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Discretization of Poincaré map

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Abstract

We analytically study the relationship between the Poincaré map and its one step discretization. Error estimates are established depending basically on the right hand side function of the investigated ODE and the given numerical scheme. Our basic tool is a parametric version of a Newton–Kantorovich type methods. As an application, in a neighborhood of a non-degenerate periodic solution a new type of step-dependent, uniquely determined, closed curve is detected for the discrete dynamics.

Keywords: discrete Poincaré map; Newton–Kantorovich Theorem; periodic solutions.

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1 Introduction

This paper is devoted to the precise analytical derivation of the numerical/discretized Poincaré map of an ordinary differential equation possessing a periodic orbit. We have been motivated by papers [11, 19], where numerical tools are used for computing the Poincaré map. On the other hand there is a nice theory studying dynamics of numerical approximations of ODE, see for instance [6–9, 17, 18]. This paper is a contribution to this direction.

The continuous Poincaré map $\mathcal{P}$ for the smooth ODE with a 1-periodic orbit γ is a well understood topic and is contained in almost every textbook on continuous dynamical systems (e.g. [14]). In order to define the discretized version of Poincaré map, designated by $\mathcal{P}_m$, for the discrete dynamical system obtained from the one-step discretization procedure ψ we have chosen a method originated in [11] (m is the number of steps realized by the discretization scheme). Our goal is to give a precise analytical meaning of $\mathcal{P}_m$ and to establish various error bounds between $\mathcal{P}$ and $\mathcal{P}_m$. It has to be noted that there are various possibilities how to define $\mathcal{P}_m$. Our approach is in some sense a natural one, it can be loosely summed up as: applying recurrently ψ with a constant step-size until the resulting elements are located on the "one side" of the Poincaré section and then establishing the suitable step-size needed to hit by ψ exactly that section. Precise setting and the corresponding analysis are treated in Section 2 and 3 (there arises a slight complication forcing us to assume $p \geq 2$ for the order p of ψ – see Remark 2 in Section 3). Error bounds related to $|\mathcal{P} - \mathcal{P}_m|$ are given in a form $\frac{C}{m^q}$ for m large enough and for a constant C essentially dependent on the right hand side of the ODE and the numerical scheme ψ (to be more precise, $q = p$ in C^0 and $q = p - 1$ in C^1 norm estimates). Achieved results, as we have anticipated, correspond to [8] where the author examined the C^j-closeness, $j \geq 0$, between the flow and its numerical approximation. Our approach uses the techniques of a moving orthonormal system (introduced rigorously in [10] and then used successfully in [1, 2, 17]) and the Newton–Kantorovich type theorem (cf. [13, 15, 20]). Hence, $\mathcal{P}_m$ is not unique but naturally depending on the choice of the Poincaré section and consequently on the corresponding tubular neighbourhood of the periodic orbit created by the mentioned moving orthonormal system. Sections 2, 3 and 4 are devoted to this topic.

In the last Section 5 we give an application of the previously developed results. It is a slight completion of [4], where two closed curves were found in a neighborhood of γ for the discrete dynamical system. The first one was found basically under the nondegeneracy of γ (that is when the trivial

Floquet multiplier 1 of γ is simple). This curve is the set of m-periodic points x, where the step h of the scheme depends on x and is close enough to $1/m$. The second, the maximal compact invariant set of the scheme in a neighborhood of γ, was derived under the hyperbolicity of γ, for any sufficiently small step (this is a historically well-known topic, it was treated for example in [1, 2, 5, 16]). We also show using the nondegeneracy of γ that in a small neighborhood of γ the set of those points, which return into themselves under the action of $\mathcal{P}_m$, forms another new type of closed curves for any m large and h close enough to $1/m$. Of course this curve in general differs from the compact maximal invariant set and depends on $\mathcal{P}_m$ and the chosen tubular neighborhood. Hence, it might be considered as somewhat artificial. However, at the end of the paper, we show a simplification which leads us to the natural curve of m-periodic points depending only on the choice of the discretization mapping. We conclude Section 5 by a short remark on spectral properties of our detected curve, which is undoubtedly an interesting application of our achieved results about the numerical Poincaré map.

Finally we note that this paper is a starting point for our future study of discretized bifurcations near periodic orbits of parametrized ODEs.

2 General settings and tools

Assumptions made here are going to be valid for the whole paper. Let us have $f \in C^3(\mathbb{R}^N), N \in \mathbb{N} \setminus \{1\}$ such that

$$\varphi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \text{ is the global flow of } \dot{x} = f(x).$$

For a numerical scheme $\psi : [0, h_0] \times \mathbb{R}^N \to \mathbb{R}^N, h_0 \in (0, 1)$ suppose for some $p \in \mathbb{N}$ that

$$\psi(h, x) = \varphi(h, x) + \Upsilon(h, x)h^{p+1}. \tag{2.1}$$

Assume again $\psi, \Upsilon \in C^3([0, h_0] \times \mathbb{R}^N, \mathbb{R}^N)$. Some technical reasons cause that we are forced to assume also $p \geq 2$ (see below Remark 2 for more details).

Let $\gamma(s) := \varphi(s, \xi_0)$ be a 1-periodic solution for fixed $\xi_0 \in \mathbb{R}^N$. Then there is a system $\{e_i(s)\}_{i=1}^{N-1}$ of vectors in $\mathbb{R}^N$ for any $s \in \mathbb{R}$ such that

$$\left.\begin{aligned} e_i \in C^3(\mathbb{R}, \mathbb{R}^N), \quad & e_i(s+1) = e_i(s), \\ \langle e_i(s), e_j(s)\rangle = \delta_{i,j}, \quad & \langle e_i(s), f(\gamma(s))\rangle = 0, \end{aligned}\right\} \tag{2.2}$$

where $i, j \in \{1, \ldots, N-1\}$, $\delta_{i,j}$ is a Kronecker's delta and $\langle \cdot, \cdot \rangle$ is the standard Euclidean scalar product. Introduce an $N \times (N-1)$ matrix

$E(s) = [e_1, \ldots, e_{N-1}]$ (i-th column is e_i, $i = 1, \ldots, N-1$). Let us also set a tubular coordinate function $\xi(s,c) := \gamma(s) + E(s)c$ for $s \in \mathbb{R}, c \in \mathbb{R}^{N-1}$. For standard Euclidean norm $|c|_2 := \sqrt{\langle c, c\rangle}$ note that $|E(s)c|_2 = |c|_2, c \in \mathbb{R}^{N-1}$. For $\delta > 0$ introduce the notation $B^{\delta}_{N-1} := \left\{c \in \mathbb{R}^{N-1} : |c|_2 < \delta\right\}$. Using the Implicit Function Theorem finite number of times we get that there is a $\delta_{\mathrm{tr}} > 0$ such that

$$\xi : [0,1) \times B^{\delta_{\mathrm{tr}}}_{N-1} \to \mathbb{R}^N \text{ is a } C^3\text{-transformation,}$$

in other words $\xi|_{[0,1)\times B^{\delta_{\mathrm{tr}}}_{N-1}}$ is a C^3-diffeomorphism between its domain and range (cf. the moving orthonormal system along γ in [10, Chapter VI.I., p. 214–219]) .

