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OSCILLATION CRITERIA FOR SECOND-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS OF MIXED TYPE

ETHIRAJU THANDAPANI* AND RENU RAMA

Ramanujan Institute for Advanced
Study in Mathematics,
University of Madras,
Chennai - 600 005, India.

Abstract. Some oscillation criteria are established for the second order nonlinear neutral differential equations of the form

$$((x(t)+ax(t-\tau_1)-bx(t+\tau_2))^{\alpha})''=q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\beta}(t+\sigma_2),$$

and

$$(x(t)-ax(t-\tau_1)+bx(t+\tau_2)^{\alpha})''=q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\beta}(t+\sigma_2)$$

where α and β are the ratios of odd positive integers with $\beta \geq 1$. Examples are provided to illustrate the main results.

1. Introduction

In this paper we study the oscillatory behavior of all solutions of neutral differential equations of the form

$$((x(t)+ax(t-\tau_1)-bx(t+\tau_2))^{\alpha})''=q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\beta}(t+\sigma_2), \tag{1.1}$$

and

$$((x(t)-ax(t-\tau_1)+bx(t+\tau_2))^{\alpha})''=q(t)x^{\beta}(t-\sigma_1)+p(t)x^{\beta}(t+\sigma_2) \tag{1.2}$$

where $t \geq t_0 \geq 0$, a and b are nonnegative constants, $\tau_1,\ \tau_2,\ \sigma_1$ and σ_2 are positive constants, $q(t),\ p(t) \in C([t_0,\infty),[t_0,\infty))$, α, and β are the ratios of odd positive integers with $\beta \geq 1$.

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*Corresponding author.

Let $\theta = \max\{\tau_1, \sigma_1\}$. By a solution of equation (1.1) or (1.2), we mean a real valued function $x(t)$ defined for all $t \geq t_0 - \theta$, and satisfying the equation (1.1) or (1.2) for all $t \geq t_0$. A nontrivial solution of equation (1.1) or (1.2) is said to be oscillatory, if it has arbitrarily large zeros; otherwise it is called nonoscillatory.

The problem of determining oscillation and nonoscillation of second order delay and neutral type differential equations has received great attention in recent years, see for example [1–21], and the references cited therein.If $a = 0$ or $b = 0$ and either $q(t) \equiv 0$ or $p(t) \equiv 0$, then the oscillatory behavior of solutions of equations (1.1) and (1.2) are studied in [1, 4, 6, 8–10, 14–18, 20]. In particular if $\alpha = \beta = 1$ and $\alpha = \beta$, then the oscillatory behavior of solutions of equations (1.1) and (1.2) are discussed in [2, 3, 5, 7, 11–13, 19, 21]. Motivated by this observation, in this paper we establish some new sufficient conditions for the oscillation of all solutions of equations (1.1) and (1.2) when $\beta \geq 1$.

In Section 2, we present some sufficient conditions for the oscillation of all solutions of equations (1.1) and (1.2). Examples are provided in Section 3 to illustrate the main results.

2. Oscillation Results

In this section we shall obtain some sufficient conditions for the oscillation of all solutions of the equations (1.1) and (1.2). Before proving the main results we state the following lemma which will be useful in proving the main results.

Lemma 2.1. *Let $A \geq 0$, $B \geq 0$, and $\gamma \geq 1$. Then*

$$A^\gamma + B^\gamma \geq \frac{1}{2^{\gamma-1}}(A+B)^\gamma. \tag{2.1}$$

If $A \geq B$, *then*

$$A^\gamma - B^\gamma \geq (A-B)^\gamma. \tag{2.2}$$

Proof. The proof may be found in [19]. ∎

First we study the oscillation of all solutions of equation (1.1).

