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HARTMAN-TYPE COMPARISON THEOREMS FOR HALF-LINEAR DIFFERENTIAL EQUATIONS OF THE SECOND ORDER

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Abstract. Comparison theorem of the Hartman type for a continuous family of nonlinear differential equations of the form

$$\big(p(t,\lambda)\varphi(u')\big)' + q(t,\lambda)\varphi(u) = 0, \quad \lambda \geq 0, \tag{E$_\lambda$}$$

where $p \in \mathrm{C}([a,b]\times[0,\infty),(0,\infty)),\ q \in \mathrm{C}([a,b]\times[0,\infty),\mathbb{R}), i = 1,...,n$, and $\varphi(s) := |s|^{\alpha-1}s$ for $s \neq 0$ and $\varphi(0) = 0$, is proved with the help of the generalized Mingarelli's identity.

Key words and phrases. Mingarelli's identity, half-linear differential equation, Sturm comparison.

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1 Introduction

Suppose that $p(t,\lambda)$ and $q(t,\lambda)$ are continuous real-valued functions with $p(t,\lambda) > 0$ on $[a,b]\times[0,\infty)$ and consider a continuous family of nonlinear differential equations of the Sturm-Liouville type

$$\big(p(t,\lambda)\varphi(u')\big)' + q(t,\lambda)\varphi(u) = 0, \quad a \leq t \leq b, \quad \lambda \geq 0, \tag{E$_\lambda$}$$

where $\varphi(s) := |s|^{\alpha-1}s$ for $s \neq 0$ and $\varphi(0) = 0$. Let $u = u(t,\lambda)$ be the solution of (E$_\lambda$) satisfying the initial conditions

$$u(a,\lambda) = 1, \quad v(a,\lambda) = c(\lambda), \quad \lambda \geq 0, \tag{A$_\lambda$}$$

where $v(t,\lambda)$ denotes the function $p(t,\lambda)\varphi\big(u'(t,\lambda)\big)$ and $c(\lambda)$ is given continuous function on $[0,\infty)$.

In the theory of half-linear differential equations the Sturm comparison theorems are usually formulated in terms of two differential equations

$$\big(p_0(t)\varphi(u_0')\big)' + q_0(t)\varphi(u) = 0, \quad a \le t \le b, \tag{E_0}$$

$$u_0(a) = 1, \quad p_0(a)\varphi(u_0')(a) = c_0,$$

and

$$\big(p_1(t)\varphi(u_1')\big)' + q_1(t)\varphi(u) = 0, \quad a \le t \le b, \tag{E_1}$$

$$u_1(a) = 1, \quad p_1(a)\varphi(u_1')(a) = c_1,$$

where it is assumed that

$$p_0(t) \ge p_1(t), \quad q_0(t) < q_1(t) \quad \text{on} \quad a \le t \le b, \tag{1.1}$$

and

$$c_1 \le c_0. \tag{1.2}$$

Comparison theorems for pairs of differential equations (E_0) and (E_1) were obtained by several authors including Elbert [2], Mirzov [9], Li and Yeh [7] and the present author et al. [4-5]. For other references and various qualitative aspects concerning differential equations of the above form see the monograph [1].

It is easy to see that (E_0) and (E_1) can be embedded in a continuous family of equations (E_λ), $\lambda \ge 0$, if we put

$$p(t,\lambda) = (1-\lambda)p_0(t) + \lambda p_1(t), \quad q(t,\lambda) = (1-\lambda)q_0(t) + \lambda q_1(t),$$

and

$$c(0) = c_0, \quad c(1) = c_1.$$

The basic Sturm's comparison results when re-formulated for a family of nonlinear equations (E_λ) read as follows.