For values

$$h \in [0, h_0],\ s \in \mathbb{R},\ c \in \mathbb{R}^{N-1},\ \Delta \in [0, h_0],$$
$$X := (x^1, x^2, \ldots, x^{m-1}) \in \mathbb{R}^{N(m-1)},\ x^i \in \mathbb{R}^N,\ m \in \mathbb{N},\ m \geq 4,$$

define the following useful functions

$$\begin{aligned}
F_m(h,s,c,X,\Delta) :=& (G_m(h,s,c,X), H_m(h,s,c,X,\Delta)),\\
G_m(h,s,c,X) :=& \big(\psi(h,\xi(s,c)) - x^1, \psi(h,x^1) - x^2, \psi(h,x^2) - x^3,\\
& \ldots, \psi(h,x^{m-2}) - x^{m-1}\big),\\
H_m(h,s,c,X,\Delta) :=& \left\langle \psi\left(\Delta, x^{m-1}\right) - \gamma(s), f(\gamma(s))\right\rangle.\\
\bar{X}_m :=& \bar{X}_m(h,s,c) := \left(\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^{m-1}\right),\\
\bar{x}^j :=& \bar{x}^j(h,s,c) := \varphi(jh, \xi(s,c)),\ j = 1, 2, \ldots, m-1.
\end{aligned}$$

Further let B be a compact set such that $\gamma(\mathbb{R})$ is contained in the interior of B. Hence there is a constant $R > 0$ such that

$$\left\{x \in \mathbb{R}^N : \min_{s\in\mathbb{R}}\{|x - \gamma(s)|\} \leq R\right\} \subset B. \tag{2.3}$$

We mean by $|\cdot|$ the standard maximum norm $|v| := \max\{|v_i| : i = 1, \ldots, l\}$ for $v \in \mathbb{R}^l, l \in \mathbb{N}$. Notation $|\cdot|$ is used also for linear operators $A : \mathbb{R}^{l_1} \to \mathbb{R}^{l_2}$ defined as $|A| := \max_{v\in\mathbb{R}^{l_1}, |v|=1} |Av|$. Further by $\mathcal{L}(X,Y)$ for Banach spaces X, Y we mean the Banach space of continuous and linear operators $A : X \to Y$, in the case $X = Y$ we set $\mathcal{L}(X) := \mathcal{L}(X,X)$. In general $|\cdot|_X$ will denote the norm in a Banach space X, however in most of the cases there are no arising confusions so we use again simply $|\cdot|$. An open ball will be denoted as $B(x,\varrho) := \{y \in X : |y - x| < \varrho\}$ for any $x \in X$ and $\varrho > 0$.

Several times we will use the following well-known result.

Lemma 2.1 (Neumann's Inversion Lemma)**.** *Suppose that X is a Banach space and $A \in \mathcal{L}(X)$ is invertible. Then for $B \in \mathcal{L}(X)$ such that $|A^{-1}B| < 1$ we have $(A+B)^{-1} \in \mathcal{L}(X)$, and*

$$(A+B)^{-1} = \sum_{n\geq 0} (A^{-1}B)^n A^{-1}, \quad \big|(A+B)^{-1}\big| \leq \frac{|A^{-1}|}{1-|A^{-1}B|}.$$

Our central tool will be the following lemma. We also give a short proof in the Appendix.

Lemma 2.2 (Newton–Kantorovich method)**.** *Let us have Banach spaces X, Y, Z and open nonempty sets $U \subset X, V \subset Y$. Let $\bar{y} : U \to V$ be any function such that*

$$\overline{B(\bar{y}(x), \varrho)} \subset V \text{ for every } x \in U \text{ and for some } \varrho > 0.$$

Let us have a function $F \in C^r(U \times V, Z)$ for $r \geq 1$. Suppose that

$$D_y F(x, \bar{y}(x))^{-1} \in \mathcal{L}(Z, Y),$$
$$|F(x, \bar{y}(x))| \leq \alpha, \quad |D_y F(x, \bar{y}(x))^{-1}| \leq \beta$$

for every $x \in U$ and for some $\alpha, \beta > 0$. Let

$$|D_y F(x, y_1) - D_y F(x, y_2)| \leq l|y_1 - y_2|, \quad x \in U, y_1, y_2 \in \overline{B(\bar{y}(x), \varrho)} \tag{2.4}$$

hold for some $l \geq 0$. For constants $\alpha, \beta, l, \varrho$ finally suppose

$$\beta l \varrho < 1, \tag{2.5}$$
$$\alpha\beta < \varrho(1 - \beta l \varrho). \tag{2.6}$$

Then there is a unique function $\mathrm{y} : U \to V$ such that

$$|\mathrm{y}(x) - \bar{y}(x)| \leq \varrho \text{ and } F(x, \mathrm{y}(x)) = 0 \text{ for all } x \in U.$$

Moreover

$$|\mathrm{y}(x) - \bar{y}(x)| < \varrho, \quad D_y F(x, \mathrm{y}(x))^{-1} \in \mathcal{L}(Z, Y)$$

for all $x \in U$ with an estimate

$$\big|D_y F(x, \mathrm{y}(x))^{-1}\big| \leq \frac{\beta}{1 - \beta l \varrho}.$$

We also get $\mathrm{y} \in C^r(U, V)$ if we additionally assume the continuity of $\bar{y}$.

3 Discretized Poincaré map

At first we state a lemma about the continuous Poincaré map, the proof can be found in the Appendix.

Lemma 3.1 (Poincaré's time return map)**.** *There is an* $\varepsilon^\star \in (0,1/2)$ *such that for every* $\varepsilon \in (0,\varepsilon^\star]$ *there is* $\delta_{\text{re}} = \delta_{\text{re}}(\varepsilon) \in (0,\delta_{\text{tr}}]$ *and a* C^3*-function*

$$\tau : \mathbb{R} \times B_{N-1}^{\delta_{\text{re}}(\varepsilon)} \to (1-\varepsilon, 1+\varepsilon)$$

such that for $t \in (1-\varepsilon,1+\varepsilon), s \in \mathbb{R}$ *and* $c \in B_{N-1}^{\delta_{\text{re}}(\varepsilon)}$ *we have*

$$z(t,s,c) = 0 \text{ for } z(t,s,c) := \langle \varphi(t,\xi(s,c)) - \gamma(s), f(\gamma(s))\rangle \tag{3.1}$$

if and only if $t = \tau(s,c)$*. In addition* $\tau(s+1,\cdot) = \tau(s,\cdot), s \in \mathbb{R}$.

In this context the usual Poincaré map is defined as

$$\mathcal{P}(s,c) := \varphi(\tau(s,c),\xi(s,c)).$$

Further for admissible values of (h,s,c) using τ from the above lemma introduce

$$\bar\Delta_m := \bar\Delta_m(h,s,c) := \tau(s,c) - (m-1)h.$$

To get the exact meaning of $\mathcal{P}_m$ mentioned informally in the introduction we have to solve the equation $F_m(h,s,c,X,\Delta) = 0$ near $(\bar X, \bar\Delta)$. Here comes the first application of Lemma 2.2. Before this let us introduce some technicalities, at first the following *positive* constants

$$\left.\begin{aligned}
\mathrm{C}_\Upsilon &\geq \max_{\substack{h\in[0,h_0],\, x\in B,\\ k\in\{0,1,2,3\}}} \left\{|D^{[k]}\Upsilon(h,x)|\right\},\\
\mathrm{C}_\varphi &\geq \max\Bigg\{ \max_{\substack{h\in[0,h_0],\, x\in B,\\ k\in\{1,2,3\}}} \left\{|D^{[k]}\varphi(h,x)|\right\},\\
&\qquad \max_{h\in[0,3/2]}\left\{|\varphi'_x(h,x)|\right\},\ \max_{h\in[0,h_0]}\left\{|\varphi^{[4]}_{txxx}(h,x)|\right\}\Bigg\},\\
\mathrm{C}_{\min} &\leq \min_{x\in\gamma(\mathbb{R})}\{|f(x)|_2^2\},\\
\mathrm{C}_\tau &\geq \max_{\substack{s\in[0,1],\, k\in\{1,2\},\\ c\in \overline{B_{N-1}^{\delta_{\text{re}}(\varepsilon^\star)/2}}}} \left\{|D^{[k]}\tau(s,c)|\right\},\\
\mathrm{C}_E &\geq \max\{|E'(s)|, s\in[0,1]\},\\
\mathrm{C}_\psi &\geq \max_{\substack{h\in[0,h_0],\, x\in B\\ k\in\{1,2,3\}}} \left\{|D^{[k]}\psi(h,x)|\right\}.
\end{aligned}\right\} \tag{3.2}$$