Theorem 2.2. *Let* $\sigma_i > \tau_i$ *for* $i = 1, 2, (1 + a^\beta - \dfrac{b^\beta}{2^{\beta-1}}) > 0$, *and* $q(t)$ *and* $p(t)$ *be positive and nonincreasing for all* $t \geq t_0$. *Assume that the differential inequalities*

$$y''(t) - \frac{p(t)}{2^{\beta-1}(1 + a^\beta - \frac{b^\beta}{2^{\beta-1}})^{\beta/\alpha}} y^{\beta/\alpha}(t + \sigma_2 - \tau_2) \geq 0, \tag{2.3}$$

$$y''(t) - \frac{q(t)}{2^{\beta-1}(1 + a^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t - \sigma_1 + \tau_1) \geq 0, \tag{2.4}$$

and

$$y''(t) + c_1 q(t) y^{\beta/\alpha}(t - \sigma_1 - \tau_2) + c_1 p(t) y^{\beta/\alpha}(t + \sigma_2 - \tau_2) \leq 0 \tag{2.5}$$

where $c_1 = \min\left\{\dfrac{1}{b^\beta}, \dfrac{1}{2^{\beta-1}}\left(\dfrac{2^{\beta-1}}{b^\beta}\right)^{\beta/\alpha}\right\}$,*have no eventually positive increasing solution, no eventually positive decreasing solution,and no eventually positive solution, respectively. Then every solution of equation* (1.1) *is oscillatory.*

Proof. Assume that $x(t)$ is a nonoscillatory solution of equation (1.1). Without loss of generality we may assume that there exists $t_1 \geq t_0$ such that $x(t-\theta) > 0$ for all $t \geq t_1$. Set

$$z(t) = (x(t) + ax(t-\tau_1) - bx(t+\tau_2))^\alpha, t \geq t_1.$$

Then $z''(t) = q(t)x^\beta(t-\sigma_1) + p(t)x^\beta(t+\sigma_2) > 0$ for all $t \geq t_1$. Therefore, both $z(t)$ and $z'(t)$ are of one sign for all $t \geq t_1$. We shall prove that $z(t) > 0$ eventually. Indeed, if $z(t) < 0$, then

$$0 < u(t) = -z(t) = (bx(t+\tau_2) - ax(t-\tau_1) - x(t))^\alpha \leq b^\alpha x^\alpha(t+\tau_2).$$

That is

$$x^{\beta}(t) \geq \frac{1}{b^{\beta}} u^{\beta/\alpha}(t-\tau_2) \geq c_1 u^{\beta/\alpha}(t-\tau_2), t \geq t_1.$$

Using the above inequality in equation (1.1), we have

$$\begin{aligned} 0 &= u''(t) + q(t)x^{\beta}(t-\sigma_1) + p(t)x^{\beta}(t+\sigma_2) \\ &\geq u''(t) + \frac{q(t)}{b^{\beta}} u^{\beta/\alpha}(t-\sigma_1-\tau_2) + \frac{p(t)}{b^{\beta}} u^{\beta/\alpha}(t+\sigma_2-\tau_2), \end{aligned}$$

or

$$u''(t) + c_1 q(t) u^{\beta/\alpha}(t-\sigma_1-\tau_2) + c_1 p(t) u^{\beta/\alpha}(t+\sigma_2-\tau_2) \leq 0.$$

Hence $u(t)$ is a positive solution of the inequality (2.5), a contradiction. Therefore $z(t) > 0$ eventually. Now we define a function $y(t)$ as

$$y(t) = z(t) + a^{\beta} z(t-\tau_1) - \frac{b^{\beta}}{2^{\beta-1}} z(t+\tau_2), t \geq t_1. \tag{2.6}$$

Then

$$\begin{aligned} y''(t) &= z''(t) + a^{\beta} z''(t-\tau_1) - \frac{b^{\beta}}{2^{\beta-1}} z''(t+\tau_2) \\ &= q(t)x^{\beta}(t-\sigma_1) + p(t)x^{\beta}(t+\sigma_2) + a^{\beta}\left(q(t-\tau_1)x^{\beta}(t-\sigma_1-\tau_1)\right. \\ &\quad \left.+p(t-\tau_1)x^{\beta}(t+\sigma_2-\tau_1)\right) - \frac{b^{\beta}}{2^{\beta-1}}\left(q(t+\tau_2)x^{\beta}(t-\sigma_1+\tau_2)\right. \\ &\quad \left.+p(t+\tau_2)x^{\beta}(t+\sigma_2+\tau_2)\right), t \geq t_1. \end{aligned} \tag{2.7}$$