Theorem A. (FUNDAMENTAL COMPARISON THEOREM OF STURM) *Suppose that for any fixed $t \in [a,b]$ the function $p(t,\lambda)$ is non-increasing and $q(t,\lambda)$ is strictly increasing in λ on $[0,\infty)$. If, for some $0 \le \lambda_1 < \lambda_2$, $u_1 = u(t,\lambda_1)$ is the solution of (E_{λ_1}) with consecutive zeros at $t = c$ and $t = d$, then for any $\lambda_2 > \lambda_1$, the solution $u_2 = u(t,\lambda_2)$ has a zero in (c,d).*

Theorem B. (FIRST COMPARISON THEOREM OF STURM) *Let, for any fixed $t \in [a,b]$, $p(t,\lambda)$ and $q(t,\lambda)$ as functions of λ be non-increasing and strictly increasing on $[0,\infty)$, respectively. Let $c(\lambda)$ be a non-increasing function on $[0,\infty)$. If, for some $\lambda_1 \ge 0$, the solution of the initial value problem (E_λ)-(A_λ) has exactly $m \ge 1$ zeros $t_j(\lambda_1), j = 1,...,m$, with*

$$a < t_1(\lambda_1) < ... < t_m(\lambda_1) \le b, \tag{1.3}$$

then, for any $\lambda_2 > \lambda_1$, the solution $u(t,\lambda_2)$ of (E_λ)-(A_λ) has $r \ge m$ zeros $t_k(\lambda_2), k = 1,...,r$, with

$$a < t_1(\lambda_2) < ... < t_r(\lambda_2) \le b, \tag{1.4}$$

and $t_j(\lambda_2) < t_j(\lambda_1)$ *for* $j = 1, ..., m$.

Theorem C. (SECOND COMPARISON THEOREM OF STURM) *Let the assumptions of Theorem* B *be satisfied. Suppose that there exists a value* $t_0 \in (a, b]$ *such that* $u(t_0, \lambda) \neq 0$ *for* $0 \leq \lambda_1 < \lambda < \lambda_2$ *and all* $u(t, \lambda), \lambda \in (\lambda_1, \lambda_2)$*, have the same number of zeros in* (a, t_0)*. Then the function*

$$p(t_0, \lambda)\varphi(u'(t_0, \lambda)/u(t_0, \lambda))$$

is a strictly decreasing function of λ *on* (λ_1, λ_2).

Proofs of the above theorems can be done easily with the help of the generalized Picone's identity which states that if, for some values of parameter $\lambda_1 < \lambda_2$, $u_i = u(t, \lambda_i)$, $i = 1, 2$, are respective solutions of (E_{λ_i}) in an interval I and $u_2(t) \neq 0$ in I, then

$$\frac{d}{dt}\left[\frac{u_1}{\varphi(u_2)}\big[\varphi(u_1)p_2\varphi(u_2') - \varphi(u_2)p_1\varphi(u_1')\big]\right] = (p_2 - p_1)|u_1'|^{\alpha+1} - (q_2 - q_1)|u_1|^{\alpha+1}$$
$$-p_2\big[|u_1'|^{\alpha+1} + \alpha|u_1u_2'/u_2|^{\alpha+1} - (\alpha+1)u_1'\varphi(u_1u_2'/u_2)\big]. \tag{1.5}$$

where $p_i = p(t, \lambda_i)$ and $q_i = q(t, \lambda_i)$ for $i = 1, 2$ (see [4-5]).

The purpose of this article is to prove analogues of the Sturm's comparison theorems for a family of half-linear differential equations with a parameter λ by comparing three underlying equations (instead of two), and replacing the assumption of monotonicity (in λ) of the coefficient functions $p(t, \lambda)$ and $q(t, \lambda)$, and the initial function $c(\lambda)$ by the assumption that for any fixed t the function $p(t, \lambda)$ is concave, $q(t, \lambda)$ is strictly convex in λ and $c(\lambda)$ is convex on $[0, \infty)$.

The main tool utilized in this work is a "half-linear" generalization of the identity obtained by A. Mingarelli [8] (see also Kuks [6]) in his extension of multiple comparison principle of the Sturm type developed by P. Hartman [3].