Here $D^{[k]}$ is the k-th Fréchet differential. Note that an upper bound of a type C_ψ could be given simply using (2.1) and constants $\mathrm{C}_\varphi, \mathrm{C}_\Upsilon$. Next, let us have $\delta > 0, \mu \in (0,1)$ and introduce

$$d_m := d_m(p,\delta,\mu) := \frac{\mu - \frac{\mathrm{C}_\tau \delta}{m^{p-1}}}{m(m-1)},$$

for

$$m \geq m_0(p,\delta,\mu) := \max\left\{ \left\lceil \frac{2}{h_0} \right\rceil, \left\lceil \left(\frac{\delta}{\delta_{\mathrm{re}}(\varepsilon^\star)} \right)^{1/p} \right\rceil, \left\lfloor \left(\frac{\mathrm{C}_\tau \delta}{\mu} \right)^{\frac{1}{p-1}} \right\rfloor + 1 \right\},$$

where $\lceil x \rceil := \min\{k \in \mathbb{Z} : k \geq x\}$ and $\lfloor x \rfloor := -\lceil -x \rceil$ for any $x \in \mathbb{R}$. Further

$$\left.\begin{aligned} \mathcal{I}_m := \mathcal{I}_m(p,\delta,\mu) &:= \left(\frac{1}{m} - d_m, \frac{1}{m} + d_m \right), \\ \mathcal{B}_m := \mathcal{B}_m(p,\delta) &:= B_{N-1}^{\delta/m^p}, \\ \mathcal{H}_m := \mathcal{H}_m(p,\delta,\mu) &:= \mathcal{I}_m \times \mathbb{R} \times \mathcal{B}_m, \end{aligned}\right\} \tag{3.3}$$

also for $m \geq m_0$.

The simple goal of these complicated assumptions is that for $(h,s,c) \in \mathcal{H}_m$ it is straightforward to show

$$d_m > 0, \quad \mathcal{I}_m \subset (0, h_0], \quad c \in B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon^\star)},$$

and

$$\frac{1-\mu}{m} < \bar{\Delta}_m < \frac{1+\mu}{m}. \tag{3.4}$$

Theorem 3.2. *Choose any constants* $\mathrm{C}_X, \mathrm{C}_\Delta$ *such that*

$$\mathrm{C}_X > \overline{\mathrm{C}}_X := \mathrm{C}_\varphi \mathrm{C}_\Upsilon, \qquad \mathrm{C}_\Delta > \overline{\mathrm{C}}_\Delta := \frac{N \mathrm{C}_\varphi^3 \mathrm{C}_\Upsilon}{\mathrm{C}_{\min}}. \tag{3.5}$$

Fix $\delta > 0$*, then for every* m *large,* μ *small enough and* $(h,s,c) \in \mathcal{H}_m(p,\delta,\mu)$ *there exists a unique pair* $(X_m, \Delta_m) = (X_m(h,s,c), \Delta_m(h,s,c))$ *such that*

$$F(X_m, \Delta_m) = F_m(h,s,c,X_m(h,s,c),\Delta_m(h,s,c)) = 0$$

and

$$|X_m - \bar{X}_m| < \mathrm{C}_X/m^p, \quad |\Delta_m - \bar{\Delta}_m| < \mathrm{C}_\Delta/m^p. \tag{3.6}$$

Moreover the functions X_m, Δ_m *are* C^3*-smooth in their arguments and*

$$(X_m, \Delta_m)(h, s+1, c) = (X_m, \Delta_m)(h,s,c), \quad (h,s,c) \in \mathcal{H}_m. \tag{3.7}$$

Proof. The proof is divided into several steps. Two main parts are the following ones:

Part 1. The solution X_m close to $\bar{X}_m$ of $G_m(h,s,c,X)=0$ is found.

Part 2. We solve $H_m(h,s,c,X_m(h,s,c),\Delta)=0$ for Δ near $\bar{\Delta}_m$.

These parts are handled using Lemma 2.2 and contain four steps.

Step 1.1. We show that

$$|G_m(h,s,c,\bar{X}_m)| \le \mathrm{C}_\Upsilon h^{p+1} \tag{3.8}$$

is valid for all $(h,s,c)\in\mathcal{H}_m$ and m large enough. From (2.1) we have for $j=1,\dots,m-1$ if m is large enough that

$$\begin{aligned}|(G_m(h,s,c,\bar{X}_m))^j| &= |(\psi(h,\bar{x}^{j-1})-\varphi(h,\bar{x}^{j-1}))|\\ &\le h^{p+1}|\Upsilon(h,\bar{x}^{j-1})| \le \mathrm{C}_\Upsilon h^{p+1}\end{aligned}$$

where $\bar{x}^0 := \xi(s,c)$. Indeed, noting that $\delta/m^p \le \min\{R/\mathrm{C}_\varphi, \delta_{\mathrm{re}}(\varepsilon^\star)/2\}$ and $jh \le (m-1)\left(\frac{1}{m}+d_m\right) \le \frac{3}{2}$ are valid for m large enough we get using (3.2) that

$$\begin{aligned}|\bar{x}^j-\gamma(jh+s)| &= \left|\int_0^1 \varphi'_x(jh,\gamma(s)+\vartheta E(s)c)E(s)c\,\mathrm{d}\vartheta\right|\\ &\le \mathrm{C}_\varphi|E(s)c| \le \mathrm{C}_\varphi|E(s)c|_2 = \mathrm{C}_\varphi|c|_2 \le \mathrm{C}_\varphi\delta/m^p \le R.\end{aligned}$$

Hence using (2.3) we have

$$\bar{x}^j = \varphi(jh,\xi) \in B \text{ for } j=0,1,\dots,m-1, \tag{3.9}$$

and so $|\Upsilon(h,\bar{x}^{j-1})| \le \mathrm{C}_\Upsilon$ and we are done.

Step 1.2. We show that for any $\mu_1\in(0,1)$

$$\left|D_X G_m(h,s,c,\bar{X}_m)^{-1}\right| \le \frac{\mathrm{C}_\varphi m}{1-\mu_1} \tag{3.10}$$

holds if $(h,s,c)\in\mathcal{H}_m$, and m is large enough (the main point is of course that the lower threshold of m-s depends also on μ_1, its limit is ∞ as $\mu_1\to 0^+$ – from now on we omit remarks of this type).

Using (2.1) again we get $D_X G_m(h,s,c,\bar{X}_m)[Y] = AY+BY$ where

$$\begin{aligned}AY &:= \big(-y^1, \varphi'_x(h,\bar{x}^1)y^1-y^2, \varphi'_x(h,\bar{x}^2)y^2-y^3,\dots\\ &\quad \dots,\varphi'_x(h,\bar{x}^{m-2})y^{m-2}-y^{m-1}\big),\\ BY &:= \left(0, h^{p+1}\Upsilon'_x(h,\bar{x}^1)y^1, h^{p+1}\Upsilon'_x(h,\bar{x}^2)y^2,\dots,h^{p+1}\Upsilon'_x(h,\bar{x}^{m-2})y^{m-2}\right).\end{aligned}$$

Now $AY = Z$ is solvable. Straightforward computation shows

$$\left.\begin{aligned} y^1 &= -z^1, \\ y^j &= -z^j - \sum_{r=1}^{j-1} \varphi'_x(rh, \bar{x}^{j-r}) z^{j-r}, \quad j = 2, \ldots, m-1. \end{aligned}\right\} \tag{3.11}$$

Therefore $|A^{-1}Z| \leq \mathrm{C}_\varphi m$ (because (3.11) implies $|y^j| \leq (1 + (m-2)\mathrm{C}_\varphi)|Z|$ for $j = 1, \ldots, m-1$, noticing $\mathrm{C}_\varphi \geq 1$ and (3.9) we arrive at the statement). Next we also obtain in a moment $|BY| \leq \mathrm{C}_\Upsilon h^{p+1}$ ((3.9) is used again). Now using

$$h < \frac{1}{m} + d_m < \frac{1+\mu}{m} \tag{3.12}$$

we get

$$|A^{-1}B| \leq \mathrm{C}_\varphi m \mathrm{C}_\Upsilon h^{p+1} < \frac{\mathrm{C}_\varphi \mathrm{C}_\Upsilon (1+\mu)^{p+1}}{m^p}$$

and so we have $|A^{-1}B| \leq \mu_1 < 1$ if m is large enough. Lemma 2.1 implies the invertibility of $A + B$ and also that

$$\left|(A+B)^{-1}\right| \leq \frac{|A^{-1}|}{1 - |A^{-1}B|} \leq \frac{\mathrm{C}_\varphi m}{1 - \mu_1}$$

and we have arrived at (3.10).