Using the monotonicity of $q(t)$ and $p(t)$ and the inequality (2.1) in (2.7), we get

$$\begin{aligned} y''(t) \geq\ & \frac{q(t)}{2^{\beta-1}}\left([x(t-\sigma_1) + ax(t-\sigma_1-\tau_1)]^{\beta} - b^{\beta}x^{\beta}(t-\sigma_1+\tau_2)\right) \\ & + \frac{p(t)}{2^{\beta-1}}\left([x(t+\sigma_2) + ax(t+\sigma_2-\tau_1)]^{\beta} - b^{\beta}x(t+\sigma_2+\tau_2)\right), t \geq t_1. \end{aligned}$$

Now using $z(t) > 0$ for $t \geq t_1$, and the inequality (2.2) in the above inequality, we obtain

$$y''(t) \geq \frac{q(t)}{2^{\beta-1}} z^{\beta/\alpha}(t-\sigma_1) + \frac{p(t)}{2^{\beta-1}} z^{\beta/\alpha}(t+\sigma_2) > 0, t \geq t_1 \tag{2.8}$$

which implies that both $y(t)$ and $y'(t)$ are of one sign, eventually. We shall prove that $y(t) > 0$ eventually. If not, then

$$0 < v(t) = -y(t) = -z(t) - a^\beta z(t-\tau_1) + \frac{b^\beta}{2^{\beta-1}} z(t+\tau_2) \le \frac{b^\beta}{2^{\beta-1}} z(t+\tau_2).$$

Hence $z(t) \ge \frac{2^{\beta-1}}{b^\beta} v(t-\tau_2)$. Using the last inequality in (2.8), we obtain

$$0 \ge v''(t) + \frac{q(t)}{2^{\beta-1}} \left(\frac{2^{\beta-1}}{b^\beta}\right)^{\beta/\alpha} v^{\beta/\alpha}(t-\sigma_1-\tau_2) + \frac{p(t)}{2^{\beta-1}} \left(\frac{2^{\beta-1}}{b^\beta}\right)^{\beta/\alpha} v^{\beta/\alpha}(t+\sigma_2-\tau_2),$$

or

$$v''(t) + c_1 q(t) v^{\beta/\alpha}(t-\sigma_1-\tau_2) + c_1 p(t) v^{\beta/\alpha}(t+\sigma_2-\tau_2) \le 0, t \ge t_1.$$

Therefore $v(t)$ is a positive solution of (2.5), contradiction. Thus $y(t) > 0$, eventually. Next we consider the following two cases:

Case:1. Let $z'(t) < 0$ for all $t \ge t_2 \ge t_1$. We claim that $y'(t) < 0$ for all $t \ge t_2$. If not, then $y(t) > 0,\ y'(t) > 0$ and $y''(t) > 0$ imply that $\lim_{t\to\infty} y(t) = \infty$. On the other hand, $z(t) > 0$ and $z'(t) < 0$ imply that $\lim_{t\to\infty} z(t) = c < \infty$. Applying limit on both the sides of equation (2.6) we obtain a contradiction. Thus $y'(t) < 0$ for all $t \ge t_2$. Using the monotonicity of $z(t)$, we obtain

$$\begin{aligned} y(t) &= z(t) + a^\beta z(t-\tau_1) - \frac{b^\beta}{2^{\beta-1}} z(t+\tau_2) \\ &\le (1+a^\beta) z(t-\tau_1), t \ge t_2. \end{aligned}$$

The above inequality together with (2.8) implies that

$$y''(t) \ge \frac{q(t)}{2^{\beta-1}} \frac{y^{\beta/\alpha}(t-\sigma_1+\tau_1)}{(1+a^\beta)^{\beta/\alpha}}, t \ge t_2.$$

Thus $y(t)$ is a positive decreasing solution of the inequality (2.4), which is a contradiction.

Case:2. Let $z'(t) > 0$ for all $t \ge t_2$. Now we consider the following two subcases:

Subcase (i): Assume that $y'(t) < 0$ for all $t \geq t_2$. Proceeding as in Case 1, and using the monotonicity of $z(t)$, we obtain

$$y(t-\sigma_1) \leq (1+a^\beta)z(t-\sigma_1).$$

Using the last inequality in (2.8) and the monotonicity of $y(t)$, we obtain

$$\begin{aligned} y''(t) &\geq \frac{q(t)}{2^{\beta-1}} z^{\beta/\alpha}(t-\sigma_1) \\ &\geq \frac{q(t)}{2^{\beta-1}(1+a^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1) \\ &\geq \frac{q(t)}{2^{\beta-1}(1+a^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1+\tau_1), \end{aligned}$$

and once again $y(t)$ is a positive decreasing solution of the inequality (2.4), which is a contradiction.