2 Generalized Mingarelli's identity

Let $p(t, \lambda)$ and $q(t, \lambda)$ be continuous functions on $[a, b] \times [0, \infty)$ with $p(t, \lambda) > 0$ and $\varphi(s) := |s|^{\alpha-1}s$ for $s \neq 0$ and $\varphi(0) = 0$, and consider nonlinear ordinary differential operators of the form

$$L_\lambda[x] = (p(t, \lambda)\varphi(x'))' + q(t, \lambda)\varphi(x), \quad \lambda \geq 0,$$

with domains $\mathcal{D}_{L_\lambda}$ defined to be the sets of all functions $x(t, \lambda)$ which are continuous on $[a, b] \times [0, \infty)$ and continuously differentiable with respect to t on (a, b) together with $p(t, \lambda)\varphi(x')$ for every fixed $\lambda \geq 0$.

Also, denote by Φ_α the form defined for $X, Y \in \mathbb{R}$ and $\alpha > 0$ by

$$\Phi_\alpha(X, Y) := |X|^{\alpha+1} + \alpha|Y|^{\alpha+1} - (\alpha + 1)X\varphi(Y).$$

From the Young inequality it follows that $\Phi_\alpha(X, Y) \geq 0$ for all $X, Y \in \mathbb{R}$ and the equality holds if and only if $X = Y$.

For a function $f(t, \lambda)$ defined on $[a, b] \times [0, \infty)$ and λ-values $0 \leq h_1 < h_2 < ...$ define

$$\Delta_i^0 f = f(t, h_i), \quad \Delta_i^1 f = f(t, h_{i+1}) - f(t, h_i)$$

and

$$\Delta_i^m f = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(t, h_{i+j}).$$

Also, for given $0 \leq h_1 < h_2 < ... < h_n$ and $t \in [a, b]$, put

$$p_i = p(t, h_i), \quad q_i = q(t, h_i), \quad x_i = x(t, h_i) \quad \text{and} \quad x_i' = \frac{\partial}{\partial t} x(t, h_i)$$

The following lemma which is the main tool in this paper can be verified easily by a direct computation.

Lemma 2.1 *Let x_i and $p_i\varphi(x_i), i = 1,,,,n$, be continuously differentiable functions on an interval I. Then, for $1 \leq m < n$ and $1 \leq i \leq n - m$,*

$$\frac{d}{dt}\left[|x_{m+i-1}|^{\alpha+1} \Delta_i^m \big(p\varphi(x'/x)\big)\right] = |x_{m+i-1}|^{\alpha+1} \Delta_i^m \left[\frac{(p\varphi(x'))'}{\varphi(x)}\right] \tag{2.1}$$

$$+ |x'_{m+i-1}|^{\alpha+1} \Delta_i^m p - \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} p_{i+j} \Phi_\alpha\big(x'_{m+i-1}, \frac{x_{m+i-1}}{x_{i+j}} x'_{i+j}\big),$$

provided that $x_{i+j}(t)$ with $j \neq m - 1$ do not vanish in I.

If, for fixed values $0 \leq h_1 < h_2 ... < h_n$, $x_i = u_i = u(t, h_i), i = 1, ..., n$, are respective solutions of half-linear differential equations (E_{h_i}), then (2.1) reduces to

$$\frac{d}{dt}\left[|u_{m+i-1}|^{\alpha+1} \Delta_i^m \big(p\varphi(u'/u)\big)\right] = |u'_{m+i-1}|^{\alpha+1} \Delta_i^m p - |u_{m+i-1}|^{\alpha+1} \Delta_i^m q$$

$$- \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} p_{i+j} \Phi_\alpha\big(u'_{m+i-1}, \frac{u_{m+i-1}}{u_{i+j}} u'_{i+j}\big). \tag{2.2}$$

If $\alpha = 1, m = n - 1$ and $i = 1$, then from (2.2) we obtain Mingarelli's identity

$$\frac{d}{dt}\left[u_{n-1}^2 \Delta_1^{n-1}(pu'/u)\right] = (u'_{n-1})^2 \Delta_1^{n-1} p - u_{n-1}^2 \Delta_1^{n-1} q \tag{2.3}$$

$$- \sum_{i=0}^{n-1} \binom{n-1}{i} (-1)^{n-i-1} p_{i+1} \big(u'_{n-1} - \frac{u_{n-1}}{u_{i+1}} u'_{i+1}\big)^2.$$