Step 1.3. We show that for any $\mu_2 > 0$ we have

$$|D_X G_m(h, s, c, X_1) - D_X G_m(h, s, c, X_2)| \leq \frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m} |X_1 - X_2| \tag{3.13}$$

for all $X_1, X_2 \in \overline{B(\bar{X}_m, R/2)}$, $(h, s, c) \in \mathcal{H}_m$ and m large enough.

At first notice that from

$$\begin{aligned} \varphi(h, x) &= \varphi(0, x) + \int_0^1 \frac{\partial}{\partial \eta}(\varphi(\eta h, x)) \mathrm{d}\eta \\ &= x + h \int_0^1 \varphi'_t(\eta h, x) \mathrm{d}\eta \end{aligned}$$

we have

$$\varphi''_{xx}(h, x) = h \int_0^1 \varphi'''_{txx}(\eta h, x) \mathrm{d}\eta$$

which readily implies (cf. (3.2))

$$|\varphi'_x(h, x_1) - \varphi'_x(h, x_2)| \leq h \mathrm{C}_\varphi |x_1 - x_2| \tag{3.14}$$

for all x_1, x_2 such that $x_1 + \vartheta(x_2 - x_1) \in B, \vartheta \in [0,1]$.

For m large enough we have that

$$\forall X_1, X_2 \in \overline{B(\bar{X}_m, R/2)} \, : \, x_1^j + \vartheta(x_2^j - x_1^j) \in B, \quad j = 1, \dots, m-1. \quad (3.15)$$

This follows from the following considerations. The condition $\delta/m^p \leq \min\{R/2\mathrm{C}_\varphi, \delta_{\mathrm{re}}(\varepsilon^\star)/2\}$ is fulfilled for m large enough, this implies that $|\bar{x}^j - \gamma(jh+s)| < R/2$ (similar considerations as we obtained (3.9)). Now

$$\begin{aligned}
&|x_1^j + \vartheta(x_2^j - x_1^j) - \gamma(jh+s)| \\
\leq &(1-\vartheta)|x_1^j - \bar{x}^j| + \vartheta|x_2^j - \bar{x}^j| + |\bar{x}^j - \gamma(jh+s)| \\
< &(1-\vartheta)\frac{R}{2} + \vartheta\frac{R}{2} + \frac{R}{2} = R
\end{aligned}$$

so from (2.3) we have $x_1^j + \vartheta(x_2^j - x_1^j) \in B$ which is exactly (3.15).

For such an X_1, X_2 using (2.1) we derive that

$$\begin{aligned}
&(D_X G_m(h,s,c,X_1) - D_X G_m(h,s,c,X_2))\,[Y] \\
&= \Big(0, \left(\varphi'_x(h,x_1^1) - \varphi'_x(h,x_2^1)\right) y^1, \left(\varphi'_x(h,x_1^2) - \varphi'_x(h,x_2^2)\right) y^2, \dots \\
&\qquad \dots, \left(\varphi'_x(h,x_1^{m-2}) - \varphi'_x(h,x_2^{m-2})\right) y^{m-2}\Big) \\
&+ \Big(0, h^{p+1}\left(\Upsilon'_x(h,x_1^1) - \Upsilon'_x(h,x_2^1)\right) y^1, h^{p+1}\left(\Upsilon'_x(h,x_1^2) - \Upsilon'_x(h,x_2^2)\right) y^2, \dots \\
&\qquad \dots, h^{p+1}\left(\Upsilon'_x(h,x_1^{m-2}) - \Upsilon'_x(h,x_2^{m-2})\right) y^{m-2}\Big).
\end{aligned}$$

Using (3.14) and (3.2) we obtain

$$|D_X G_m(h,s,c,X_1) - D_X G_m(h,s,c,X_2)| \leq h(\mathrm{C}_\varphi + h^p \mathrm{C}_\Upsilon)|X_1 - X_2|.$$

Note again that (3.12) is valid, therefore for every m large enough we have

$$h(\mathrm{C}_\varphi + h^p \mathrm{C}_\Upsilon) < \frac{(1+\mu)\mathrm{C}_\varphi + \frac{(1+\mu)^{p+1}\mathrm{C}_\Upsilon}{m^p}}{m} \leq \frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m}$$

and we have obtained exactly (3.13).

Step 1.4. Now the final step of the first part is coming. To fit into the framework of Lemma 2.2 with an equation $G_m(h,s,c,X) = 0$ set

$$\left.\begin{aligned}
&U := \mathcal{H}_m, V := \mathbb{R}^{N(m-1)}, x = (h,s,c), \bar{y}(x) := \bar{X}_m(h,s,c), \\
&\alpha := \frac{\mathrm{C}_\Upsilon}{m^{p+1}}, \beta := \frac{\mathrm{C}_\varphi m}{1-\mu_1}, l := \frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m}, \varrho := \frac{\mathrm{C}_X}{m^p}.
\end{aligned}\right\} \quad (3.16)$$

It has to be noted that for large m, $\mathrm{C}_X/m^p \le R$ is valid and so (2.4) holds on $\overline{B(\bar{y}(x),\varrho)}$. Conditions (2.5) and (2.6) have to be fulfilled. For (2.5) pick $\mu_3 \in (0,1)$, then for m large enough we get

$$\beta l \varrho = \frac{(1+\mu)\mathrm{C}_\varphi^2 + \mu_2 \mathrm{C}_X \mathrm{C}_\varphi}{m^p} \le \mu_3 < 1.$$

Further using (3.12) we get

$$\frac{\alpha\beta}{\varrho(1-\beta l\varrho)} < \frac{\mathrm{C}_\varphi \mathrm{C}_\Upsilon (1+\mu)^{p+1}}{\mathrm{C}_X(1-m_1)(1-\mu_3)},$$

so (2.6) in this setting will be valid if

$$\mathrm{C}_\varphi \mathrm{C}_\Upsilon \frac{(1+\mu)^{p+1}}{(1-\mu_1)(1-\mu_3)} < \mathrm{C}_X. \tag{3.17}$$

According to the assumption $\overline{\mathrm{C}}_X < \mathrm{C}_X$ and that $\frac{(1+\mu)^{p+1}}{(1-\mu_1)(1-\mu_3)} \to 1^+$ as $\mu, \mu_1, \mu_3 \to 0^+$, there are always such suitably small parameters $\mu, \mu_1, \mu_3 \in (0,1)$ that (3.17) is valid. Therefore Lemma 2.2 can be used (the remaining assumptions are trivially satisfied) and it gives a unique element $X_m(h,s,c) \in \overline{B(\bar{X}_m, \mathrm{C}_X/m^p)}$ such that

$$G_m(h,s,c,X_m(h,s,c)) = 0.$$

Moreover X_m is C^3-smooth, $|X_m - \bar{X}_m| < \mathrm{C}_X/m^p$ and

$$|D_X G_m(h,s,c,X_m)^{-1}| \le \frac{\beta}{1-\beta l\varrho} \le \frac{\mathrm{C}_\varphi m}{(1-\mu_1)(1-\mu_3)}.$$