Subcase (ii): Assume that $y'(t) > 0$ for all $t \geq t_2$. Then we have $y(t+) \leq (1+a^\beta - \frac{b^\beta}{2^{\beta-1}})z(t+\tau_2)$, and this with (2.8) implies

$$y''(t) \geq \frac{p(t)}{2^{\beta-1}} z^{\beta/\alpha}(t+\sigma_2) \geq \frac{p(t)}{2^{\beta-1}(1+a^\beta - \frac{b^\beta}{2^{\beta-1}})^{\beta/\alpha}} y^{\beta/\alpha}(t+\sigma_2-\tau_2).$$

That is, $y(t)$ is a positive increasing solution of the inequality (2.3), which is a contradiction. The proof is now complete. ■

Remark 2.1. Theorem 2.2 permits us to get various oscillation criteria for equation (1.1). Also we are able to study the asymptotic properties of solutions of equation (1.1) even if some of the assumptions of Theorem 2.2 are not satisfied. If the differential inequality (2.3) has eventually positive increasing solution then the conclusion of Theorem 2.2 will be replaced by "every solution $x(t)$ of equation (1.1) is either oscillatory or $x(t)$ tends to ∞ as $t \to \infty$."

Next we present a ready to verify conditions for the oscillation of all solutions of equation (1.1).

Corollary 2.3. *Let* $\sigma_i > \tau_i$ *for* $i = 1, 2, (1 + a^\alpha - \frac{b^\alpha}{2^{\alpha-1}}) > 0$, *and* $\beta = \alpha$. *If*

$$\limsup_{t\to\infty} \int_t^{t+\sigma_2-\tau_2} (s-t)p(s)ds > (1 + a^\alpha - \frac{b^\alpha}{2^{\alpha-1}})2^{\alpha-1}, \tag{2.9}$$

$$\limsup_{t\to\infty} \int_{t-\sigma_1+\tau_1}^{t} (t-s)q(s)ds > (1 + a^\alpha)2^{\alpha-1}, \tag{2.10}$$

and

$$\liminf_{t\to\infty} \int_{t-\sigma_1-\tau_1}^{t} (s - \sigma_1 - \tau_2)(p(s) + q(s))ds > \frac{2b^\alpha}{e} \tag{2.11}$$

then every solution of equation (1.1) *is oscillatory.*

Proof. Let $y(t)$ be a positive solution of (2.5), for $t \geq t_1 \geq t_0$. Then we have $y''(t) \leq 0$ for all $t \geq t_1$. Further $y'(t) > 0$ for all $t \geq t_1$, otherwise $y(t) \to -\infty$ as $t \to -\infty$. Hence we have $y(t) > 0$, $y'(t) > 0$ and $y''(t) \leq 0$, for $t \geq t_1$. Then we obtain

$$y(t) \geq \frac{t}{2}y'(t) \ \text{ for } \ \mathrm{t} \geq \mathrm{t}_2 \geq 2\mathrm{t}_1.$$

From (2.5) and the monotonicity of $y(t)$,we have

$$y''(t) + \frac{1}{b^\alpha}(p(t) + q(t))y(t - \sigma_1 - \tau_2) \leq 0, \quad \mathrm{t} \geq \mathrm{t}_2.$$

Combining the last two inequalities,we obtain

$$y''(t) + \frac{1}{2b^\alpha}(p(t) + q(t))(t - \sigma_1 - \tau_2)y'(t - \sigma_1 - \tau_2) \leq 0, \quad \mathrm{t} \geq \mathrm{t}_2.$$

Let $w(t) = y'(t)$. Then we see that $w(t)$ is a positive solution of

$$w'(t) + \frac{1}{2b^\alpha}(p(t) + q(t))(t - \sigma_1 - \tau_2)w(t - \sigma_1 - \tau_2) \leq 0, \quad \mathrm{t} \geq \mathrm{t}_2.$$

which is a contradiction by condition(2.11) and Theorem 2.1.1 in [15].Hence (2.5) has no eventually positive solution. More over condition (2.9) is sufficient for the inequality (2.3) to have no positive increasing solution and condition (2.10) is sufficient for the inequality (2.4) to have no positive decreasing solution,see [1, Lemma 2.2.12]. Then the proof follows from Theorem 2.2. ■

Next we consider the equation (1.2), and present sufficient conditions for the oscillation of all solutions.