If $m = 1$, then for $i = 1, ..., n - 1$ we get the $(n - 1)$-tuple of generalized Picone's identities

$$\frac{d}{dt}\big[|u_i|^{\alpha+1}\Delta_i^1\big(p\varphi(u'/u)\big)\big] \tag{2.4}$$
$$= |u_i'|^{\alpha+1}\Delta_i^1 p - |u_i|^{\alpha+1}\Delta_i^1 q - p_{i+1}\Phi_\alpha\big(u_i', \frac{u_i}{u_{i+1}}u_{i+1}'\big),$$

derived by the present author et al. in [4-5] and, finally, if $\alpha = 1, m = 1$ and $i = 1$, then (2.4) reduces to the classical Picone's formula

$$\frac{d}{dt}\left[\frac{u_1}{u_2}(u_1 p_2 u_2' - u_2 p_1 u_1')\right] = (p_2 - p_1)(u_1')^2 - (q_2 - q_1)u_1^2 - p_2\big(u_1' - \frac{u_1}{u_2}u_2'\big)^2 \tag{2.5}$$

(see [10]). Development of the qualitative theory of linear and half-linear differential equations in the last decades has proven that the identities (2.4) and (2.5) are very useful tools in obtaining comparison theorem, uniqueness, factorization of operators and bounds for eigenvalues for equations under study, and they have been generalized and extended to various classes of ordinary and partial differential equations of the second and the higher (even) orders (see [1]).

3 Comparison theorems

An analogue of the Sturm's fundamental comparison theorem is the following result.

THEOREM 3.1. (FUNDAMENTAL COMPARISON THEOREM) *Suppose that for any fixed t the function $p(t,\lambda)$ is concave and $q(t,\lambda)$ is strictly convex in λ on $[0,\infty)$. If, for some $\lambda_2 > 0$, $u_2 = u(t,\lambda_2)$ is the solution of (E_{λ_2}) with consecutive zeros at $t = c$ and $t = d$, then for any $\varepsilon > 0$ with $\lambda_2 - \varepsilon > 0$, at least one of the solutions $u_1 = u(t,\lambda_2 - \varepsilon)$ and $u_3 = u(t,\lambda_2 + \varepsilon)$ has a zero in (c,d).*

Remark 3.1. If the function $p(t,\lambda)$ is concave in λ for a fixed $t \in [a,b]$ and $h_i = h_1 + (i-1)\delta \geq 0$ for $i = 1,2,3$ and $\delta > 0$, then

$$\Delta_1^2 p \equiv p(t,h_3) - 2p(t,h_2) + p(t,h_1) \leq 0. \tag{3.1}$$

(the so-called midpoint concavity property).

Similarly, if $q(t,\lambda)$ is strictly convex in λ on $[0,\infty)$ for a fixed $t \in [a,b]$ and $h_i = h_1 + (i-1)\delta \geq 0, i = 1,2,3, \delta > 0$, then

$$\Delta_1^2 q \equiv q(t,h_3) - 2q(t,h_2) + q(t,h_1) > 0. \tag{3.2}$$

Proof of Theorem 3.1. Suppose that for some value of parameter $\lambda_2 > 0$ the solution $u_2 = u(t,\lambda_2)$ has consecutive zeros at $t = c$ and $t = d$, but there is an $\varepsilon_0 > 0$ with $\lambda_2 - \varepsilon_0 > 0$, such that none of $u_1 = u(t,\lambda_2 - \varepsilon_0)$ and $u_3 = u(t,\lambda_2 + \varepsilon_0)$ vanish in $[c,d]$. Then, integrating the identity (2.2) with $n = 3, m = 2, i = 1, h_1 = \lambda_2 - \varepsilon_0, h_2 = \lambda_2$ and $h_3 = \lambda_2 + \varepsilon_0$ over $[c,d]$, we obtain

$$\big[|u_2|^{\alpha+1}\Delta_1^2\big(p\varphi(u'/u)\big)\big]_c^d \tag{3.3}$$

$$= \int_c^d \big[|u_2'|^{\alpha+1}\Delta_1^2 p - |u_2|^{\alpha+1}\Delta_1^2 q - p_1\Phi_\alpha\big(u_2', \frac{u_2}{u_1}u_1'\big) - p_3\Phi_\alpha\big(u_2', \frac{u_2}{u_3}u_3'\big)\big]\,dt.$$