Step 2.1. Set

$$\left.\begin{aligned} z(h,s,c,\Delta) : &= H_m(h,s,c,X_m(h,s,c),\Delta) \\ &= \left\langle \psi(\Delta, x_m^{m-1}) - \gamma(s), f(\gamma(s)) \right\rangle. \end{aligned}\right\} \tag{3.18}$$

We show that for any $\mu_4 > 0$ we have

$$|z(h,s,c,\bar{\Delta}_m)| \le \frac{N\mathrm{C}_\varphi^2 \mathrm{C}_X + \mu_4}{m^p} \tag{3.19}$$

for all $(h,s,c) \in \mathcal{H}_m$ and m large enough. At first note that

$$\begin{gathered} z(h,s,c,\bar{\Delta}_m) = \left\langle \varphi(\bar{\Delta}_m, \bar{x}^{m-1}) - \gamma(s), f(\gamma(s)) \right\rangle \\ + \left\langle \varphi(\bar{\Delta}_m, x_m^{m-1}) - \varphi(\bar{\Delta}_m, \bar{x}^{m-1}) + \bar{\Delta}_m^{p+1} \Upsilon(\bar{\Delta}_m, x_m^{m-1}), f(\gamma(s)) \right\rangle \end{gathered}$$

where the first term vanishes because of Lemma 3.1. From (3.4) we infer $\bar{\Delta}_m \in (0, h_0/2)$ for m large enough. Next

$$|\varphi(\bar{\Delta}_m, x_m^{m-1}) - \varphi(\bar{\Delta}_m, \bar{x}^{m-1})| \leq \mathrm{C}_\varphi |x_m^{m-1} - \bar{x}^{m-1}| < \mathrm{C}_\varphi \mathrm{C}_X / m^p,$$
$$|\bar{\Delta}_m^{p+1} \Upsilon(\bar{\Delta}_m, x_m^{m-1})| \leq \frac{(1+\mu)^{p+1}\mathrm{C}_\Upsilon}{m^{p+1}}.$$

From $|\langle a, b\rangle| \leq N|a||b|$ and $\varphi_t'(0, x) = f(x)$ we obtain

$$|z(h, s, c, \bar{\Delta}_m)| \leq \frac{N\mathrm{C}_\varphi \left(\mathrm{C}_\varphi \mathrm{C}_X + \frac{(1+\mu)^{p+1}\mathrm{C}_\Upsilon}{m}\right)}{m^p}.$$

For m large enough $\frac{N\mathrm{C}_\varphi (1+\mu)^{p+1}\mathrm{C}_\Upsilon}{m} \leq \mu_4$ is valid, therefore (3.19) holds.

Step 2.2. We show for any $\mu_5 > 0$ that

$$|D_\Delta z(h, s, c, \bar{\Delta}_m)^{-1}| \leq \frac{1+\mu_5}{\mathrm{C}_{\min}} \tag{3.20}$$

where $(h, s, c) \in \mathcal{H}_m$ and m is large enough. Straightforward computation yields $D_\Delta z(h, s, c, \Delta_m) = |f(\gamma(s))|_2^2 + w_m(h, s, c)$ where

$$\begin{aligned} w_m(h, s, c) :=& \Big\langle f(\varphi(\bar{\Delta}_m, x_m^{m-1})) - f(\varphi(\bar{\Delta}_m^0, \bar{x}^{m-1,0})) \\ &+ \bar{\Delta}_m^{p+1} \Upsilon_h'(\bar{\Delta}_m, x_m^{m-1}), f(\gamma(s))\Big\rangle, \\ \bar{\Delta}_m^0 :=& \bar{\Delta}_m(h, s, 0) = 1 - (m-1)h, \\ \bar{x}^{m-1,0} :=& \bar{x}^{m-1}(h, s, 0) = \gamma(s + (m-1)h). \end{aligned}$$

Elementary considerations show that

$$|w_m| \leq \frac{N\mathrm{C}_\varphi^2 \delta(\mathrm{C}_X + \delta(\mathrm{C}_\tau + \sqrt{N}\mathrm{C}_\varphi))}{m^p},$$

therefore for m large enough we obtain

$$|D_\Delta z(h, s, c, \bar{\Delta}_m)| \geq \frac{|f(\gamma(s))|_2^2}{1+\mu_5} \geq \frac{\mathrm{C}_{\min}}{1+\mu_5}.$$

This shows (3.20) and we are done.

Step 2.3. We have that

$$|D_\Delta z(h, s, c, \Delta_1) - D_\Delta z(h, s, c, \Delta_2)| \leq N\mathrm{C}_\varphi \mathrm{C}_\psi |\Delta_1 - \Delta_2| \tag{3.21}$$

is valid for all $(h,s,c)\in\mathcal{H}_m, \Delta_1,\Delta_2\in[0,h_0]$ and m large. We easily derive that

$$\begin{aligned}&D_\Delta z(h,s,c,\Delta_1)-D_\Delta z(h,s,c,\Delta_2)\\&=\left\langle\psi'_h(\Delta_1,x_m^{m-1})-\psi'_h(\Delta_2,x_m^{m-1}),f(\gamma(s))\right\rangle\\&=\left\langle\int_0^1\psi''_{hh}(\Delta_2+\vartheta(\Delta_1-\Delta_2),x_m^{m-1})\mathrm{d}\vartheta,f(\gamma(s))\right\rangle(\Delta_1-\Delta_2)\end{aligned}$$

which immediately yields (3.21).

Step 2.4. Finally we solve $z(h,s,c,\Delta)$ with Lemma 2.2 (see (3.18)). Set

$$\left.\begin{aligned}U:=\mathcal{H}_m,\ V:=(0,h_0),\ x:=(h,s,c),\ \bar{y}(x):=\bar{\Delta}_m(h,s,c),\\ \alpha:=\frac{N\mathrm{C}_\varphi^2\mathrm{C}_X+\mu_4}{m^p},\ \beta:=\frac{1+\mu_5}{\mathrm{C}_{\min}},\ l:=N\mathrm{C}_\varphi\mathrm{C}_\psi,\ \varrho:=\mathrm{C}_\Delta/m^p.\end{aligned}\right\}\tag{3.22}$$

Note (3.4) again, so $\overline{B(\bar{\Delta}_m,\varrho)}\subset V$ holds for m large enough. Now

$$\beta l\varrho=\frac{(1+\mu_5)N\mathrm{C}_\varphi\mathrm{C}_\psi\mathrm{C}_\Delta}{m^p}\le\mu_6<1$$

is valid for any $\mu_6\in(0,1)$ if m is sufficiently large which fulfills (2.5). Now

$$\frac{\alpha\beta}{\varrho(1-\beta l\varrho)}\le\frac{(N\mathrm{C}_\varphi^2\mathrm{C}_X+\mu_4)(1+\mu_5)}{\mathrm{C}_\Delta(1-\mu_6)},$$

therefore (2.6) holds if

$$\frac{(N\mathrm{C}_\varphi^2\mathrm{C}_X+\mu_4)(1+\mu_5)}{\mathrm{C}_\Delta(1-\mu_6)}<1.\tag{3.23}$$

Because of $\overline{\mathrm{C}}_\Delta<\mathrm{C}_\Delta$ and the already proven part of our theorem – that is C_X can be chosen arbitrarily close to $\overline{\mathrm{C}}_X$ for m large enough – we conclude that (3.23) can be fulfilled (with sufficiently small $\mu,\mu_4,\mu_5,\mu_6>0$). Now Lemma 2.2 gives a unique element $\Delta_m\in\overline{B(\bar{\Delta}_m,\mathrm{C}_\Delta/m^p)}$ with $z(h,s,c,\Delta_m)=0$. Moreover

$$|\Delta_m-\bar{\Delta}_m|<\mathrm{C}_\Delta/m^p,\quad|D_\Delta z(h,s,c,\Delta_m)^{-1}|\le\frac{\beta}{1-\beta l\varrho}\le\frac{1+\mu_5}{\mathrm{C}_{\min}(1-\mu_4)}$$

are valid and the proof is finished ((3.7) is a straightforward consequence of the 1-periodicity of $G_m,\bar{X}_m,H_m,z,\bar{\Delta}_m$ in the variable s, and the uniqueness parts of the steps 1.4. and 2.4.). □