Theorem 2.4. *Assume that $\sigma_i > \tau_i$ for $i = 1, 2$, $q(t)$ and $p(t)$ are positive and nondecreasing functions for $t \geq t_0$. If the differential inequality*

$$y''(t) - \frac{p(t)}{2^{\beta-1}} \frac{y^{\beta/\alpha}(t + \sigma_2 - \tau_2)}{(1 + b^\beta)^{\beta/\alpha}} \geq 0 \tag{2.12}$$

has no positive increasing solution, the differential inequality

$$y''(t) - \frac{q(t)}{2^{\beta-1}} \frac{y^{\beta/\alpha}(t - \sigma_1 + \tau_1)}{(1 + b^\beta)^{\beta/\alpha}} \geq 0 \tag{2.13}$$

has no positive decreasing solution, and the differential inequality

$$y''(t) + c_2 q(t) y^{\beta/\alpha}(t + \tau_1 - \sigma_1) + c_2 p(t) y^{\beta/\alpha}(t + \tau_1 + \sigma_2) \leq 0 \tag{2.14}$$

where $c_2 = \min\left\{\frac{1}{a^\beta}, \frac{1}{2^{\beta-1}}\left(\frac{2^{\beta-1}}{a^\beta}\right)^{\beta/\alpha}\right\}$, has no positive solution, then every solution of equation (1.2) is oscillatory.

Proof. Let $x(t)$ be a nonoscillatory solution of equation (1.2). Without loss of generality, we may assume that there exists a $t_1 \geq t_0$ such that $x(t - \theta) > 0$ for all $t \geq t_1$. By setting

$$z(t) = (x(t) - ax(t - \tau_1) + bx(t + \tau_2))^\alpha, t \geq t_1$$

we have $z''(t) = q(t)x^\beta(t-\sigma_1) + p(t)x^\beta(t+\sigma_2) > 0$ for all $t \geq t_1$. Therefore, both $z(t)$ and $z'(t)$ are of one sign for all $t \geq t_1$. We shall prove that $z(t) > 0$ for all $t \geq t_1$. If not, then $z(t) < 0$ and

$$\begin{aligned} 0 < u(t) = -z(t) &= (ax(t-\tau_1) - bx(t+\tau_2) - x(t))^\alpha \\ &\leq a^\alpha x^\alpha(t-\tau_1) \end{aligned}$$

which implies that

$$x^\beta(t) \geq \frac{u^{\beta/\alpha}(t+\tau_1)}{a^\beta} \geq c_2 u^{\frac{\beta}{\alpha}}(t+\tau_1) \text{ for all } t \geq t_1.$$

From equation (1.2), we obtain

$$\begin{aligned} 0 &= u''(t) + q(t)x^\beta(t-\sigma_1) + p(t)x^\beta(t+\sigma_2) \\ &\geq u''(t) + c_2 q(t)u^{\beta/\alpha}(t-\sigma_1+\tau_1) + c_2 p(t)u^{\beta/\alpha}(t+\tau_1+\sigma_2). \end{aligned}$$

Thus $u(t)$ is a positive solution of the inequality (2.14), which is a contradiction. Hence $z(t) > 0$ for all $t \geq t_1$. Now define a function $y(t)$ by

$$y(t) = z(t) - \frac{a^\beta}{2^{\beta-1}} z(t-\tau_1) + b^\beta z(t+\tau_2), \text{ for all } t \geq t_1. \tag{2.15}$$