The left-hand side of (3.3) is zero, while the integral on the right-hand side is positive. This contradiction proves that at least one of $u_1 = u(t, \lambda_2 - \varepsilon_0)$ and $u_3 = u(t, \lambda_2 + \varepsilon_0)$ must have a zero in (c, d).

Remark 3.2. The theorem remains true if u_1 and/or u_3 are zero at one or both of the end-points of the interval $[c, d]$. Let, for example, $u_1(c) = u_3(c) = 0$. Then, due to the fact that zeros of nontrivial solutions of half-linear differential equations of the second order are simple (see [7, Lemma 2.3]), $u_1'(c)$ and $u_3'(c)$ must be nonzero finite values. Then the functions $\varphi(u_i/u_2), i = 1, 3$, have at $t = c$ the nonzero finite limits equal to $\varphi\big(\lim_{t\to c+}(u_i'/u_2')\big)$ by l'Hospital rule and since, obviously, $\lim_{t\to c+} p_i(t)u_2(t)\varphi\big(u_i'(t)\big) = 0, i = 1, 3$, it follows that

$$\lim_{t\to c+} |u_2(t)|^{\alpha+1}\Delta_1^2\big(p\varphi(u'/u)\big)(t) = 0.$$

THEOREM 3.2. (FIRST COMPARISON THEOREM) *Let, for any fixed $t \in [a, b]$, $p(t, \lambda)$ and $q(t, \lambda)$ as functions of λ be concave and strictly convex on $[0, \infty)$, respectively. Let $c(\lambda)$ be convex on $[0, \infty)$. If, for some $\lambda_0 \geq 0$ and $\varepsilon > 0$, the initial value problem* (E_λ)-(A_λ) *has the solutions $u_0 = u(t, \lambda_0)$ and $u_1 = u(t, \lambda_1), \lambda_1 = \lambda_0 + \varepsilon$, such that*

$$u(t, \lambda_0) > 0 \quad on \quad a \leq t \leq b, \tag{3.4}$$

and $u(t, \lambda_1)$ has exactly $m \geq 1$ zeros $t_j(\lambda_1),\ j = 1, ..., m$, with

$$a < t_1(\lambda_1) < ... < t_m(\lambda_1) \leq b, \tag{3.5}$$

then, for $\lambda_2 = \lambda_0 + 2\varepsilon$, the solution $u(t, \lambda_2)$ of (E_λ)-(A_λ) *has $r \geq m$ zeros $t_k(\lambda_2),\ k = 1, ..., r$, with*

$$a < t_1(\lambda_2) < ... < t_r(\lambda_2) \leq b, \tag{3.6}$$

and $t_j(\lambda_2) < t_j(\lambda_1)$ for $j = 1, ..., m$.

Proof. Let $t_j(\lambda_1), j = 1, ..., m$, be the zeros of $u(t, \lambda_1)$ satisfying (3.5). By Theorem 3.1 and the assumption (3.4), between each pair of consecutive zeros $t_j(\lambda_1)$ and $t_{j+1}(\lambda_1)$ there is at least one zero of $u(t, \lambda_2)$. It remains to prove that at least one zero of $u(t, \lambda_2)$ lies also between a and $t_1(\lambda_1)$.