Remark 1. In the framework of Theorem 3.2 a natural approximation of $\mathcal{P}$ is
$$\mathcal{P}_m(h,s,c) := \psi\left(\Delta_m(h,s,c), x_m^{m-1}(h,s,c)\right).$$
Now
$$|\mathcal{P}(s,c) - \mathcal{P}_m(h,s,c)| \leq \left|\varphi(\tau,\xi) - \varphi(\Delta_m, x_m^{m-1})\right| + \left|\Delta_m^{p+1}\Upsilon(\Delta_m, x_m^{m-1})\right|.$$
Notice that
$$\begin{aligned}
&\left|\varphi(\tau,\xi) - \varphi(\Delta_m, X_m^{m-1})\right| = \left|\varphi(\bar{\Delta}_m, \bar{x}^{m-1}) - \varphi(\Delta_m, x_m^{m-1})\right| \\
&\leq \left|\varphi(\bar{\Delta}_m, \bar{x}^{m-1}) - \varphi(\Delta_m, \bar{x}^{m-1})\right| + \left|\varphi(\Delta_m, \bar{x}^{m-1}) - \varphi(\Delta_m, x_m^{m-1})\right| \\
&\leq \int_0^1 |\varphi_t'(\Delta_m + \vartheta(\bar{\Delta}_m - \Delta_m), \bar{x}^{m-1})|\mathrm{d}\vartheta|\bar{\Delta}_m - \Delta_m| \\
&\quad + \int_0^1 |\varphi_x'(\Delta_m, x_m^{m-1} + \vartheta(\bar{x}^{m-1} - x_m^{m-1}))|\mathrm{d}\vartheta|\bar{x}^{m-1} - x_m^{m-1}|,
\end{aligned}$$
therefore $|\varphi(\tau,\xi) - \varphi(\Delta_m, X_m^{m-1})| \leq \mathrm{C}_\varphi(\mathrm{C}_X + \mathrm{C}_\Delta)/m^p$ (we used (3.2) and (3.15)). In addition from (3.4) and (3.6) we have
$$|\Delta_m| \leq |\bar{\Delta}_m| + |\Delta_m - \bar{\Delta}_m| \leq \frac{1+\mu}{m} + \frac{\mathrm{C}_\Delta}{m^p}$$
so
$$\left|\Delta_m^{p+1}\Upsilon(\Delta_m, x_m^{m-1})\right| \leq \frac{\left(1 + \mu + \frac{\mathrm{C}_\Delta}{m^{p-1}}\right)^{p+1}\mathrm{C}_\Upsilon}{m^{p+1}}.$$
Hence for any fixed $\mu_7 > 0$ we have $\left|\Delta_m^{p+1}\Upsilon(\Delta_m, x_m^{m-1})\right| \leq \frac{\mu_7}{m^p}$ for every m sufficiently large.

Putting all this together we arrive at
$$|\mathcal{P}(s,c) - \mathcal{P}_m(h,s,c)| \leq \kappa/m^p, \tag{3.24}$$
where $\kappa > \overline{\kappa} := \mathrm{C}_\varphi(\overline{\mathrm{C}}_X + \overline{\mathrm{C}}_\Delta)$ is an arbitrary constant, m is sufficiently large and μ, μ_7 are small enough (c.f. (3.5)).

Remark 2. With minor modifications in our settings $p \geq 1$ would be possible until now (basically to tackle the additional case $p = 1$ we would need: the extension ψ to be a function defined on $[-h_0, h_0] \times \mathbb{R}^N$; enlarging constants in (3.2) by replacing $[0, h_0]$ with $[-h_0, h_0]$; suitable changes in the definitions of $d_m, m_0, \mathcal{I}_m, \mathcal{B}_m$). The fundamental difference in the case $p = 1$ would be that the natural requirement $0 < \Delta_m < 2h$ is generally *not* satisfied, even for m large. So the last step-size is inappropriate. Possible correction would be to find the right number of iterations of $\psi(h,\cdot)$ to ensure that the next iteration with a step $\hat{\Delta}$ near h (at least satisfying $0<\hat{\Delta}<2h$) we hit the Poincaré section. This procedure does not fit to our approach based on Lemma 2.2 therefore we are not going to specify the details.

4 Closeness of differentials

Now we would like to get an upper bound in the spirit of (3.24) but for various differentials $|D_v[\mathcal{P}(s,c)-\mathcal{P}_m(h,s,c)]|$ for $v\in\{h,s,c\}$. At first we upgrade Lemma 2.2. Undoubtedly it is of its own interest in this abstract setting.

Lemma 4.1. *Suppose all the assumption of Lemma 2.2. Moreover let us have $\alpha_1,\alpha_2,l_1\ge 0$ such that*

$$\left.\begin{array}{c}\bar{y}\in C^1(U,V) \text{ and } |\bar{y}'(x)|\le\alpha_1,\\ |\vartheta'(x)|\le\alpha_2,\ x\in U,\ \text{ for } \vartheta(x):=F(x,\bar{y}(x)),\\ |F'_x(x,y_1)-F'_x(x,y_2)|\le l_1|y_1-y_2| \text{ for } x\in U, y_1,y_2\in\overline{B(\bar{y}(x),\varrho)}.\end{array}\right\}\tag{4.1}$$

Then we are able to extend the results of Lemma 2.2 by an estimate

$$|\mathrm{y}'(x)-\bar{y}'(x)|\le\varrho_1,\quad x\in U,\tag{4.2}$$

where

$$\varrho_1:=\frac{\beta}{1-\beta l\varrho}\left(l\varrho\alpha_1+l_1\varrho+\alpha_2\right).\tag{4.3}$$

Proof. From the equations $F(x,\mathrm{y}(x))=0$ and $F(x,\bar{y}(x))=\vartheta(x)$ after differentiation we infer for $x\in U$ that

$$\begin{aligned}\mathrm{y}'(x)&=-(F'_y(x,\mathrm{y}(x)))^{-1}F'_x(x,\mathrm{y}(x)),\\ \bar{y}'(x)&=(F'_y(x,\bar{y}(x)))^{-1}(\vartheta'(x)-F'_x(x,\bar{y}(x))).\end{aligned}$$

From now we omit $(x,\mathrm{y}(x))$ and $(x,\bar{y}(x))$, the superscript $^-$ above F will indicate the substitution of $(x,\bar{y}(x))$, otherwise we substitute $(x,\mathrm{y}(x))$. We have

$$\begin{aligned}\mathrm{y}'-\bar{y}'&=(F'_y)^{-1}\left(-F'_x-F'_y\bar{y}'\right)=(F'_y)^{-1}\left((\bar{F}'_y-F'_y)\bar{y}'-\bar{F}'_y\bar{y}'-F'_x\right)\\ &=(F'_y)^{-1}\left((\bar{F}'_y-F'_y)\bar{y}'+\bar{F}'_x-F'_x-\vartheta'\right),\end{aligned}$$

from which we get exactly (4.2) (using (4.1) and the assumptions and results of Lemma 2.2) and the proof is finished. □

Adopting the notations of Theorem 3.2 and applying the previous lemma we may obtain the following statement, which is a continuation of Theorem 3.2.

Theorem 4.2. *There are constants* $\mathrm{C}_{V,v}$ *for* $V \in \{X, \Delta\}$ *and* $v \in \{h, s, c\}$ *such that*

$$\left.\begin{aligned} \left|D_v[V_m - \bar{V}_m]\right| \leq & \mathrm{C}_{V,v}/m^p, \quad V \in \{X, \Delta\}, v \in \{s, c\}, \\ \left|D_h[V_m - \bar{V}_m]\right| \leq & \mathrm{C}_{V,h}/m^{p-1}, \quad V \in \{X, \Delta\}, \end{aligned}\right\} \tag{4.4}$$

where $\delta > 0$ *is an arbitrary constant,* m *is large enough,* μ *is sufficiently small and* $(h, s, c) \in \mathcal{H}_m(p, \delta, \mu)$.