Differentiating (2.15) twice, and using the equation (1.2), we obtain

$$\begin{aligned} y''(t) &= z''(t) - \frac{a^\beta}{2^{\beta-1}} z''(t-\tau_1) + b^\beta z''(t+\tau_2) \\ &= q(t)x^\beta(t-\sigma_1) + p(t)x^\beta(t+\sigma_2) - \frac{a^\beta}{2^{\beta-1}} q(t-\tau_1)x^\beta(t-\tau_1-\sigma_1) \\ &\quad - \frac{a^\beta}{2^{\beta-1}} p(t-\tau_1)x^\beta(t-\tau_1+\sigma_2) + b^\beta q(t+\tau_2)x^\beta(t+\tau_2-\sigma_1) \\ &\quad + b^\beta p(t+\tau_2)x^\beta(t+\tau_2+\sigma_2). \end{aligned}$$

Using the monotonicity of $q(t)$ and $p(t)$ in the above inequality, we obtain

$$\begin{aligned} y''(t) &\geq q(t)\left[x^\beta(t-\sigma_1) - \frac{a^\beta}{2^{\beta-1}}x^\beta(t-\tau_1-\sigma_1) + b^\beta x^\beta(t+\tau_2-\sigma_1)\right] \\ &\quad + p(t)\left[x^\beta(t+\sigma_2) - \frac{a^\beta}{2^{\beta-1}}x^\beta(t-\tau_1+\sigma_2) + b^\beta x^\beta(t+\tau_2+\sigma_2)\right]. \end{aligned} \tag{2.16}$$

Now using the inequalities (2.1), (2.2) and $z(t) > 0$ for all $t \geq t_1$, we obtain

$$y''(t) \geq \frac{q(t)}{2^{\beta-1}} z^{\beta/\alpha}(t-\sigma_1) + \frac{p(t)}{2^{\beta-1}} z^{\beta/\alpha}(t+\sigma_2) > 0, t \geq t_1. \tag{2.17}$$

Therefore, both $y(t)$ and $y'(t)$ are of one sign eventually. We prove that $y(t) > 0$, eventually. If not, then $y(t) < 0$ and

$$0 < v(t) = -y(t) = \frac{a^\beta}{2^{\beta-1}} z(t-\tau_1) - b^b z(t+\tau_2) - z(t) \leq \frac{a^\beta}{2^{\beta-1}} z(t-\tau_1).$$

Hence $z(t) \geq \dfrac{2^{\beta-1}}{a^\beta} v(t+\tau_1)$. Using the last inequality in (2.17), we obtain

$$0 \geq v''(t) + \frac{q(t)}{2^{\beta-1}}\left(\frac{2^{\beta-1}}{a^\beta}\right)^{\beta/\alpha} v^{\beta/\alpha}(t+\tau_1-\sigma_1) + \frac{p(t)}{2^{\beta-1}}\left(\frac{2^{\beta-1}}{a^\beta}\right)^{\beta/\alpha} v^{\beta/\alpha}(t+\tau_1+\sigma_2).$$

or

$$v''(t) + c_2 q(t) v^{\beta/\alpha}(t+\tau_1-\sigma_1) + c_2 p(t) v^{\beta/\alpha}(t+\tau_1+\sigma_2) \leq 0, t \geq t_1.$$

Thus $v(t)$ is a positive solution of the inequality (2.14),a contradiction. Hence $y(t) > 0$, eventually. Now we consider the following two cases.

Case:1 Assume that there exists a t_2 such that $z'(t) < 0$ for all $t \geq t_2 \geq t_1$. Then we prove that $y'(t) < 0$. Suppose $y'(t) > 0$. Then $y(t) > 0,\ y'(t) > 0$ and $y''(t) > 0$ imply that $\lim_{t\to\infty} y(t) = \infty$. On the other hand, $z(t) > 0$ and $z'(t) < 0$ imply that $\lim_{t\to\infty} z(t) = c < \infty$. Letting $t \to \infty$ on both sides of (2.15), we obtain a contradiction. Hence $y'(t) < 0$ for all $t \geq t_2$. Now using the monotonicity of $z(t)$, we get

$$\begin{aligned} y(t) &= z(t) - \frac{a^\beta}{2^{\beta-1}} z(t-\tau_1) + b^\beta z(t+\tau_2) \\ &\leq z(t) + b^\beta z(t+\tau_2) \leq (1+b^\beta) z(t). \end{aligned}$$

Using the last inequality in (2.17) and the monotonicity of $y(t)$, we have

$$y''(t) \geq \frac{q(t)}{2^{\beta-1}(1+b^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1) \geq \frac{q(t)}{2^{\beta-1}(1+b^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1+\tau_1).$$

Thus $y(t)$ is a positive decreasing solution of the inequality (2.13), a contradiction.