Suppose this is not true. Then, integrating the identity (2.2) (with $n = 3, m = 2, i = 1, h_j = \lambda_0 + (j-1)\varepsilon, j = 1, 2, 3$, and u_j replaced by u_{j-1}) over $[a, t_1(\lambda_1)]$, we get

$$\big[|u_1|^{\alpha+1}\Delta_1^2\big(p\varphi(u'/u)\big)\big]_a^{t_1} \tag{3.7}$$

$$= \int_a^{t_1} \big[|u_1'|^{\alpha+1}\Delta_1^2 p - |u_1|^{\alpha+1}\Delta_1^2 q - p_0\Phi_\alpha\big(u_1', \frac{u_1}{u_0}u_0'\big) - p_2\Phi_\alpha\big(u_1', \frac{u_1}{u_2}u_2'\big)\big]\,dt.$$

Since the assumption of convexity of $c(\lambda)$ implies

$$\Delta_1^2 c \equiv c(h_3) - 2c(h_2) + c(h_1) \geq 0 \tag{3.8}$$

for $h_j = \lambda_0 + (j-1)\varepsilon,\ j = 1, 2, 3$, the left-hand side of (3.7) is non-positive, while the right-hand side is positive. This is a contradiction, and so $u(t, \lambda_2)$ must have at least one zero between a and $t_1(\lambda_1)$.

Theorem 3.3. (SECOND COMPARISON THEOREM) *Let the assumptions of Theorem 3.2 be satisfied. Suppose that there exists a value $t_0 \in (a, b]$ such that $u(t_0, \lambda) \neq 0$ for $0 \leq \lambda_1 < \lambda < \lambda_2$ and $u(t, \lambda), \lambda \in (\lambda_1, \lambda_2)$, have the same number $m \geq 1$ of zeros in (a, t_0). Then the function $p(t_0, \lambda)\varphi(u'(t_0, \lambda)/u(t_0, \lambda))$ is strictly convex on (λ_1, λ_2) in the sense that for any $h_i = h_1 + (i-1)\delta \in (\lambda_1, \lambda_2), \delta > 0, i = 1, 2, 3$,*

$$\Delta_1^2(p\varphi(u'/u))(t_0) \equiv \frac{p_3(t_0)\varphi(u_3'(t_0))}{\varphi(u_3(t_0))} - 2\frac{p_2(t_0)\varphi(u_2'(t_0))}{\varphi(u_2(t_0))} + \frac{p_1(t_0)\varphi(u_1'(t_0))}{\varphi(u_1(t_0))} > 0, \tag{3.9}$$

where, as before, $p_i(t_0) = p(t_0, h_i)$ and $u_i = u(t, h_i), i = 1, 2, 3$.

Proof. Let $h_i = h_1 + (i-1)\delta \in (\lambda_1, \lambda_2), i = 1, 2, 3, \delta > 0$, be fixed and let t_m be the zero next before t_0. Then it must be a zero of $u_2 = u(t, h_2)$ and not of $u_3 = u(t, h_3)$, because between a and t_m there are not less than m (and by assumption exactly m) zeros of $u(t, h_3)$. The formula (2.2) (with $n = 3, m = 1$ and $i = 1$) integrated between t_m and t_0 shows that

$$\Big[|u_2|^{\alpha+1}\big(\Delta_1^2\big(p\varphi(u'/u)\big)\Big]_{t_m}^{t_0} > 0, \tag{3.10}$$

from which the desired inequality readily follows. If $u(t, \lambda), \lambda_1 < \lambda < \lambda_2$, have no zeros in the interval (a, t_0), then the proof of the theorem can be done in a similar way by integrating the identity (2.2) between a and t_0.

Remark 3.3. As in the classical linear Sturmian theory, under some additional conditions, Theorems 3.1-3.3 can be used to prove that the eigenvalue problem consisting of the differential equation (E_λ) and the two-point boundary conditions

$$u'(a) - c(\lambda)u(a) = 0, \quad u'(b) - d(\lambda)u(b) = 0,$$

depending on parameter λ, where $c(\lambda)$ is convex and $d(\lambda)$ is concave on $[0, \infty)$, has an increasing sequence of eigenvalues $0 < \mu_1 < \lambda_1 < \mu_2 < \lambda_2 < \ldots$, and that the k-th eigenfunction has exactly k zeros in the interval (a, b). The results of this sort will be the subject of the forthcoming paper.

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