Proof. To be able to apply Lemma 4.1 twice with frameworks described in (3.16) and (3.22) we have to find additional constants (for the sake of (4.1))

$$\alpha_1 = \alpha_1[V, v], \quad \alpha_2 = \alpha_2[V, v], \quad l_1 = l_1[V, v]$$

for all $V \in \{X, \Delta\}, v \in \{h, s, c\}$. This will be a bit sweating task.

Part 1.1 – about $\alpha_1[X, v]$ *for* $v \in \{h, s, c\}$. After differentiation we get

$$D_h(\bar{x}^j) = f(\bar{x}^j)j, \quad D_s(\bar{x}^j) = \varphi'_x(jh, \xi)(f(\gamma(s)) + E'(s)c),$$
$$D_c(\bar{x}^j) = \varphi'_x(jh, \xi)E(s)$$

for $j = 1, 2, \ldots, m-1$. Therefore (using (3.2) and that $|E(s)| \leq \sqrt{N}$)

$$|D_h(\bar{X}_m)| \leq \mathrm{C}_\varphi m, \quad |D_s(\bar{X}_m)| \leq \mathrm{C}_\varphi^2 + \mu_9, \quad |D_c(\bar{X}_m)| \leq \mathrm{C}_\varphi\sqrt{N}$$

where $\mu_9 > 0$ is an arbitrary parameter and m is large enough ($\mathrm{C}_\varphi \mathrm{C}_E \delta/m^p \leq \mu_9$ is valid for m large enough). So

$$\alpha_1[X, h] := \mathrm{C}_\varphi m, \quad \alpha_1[X, s] := \mathrm{C}_\varphi^2 + \mu_9, \quad \alpha_1[X, c] := \mathrm{C}_\varphi\sqrt{N}. \tag{4.5}$$

Part 1.2 – about $\alpha_2[X, v]$ *for* $v \in \{h, s, c\}$. Note that

$$\begin{aligned} \bar{G}_m^j := & G_m(h, s, c, \bar{X}_m(h, s, c))^j = \psi(h, \bar{x}^{j-1}) - \varphi(h, \bar{x}^{j-1}) \\ = & h^{p+1}\Upsilon(h, \bar{x}^{j-1}), \qquad j = 1, 2, \ldots, m-1. \end{aligned}$$

This implies

$$\begin{aligned} D_h(\bar{G}_m^j) = & h^p[(p+1)\Upsilon(h, \bar{x}^{j-1}) \\ & + h(\Upsilon'_h(h, \bar{x}^{j-1}) + \Upsilon'_x(h, \bar{x}^{j-1})D_h(\bar{x}^{j-1}))], \\ D_s(\bar{G}_m^j) = & h^{p+1}\Upsilon'_x(h, \bar{x}^{j-1})D_s(\bar{x}^{j-1}), \\ D_c(\bar{G}_m^j) = & h^{p+1}\Upsilon'_x(h, \bar{x}^{j-1})D_c(\bar{x}^{j-1}). \end{aligned}$$

Using Part 1.1. of this proof and $h < \frac{1+\mu}{m}$ for $\bar{G}_m := (\bar{G}_m^1, \bar{G}_m^2, \dots, \bar{G}_m^{m-1})$ we infer

$$|D_h(\bar{G}_m)| \le \frac{\mathrm{C}_\Upsilon(\mathrm{C}_\varphi + p + 1) + \mu_{10}}{m^p},$$
$$|D_s(\bar{G}_m)| \le \frac{\mathrm{C}_\Upsilon \mathrm{C}_\varphi^2 + \mu_{10}}{m^{p+1}}, \qquad |D_c(\bar{G}_m)| \le \frac{\mathrm{C}_\Upsilon \mathrm{C}_\varphi \sqrt{N} + \mu_{10}}{m^{p+1}}.$$

for any fixed $\mu_{10} > 0$, every m large enough and μ sufficiently small. This yields

$$\left.\begin{aligned} \alpha_2[X,h] := \frac{\mathrm{C}_\Upsilon(\mathrm{C}_\varphi + p + 1) + \mu_{10}}{m^p}, \qquad \alpha_2[X,s] := \frac{\mathrm{C}_\Upsilon \mathrm{C}_\varphi^2 + \mu_{10}}{m^{p+1}}, \\ \alpha_2[X,c] := \frac{\mathrm{C}_\Upsilon \mathrm{C}_\varphi \sqrt{N} + \mu_{10}}{m^{p+1}}. \end{aligned}\right\} \tag{4.6}$$

Part 1.3 – about $l_1[X,v]$ for $v \in \{h,s,c\}$. We have in a moment that $l_1[X,v] = 0$ for $v \in \{s,c\}$. Further note at first that

$$D_h G_m(h,s,c,X_i) = (\psi_h(h,\xi), \psi_h(h,x_i^1), \dots, \psi_h(h,x_i^{m-1}))$$

for $X_i \in \overline{B(\bar{X}_m, \mathrm{C}_X/m^p)}$, $i \in \{1,2\}$. Now for x_1, x_2 such that $x_1 + \vartheta(x_2 - x_1) \in B$ for all $\vartheta \in [0,1]$ we have

$$\begin{aligned} |\psi_h(h,x_1) - \psi_h(h,x_2)| \le & \int_0^1 |\psi''_{hx}(h, x_2 + \vartheta(x_1 - x_2))| \mathrm{d}\vartheta |x_1 - x_2| \\ \le & \mathrm{C}_\psi |x_1 - x_2| \end{aligned}$$

which implies that $l_1[X,h] := \mathrm{C}_\psi$ is a good choice. Therefore

$$l_1[X,h] := \mathrm{C}_\psi, \quad l_1[X,s] := 0, \quad l_1[X,c] := 0. \tag{4.7}$$

Part 1.4 – determining $\mathrm{C}_{X,v}$ for $v \in \{h,s,c\}$. Now we are ready to apply Lemma 4.1 in a setting (3.16) extended with (4.5),(4.6) and (4.7). From (4.2) we obtain exactly (4.4) in a case $V = X, v \in \{h,s,c\}$ with

$$\begin{aligned} \mathrm{C}_{X,h} > & \overline{\mathrm{C}}_{X,h} := \mathrm{C}_\varphi[\mathrm{C}_\varphi^2 \overline{\mathrm{C}}_X + \mathrm{C}_\psi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon(\mathrm{C}_\varphi + p + 1)], \\ \mathrm{C}_{X,s} > & \overline{\mathrm{C}}_{X,s} := \mathrm{C}_\varphi^3[\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon], \\ \mathrm{C}_{X,c} > & \overline{\mathrm{C}}_{X,c} := \sqrt{N}\mathrm{C}_\varphi^2[\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon] \end{aligned}$$

for every m large enough. Indeed, for example in the case $v = h$ (others are treated similarly) we get from (4.3) for $\mu_{11} > 0$ that

$$\begin{aligned}\big|D_h(X_m - \bar{X}_m)\big| &\le \frac{\beta}{1-\beta l\varrho}\Big[l\varrho\alpha_1[X,h] + l_1[X,h]\varrho + \alpha_2[X,h]\Big] \\ &= \frac{\mathrm{C}_\varphi m}{(1-\mu_1)(1-\mu_3)}\left[\frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m}\frac{\mathrm{C}_X}{m^p}\mathrm{C}_\varphi m + \mathrm{C}_\psi\frac{\mathrm{C}_X}{m^p}\right. \\ &\qquad \left. + \frac{\mathrm{C}_\Upsilon(\mathrm{C}_\varphi + p + 1) + \mu_{10}}{m^p}\right] \le \frac{\overline{\mathrm{C}}_{X,h} + \mu_{11}}{m^{p-1}}\end{aligned}$$

for m large and μ small enough (we have also used (3.5) from Theorem 3.2).