Case:2 Let $z'(t) > 0$ for all $t \geq t_2 \geq t_1$. Now we consider the following two subcases.

Subcases (i): Assume that $y'(t) < 0$ for all $t \geq t_2$. Then proceeding as in Case 1 and using the monotonicity of $z(t)$, we obtain

$$y(t) = z(t) - \frac{a^\beta}{2^{\beta-1}} z(t-\tau_1) + b^\beta z(t+\tau_2) \leq (1+b^\beta) z(t+\tau_2).$$

Using last inequality in (2.17) and the monotonicity of $y(t)$, we get

$$y''(t) \geq \frac{q(t)}{2^{\beta-1}(1+b^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1-\tau_2) \geq \frac{q(t)}{2^{\beta-1}(1+b^\beta)^{\beta/\alpha}} y^{\beta/\alpha}(t-\sigma_1+\tau_1).$$

Thus $y(t)$ is a positive decreasing solution of the inequality (2.13), a contradiction.

Subcases (ii): Assume that $y'(t) > 0$ for all $t \geq t_2$. Then using the monotonicity of $z(t)$, we have

$$\begin{aligned} y(t) &= z(t) - \frac{a^\beta}{2^{\beta-1}} z(t-\tau_1) + b^\beta z(t+\tau_2) \\ &\leq z(t) + b^\beta z(t+\tau_2) \leq (1+b^\beta) z(t+\tau_2). \end{aligned}$$

Using the last inequality in (2.17), we obtain

$$y''(t) \geq \frac{p(t)}{2^{\beta-1}} \frac{y^{\beta/\alpha}(t+\sigma_2-\tau_2)}{(1+b^\beta)^{\beta/\alpha}}.$$

Therefore $y(t)$ is a positive increasing solution of the inequality (2.12), a contradiction. This completes the proof. ■

Remark 2.2. *Theorem 2.4 permits us to get various oscillation criteria for equation* (1.2). *Also we are able to study the asymptotic behavior of solutions of* (1.2) *if some of the assumptions of Theorem 2.4 are not satisfied.*

Corollary 2.5. *Let* $\sigma_i > \tau_i$, *for* $i = 1, 2$, *and* $\beta = \alpha$. *Assume*

$$\limsup_{t\to\infty} \int_t^{t+\sigma_2-\tau_2} (s-t)p(s)ds > (1+b^\alpha)2^{\alpha-1}, \tag{2.18}$$

$$\limsup_{t\to\infty} \int_{t-\sigma_1+\tau_1}^{t} (t-s)q(s)ds > (1+b^\alpha)2^{\alpha-1}, \tag{2.19}$$

and

$$\liminf_{t\to\infty} \int_{t+\tau_1-\sigma_1}^{t} (s+\tau_1-\sigma_1)(p(s)+q(s))ds > \frac{2a^\alpha}{e}. \tag{2.20}$$

Then every solution of equation (1.2) *is oscillatory.*

Proof. The proof is similar to that of Corollary 2.3, and hence the details are omitted. ∎

3. Examples

Now we present some examples to illustrate the main results.

Example 3.1. *Consider the differential equation*

$$\left(\left(x(t)+\frac{1}{2}x(t-1)-\frac{1}{3}x(t+1)\right)^3\right)'' = \frac{231}{36(t-3)^2}x^3(t-3)+\frac{28}{9(t+4)^2}x^3(t+4), t \geq 4. \tag{3.1}$$

Here $a = \frac{1}{2}$, $b = \frac{1}{3}$, $\alpha = \beta = 3, \tau_1 = \tau_2 = 1, \sigma_1 = 3, \sigma_2 = 4$, $q(t) = \frac{231}{36(t-3)^2}$, *and* $p(t) = \frac{28}{9(t+4)^2}$. *Then it is easy to check that condition* (2.9) *of Corollary 2.3 is not satisfied. Therefore equation* (3.1) *has a nonoscillatory solution. In fact* $x(t) = t$ *is one such nonoscillatory solution, since it satisfies the equation* (3.1).

