1}+\frac{\alpha_{m-1}(p_m-N)}{p_m-1}} dt \right)^{\frac{1}{p_{m-1}-1}} ds \\ &= \int_{r_1+1}^{r} \left(s^{1-N} \int_{r_1}^{s} t^{N-1-\min\{p_{m-1},N\}} dt \right)^{\frac{1}{p_{m-1}-1}} ds. \end{aligned}$$

Therefore we see that for $r \geq r_2 > r_1 + 1$

$$u_{m-1}(r) \geq \begin{cases} C \log r & \text{if } \ p_{m-1} < N, \\ C(\log r)^{\frac{p_{m-1}}{p_{m-1}-1}} & \text{if } \ p_{m-1} = N, \\ Cr^{\frac{p_{m-1}-N}{p_{m-1}-1}} (\log r)^{\frac{1}{p_{m-1}-1}} & \text{if } \ p_{m-1} > N. \end{cases}$$

On the other hand, from (4.4) with $i = m-1$ and the definition of β_{m-1} we see that

$$u_{m-1}(r) \leq \begin{cases} C & \text{if } \ p_{m-1} \leq N, \\ Cr^{\frac{p_{m-1}-N}{p_{m-1}-1}} & \text{if } \ p_{m-1} > N, \end{cases}$$

for large $r \geq r_*$. This is a contradiction.

Using similar arguments as in (i), we can get a contradiction for the case that

$$\Lambda_{i_0} = \frac{A-P}{P} \max\{0, p_{i_0-1} - N\} \quad \text{for some} \quad i_0 \in \{1, 2, \cdots, m\},$$

and

$$\Lambda_i > \frac{A-P}{P} \max\{0, p_{i-1} - N\}, \quad i = i_0+1, i_0+2, \cdots, m-1.$$

The proof is finished.□

Proof of Theorem 4.2. Let $(u_1, u_2, \cdots, u_m)$ be a nonnegative nontrivial radial entire solution of (1.1). From Theorem 3.2 and its proof we see that $u_i(r) > 0,\ r \geq r_*,\ i = 1, 2, \cdots, m$, for some $r_* > r_0$ and $u_i,\ i = 1, 2, \cdots, m$, satisfy

$$u_i(r) \leq C_i (\log r)^{\beta_i} \quad \text{at} \quad \infty, \tag{4.9}$$

where $C_i > 0,\ i = 1, 2, \cdots, m$, are constants. If there exists an $i_0 \in \{1, 2, \cdots, m\}$ such that

$$\Lambda_{i_0} < \frac{A - (N-1)^m}{(N-1)^{m-1}},$$

then we see that $\beta_{i_0} < 1$ by the definition of β_{i_0}. On the other hand, integrating (3.5) on $[0, r]$ twice, we have

$$
\begin{aligned}
(4.10) \qquad u_{i_0}(r) &= u_{i_0}(0) + \int_0^r s^{-1}\left(\int_0^s t^{N-1} H_{i_0}(t) u_{i_0+1}(t)^{\alpha_{i_0}}\, dt\right)^{\frac{1}{N-1}} ds \\
&\geq \int_{r_*}^r s^{-1} ds \left(\int_0^{r_*} t^{N-1} H_{i_0} u_{i_0+1}(t)^{\alpha_{i_0}}\, dt\right)^{\frac{1}{N-1}} \\
&\geq C \log r, \quad r \geq r_1 > r_*
\end{aligned}
$$

for some constant $C > 0$. This contradicts to (4.9) with $\beta_{i_0} < 1$. It remains to discuss the case

$$\Lambda_i \geq \frac{A - (N-1)^m}{(N-1)^{m-1}}, \quad i = 1, 2, \cdots, m.$$

From the assumption of Λ_i there exists an $i_0 \in \{1, 2, \cdots, m\}$ such that

$$\Lambda_{i_0} = \frac{A - (N-1)^m}{(N-1)^{m-1}}.$$

Without loss of generality we may assume that $i_0 = m$, that is,

$$\Lambda_i \geq \frac{A - (N-1)^m}{(N-1)^{m-1}}, \quad i = 1, 2, \cdots, m-1$$

and

$$\Lambda_m = \frac{A - (N-1)^m}{(N-1)^{m-1}}.$$

A similar computation as was used in the proof of Theorem 4.1 shows that
(4.11)

$$\lambda_i \leq -\sum_{j=1}^{m-i-1}\left\{\frac{(\lambda_{i+j} - N)}{(N-1)^j}\prod_{k=0}^{j-1}\alpha_{i+k}\right\} + \frac{\prod_{k=0}^{m-i-1}\alpha_{i+k}}{(N-1)^{m-i-1}} + 1, \quad i = 1, 2, \cdots, m-2,$$

and

$$(4.12) \qquad \lambda_{m-1} \leq \alpha_{m-1} + 1.$$

We notice that "$<$" holds in (4.11) and (4.12) if

$$\Lambda_i > \frac{A - (N-1)^m}{(N-1)^{m-1}},$$

and "=" holds in (4.11) and (4.12) if

$$\Lambda_i = \frac{A-(N-1)^m}{(N-1)^{m-1}}.$$

First we consider the case that

$$\Lambda_{m-1} = \frac{A-(N-1)^m}{(N-1)^{m-1}}.$$

Using the same computation as (4.10), we have

$$u_m(r) \geq C \log r, \quad r \geq r_1 > r_*$$

for some constant $C > 0$. From this estimate, (4.7) with $i = m-1$, (4.12) we have

$$\begin{aligned} u_{m-1}(r) \geq\ & C \int_{r_1+1}^{r} s^{-1} \left(\int_{r_1}^{s} t^{-1} (\log t)^{-\lambda_{m-1}+\alpha_{m-1}} dt \right)^{\frac{1}{N-1}} ds \\ =\ & C \int_{r_1+1}^{r} s^{-1} \left(\int_{r_1}^{s} t^{-1} (\log t)^{-1} dt \right)^{\frac{1}{N-1}} ds \\ \geq\ & C \int_{r_1+1}^{r} s^{-1} (\log(\log s))^{\frac{1}{N-1}} ds \\ \geq\ & C \log r (\log(\log r))^{\frac{1}{N-1}}, \quad r \geq r_2 > r_1 + 1 \end{aligned}$$

for some constant $C > 0$. On the other hand, from (4.9) with $i = m-1$ and the definition of β_{m-1} we see that

$$u_{m-1}(r) \leq C_{m-1} \log r \quad \text{at} \ \infty.$$

This is a contradiction.

Using similar arguments as in the proof of Theorem 4.1, we get a contradiction for the case that

$$\Lambda_{i_0} = \frac{A-(N-1)^m}{(N-1)^{m-1}} \quad \text{for some} \ \ i_0 \in \{1, 2, \cdots, m\}$$

and

$$\Lambda_i > \frac{A-(N-1)^m}{(N-1)^{m-1}}, \quad i = i_0+1, i_0+2, \cdots, m-1.$$

The proof is completed.□

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