Example 3.2. *Consider the differential equation*

$$\left(x(t)+\frac{3}{2}x(t-\pi/2)-\frac{1}{2}x(t+\pi)\right)''=\frac{3}{2}x(t-3\pi)+\frac{3}{2}x(t+5\pi/2),\ t\geq 7\pi. \quad (3.2)$$

Here $a=\frac{3}{2},\ b=\frac{1}{2},\ \tau_1=\pi/2,\tau_2=\pi,\ \sigma_1=3\pi,\ \sigma_2=5\pi/2,\ q(t)=p(t)=\frac{3}{2}$ *and* $\alpha=\beta=1$. *Then one can see that all the conditions of Corollary 2.3 are satisfied. Hence all the solutions of equation* (3.2) *are oscillatory. In fact* $x(t)=sint$ *is one such solution of equation* (3.2)*, since it satisfies the equation* (3.2)*.*

Example 3.3. *Consider the differential equation*

$$\left(\left(x(t)+e^{\tau_1}x(t-\tau_1)-e^{-\tau_2}x(t+\tau_2)\right)^3\right)''=\frac{9e^{3\sigma_1}}{2}x^3(t-\sigma_1)+\frac{9e^{-3\sigma_2}}{2}x^3(t+\sigma_2),t\geq t_0, \quad (3.3)$$

with $\sigma_1>\tau_1$ *and* $\sigma_2>\tau_2$*.Here* $a=e^{\tau_1},\ b=e^{-\tau_2},\ \alpha=\beta=3,\ q(t)=\frac{9e^{3\sigma_1}}{2},\ p(t)=\frac{9e^{-3\sigma_2}}{2}$. *Then one can see that all conditions of Corollary 2.3 are satisfied except condition (2.9). Therefore all the solutions of equation* (3.3) *are not necessarily oscillatory. In fact* $x(t)=e^t$ *is one such nonoscillatory solution, since it satisfies equation* (3.3) *.*

Example 3.4. *Consider the differential equation*

$$\left((x(t)-ax(t-\pi)+bx(t+2\pi))^3\right)''=qx^3\left(t-3\pi/2\right)+px^3\left(t+5\pi/2\right),t\geq 3\pi. \quad (3.4)$$

Here $a=\frac{1}{2}e^{\pi/3},\ b=\frac{3}{2}e^{-\pi/3},\ \tau_1=\pi,\ \tau_2=\pi,\ \sigma_1=3\pi/2,\ \sigma_2=3\pi/2,\ q=(8e^{3\pi}+27e^{2\pi}),p=(8+27e^{-\pi}),$ *and* $\alpha=\beta=3$. *Then one can easily verify that all the conditions of Corollary 2.5 are satisfied. Hence all the solutions of equation* (3.4) *are oscillatory. In fact* $x(t)=e^{t/3}sin^{1/3}t$ *is one such oscillatory solution of equation* (3.4)*.*

Example 3.5. *Consider the differential equation*

$$\left(\left(x(t) - e^{-\tau_1}x(t-\tau_1) + e^{\tau_2}x(t+\tau_2)\right)^5\right)'' = \frac{25}{2}e^{-5\sigma_1}x^5(t-\sigma_1) + \frac{25}{2}e^{5\sigma_2}x^5(t+\sigma_2) \tag{3.5}$$

with $\sigma_1 > \tau_1$ *and* $\sigma_2 > \tau_2$*. Here* $a = e^{-\tau_1}$, $b = e^{\tau_2}$, $q(t) = \dfrac{25}{2}e^{-5\sigma_1}$, $p(t) = \dfrac{25}{2}e^{5\sigma_2}$, *and* $\alpha = \beta = 5$*. Condition* (2.19) *of Corollary 2.5 is not satisfied. Therefore all the solutions of equation* (3.4) *are not necessarily oscillatory. In fact* $x(t) = e^{-t}$ *is one such nonoscillatory solution, since it satisfies the equation* (3.5)*.*

We conclude this paper with the following remark.

Remark: It would be interesting to study the oscillatory behavior of all solutions of equations (1.1) and (1.2) when $\beta < 1$.

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Ramanujan Institute for Advanced Study in Mathematics, University of Madras, Chennai -600 005, India.

E-mail address: ethandapani@yahoo.co.in [E.Thandapani]

E-mail address: renurama68@gmail.com [R.Rama]