Part 2.1 – about $\alpha_1[\Delta, v]$ for $v \in \{h, s, c\}$. We easily get

$$D_h(\bar{\Delta}_m) = -m + 1, \quad D_s(\bar{\Delta}_m) = \tau_s', \quad D_c(\bar{\Delta}_m) = \tau_c'.$$

Therefore

$$\alpha_1[\Delta, h] := m, \quad \alpha_1[\Delta, s] := \mathrm{C}_\tau, \quad \alpha_1[X, c] := \mathrm{C}_\tau. \tag{4.8}$$

Part 2.2 – about $\alpha_2[\Delta, v]$ for $v \in \{h, s, c\}$. Lemma 3.1 implies (see also the definition (3.18))

$$\begin{aligned}z(h,s,c,\bar{\Delta}_m) &= \big\langle \varphi(\bar{\Delta}_m, \bar{x}^{m-1}) - \gamma(s), f(\gamma(s))\big\rangle + \langle w_m(h,s,c), f(\gamma(s))\rangle \\ &= \langle w_m(h,s,c), f(\gamma(s))\rangle\,,\end{aligned}$$

where

$$w_m := \varphi(\bar{\Delta}_m, x_m^{m-1}) - \varphi(\bar{\Delta}_m, \bar{x}^{m-1}) + \bar{\Delta}_m^{p+1}\Upsilon(\bar{\Delta}_m, x_m^{m-1}).$$

Now

$$\begin{aligned}D_v z(h,s,c,\Delta_m) =& \langle D_v w_m, f(\gamma(s))\rangle, \quad s \in \{h, c\}, \\ D_s z(h,s,c,\Delta_m) =& \langle D_s w_m, f(\gamma(s))\rangle + \langle w_m, f_x'(\gamma(s))f(\gamma(s))\rangle.\end{aligned}$$

So at first we handle terms $D_v w_m$ for $v \in \{h, s, c\}$. Straightforward computation shows that

$$\left.\begin{aligned}D_v w_m =& (A_1 + A_2)D_v\bar{\Delta}_m + (A_3 + A_4)D_v\bar{x}^{m-1} \\ & + (A_5 + A_4)D_v(x_m^{m-1} - \bar{x}^{m-1}), \quad v \in \{h, s, c\}\end{aligned}\right\} \tag{4.9}$$

where

$$
\begin{aligned}
A_1 :=& \varphi_t'(\bar{\Delta}_m, x_m^{m-1}) - \varphi_t'(\bar{\Delta}_m, \bar{x}^{m-1}),\\
A_2 :=& (p+1)\bar{\Delta}_m^p \Upsilon(\bar{\Delta}_m, x_m^{m-1}) + \bar{\Delta}_m^{p+1}\Upsilon_h'(\bar{\Delta}_m, x_m^{m-1}),\\
A_3 :=& \varphi_x'(\bar{\Delta}_m, x_m^{m-1}) - \varphi_x'(\bar{\Delta}_m, \bar{x}^{m-1}),\\
A_4 :=& \bar{\Delta}_m^{p+1}\Upsilon_x'(\bar{\Delta}_m, x_m^{m-1}),\\
A_5 :=& \varphi_x'(\bar{\Delta}_m, x_m^{m-1}).
\end{aligned}
$$

Let us have $\mu_{12} > 0$, then computations as in the previous parts show that for m large and μ small enough we have

$$
|A_1 + A_2| \le \frac{\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon(p+1) + \mu_{12}}{m^p},
$$

$$
|A_3 + A_4| \le \frac{\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mu_{12}}{m^p}, \qquad |A_5 + A_4| \le \mathrm{C}_\varphi + \mu_{12}
$$

For the remaining parts of the right side of (4.9) we have upper bounds in (4.5), (4.8) and in the already proved case of (4.4) (c.f. Part 1.4). Putting this together we get for any $\mu_{13} > 0$ that

$$
|D_h w_m| \le \frac{C_1 + \mu_{13}}{m^{p-1}}, \quad |D_s w_m| \le \frac{C_2 + \mu_{13}}{m^p}, \quad |D_c w_m| \le \frac{C_3 + \mu_{13}}{m^p},
$$

where m is sufficiently large, μ is small enough and

$$
\begin{aligned}
C_1 :=& \mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon(p+1) + \mathrm{C}_\varphi^2 \overline{\mathrm{C}}_X + \overline{\mathrm{C}}_{X,h}\mathrm{C}_\varphi,\\
C_2 :=& \left(\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon(p+1)\right)\mathrm{C}_\tau + \mathrm{C}_\varphi^3 \overline{\mathrm{C}}_X + \overline{\mathrm{C}}_{X,s}\mathrm{C}_\varphi,\\
C_3 :=& \left(\mathrm{C}_\varphi \overline{\mathrm{C}}_X + \mathrm{C}_\Upsilon(p+1)\right)\mathrm{C}_\tau + \sqrt{N}\mathrm{C}_\varphi^2 \overline{\mathrm{C}}_X + \overline{\mathrm{C}}_{X,c}\mathrm{C}_\varphi.
\end{aligned}
$$

Furthermore, for $C_4 := \mathrm{C}_\varphi \overline{\mathrm{C}}_X$ similar computations show also $|w_m| \le (C_4 + \mu_{13})/m^p$. Therefore we can finish this step with the following choices

$$
\left.
\begin{aligned}
\alpha_2[\Delta, h] :=& \frac{N\mathrm{C}_\varphi C_1 + \mu_{14}}{m^{p-1}},\\
\alpha_2[\Delta, s] :=& \frac{N\mathrm{C}_\varphi (C_2 + C_4) + \mu_{14}}{m^p},\\
\alpha_2[\Delta, c] :=& \frac{N\mathrm{C}_\varphi C_3 + \mu_{14}}{m^p},
\end{aligned}
\right\} \tag{4.10}
$$

where $\mu_{14} > 0$ is an arbitrary parameter, m is large and μ is small enough.

Part 2.3 – about $l_1[\Delta, v]$ *for* $v \in \{h, s, c\}$. For $\Delta \in \overline{B(\bar{\Delta}_m, \mathrm{C}_\Delta/m^p)}$ differentiating yields

$$\begin{aligned} D_v z(h,s,c,\Delta) =& \langle \psi'_x(\Delta, x_m^{m-1}) D_v x_m^{m-1}, f(\gamma(s))\rangle, \quad v \in \{h, c\}, \\ D_s z(h,s,c,\Delta) =& \langle \psi'_x(\Delta, x_m^{m-1}) D_s x_m^{m-1}, f(\gamma(s))\rangle \\ &+ \langle \psi(\Delta, x_m^{m-1}) - \gamma(s), \varphi''_{tx}(s, \xi_0)\rangle. \end{aligned}$$

Note that from a triangle inequality we have

$$|D_h x_m^{m-1}| \le |D_h \bar{x}^{m-1}| + |D_h(x_m^{m-1} - \bar{x}^{m-1})| \le \mathrm{C}_\varphi m + \mathrm{C}_{X,h}/m^p,$$
$$|D_s x_m^{m-1}| \le \mathrm{C}_\varphi^2 + \mu_9 + \mathrm{C}_{X,s}/m^p, \qquad |D_c x_m^{m-1}| \le \sqrt{N}\mathrm{C}_\varphi + \mathrm{C}_{X,c}/m^p.$$

Employing Newton–Leibniz formula straightforward computation implies that for any $\mu_{15} > 0$ and m large enough we have the choices

$$\left.\begin{aligned} l_1[\Delta, h] := N\mathrm{C}_\psi \mathrm{C}_\varphi^2 m + \mu_{15}, \quad l_1[\Delta, s] := N\mathrm{C}_\psi \mathrm{C}_\varphi(1 + \mathrm{C}_\varphi^2) + \mu_{15}, \\ l_1[\Delta, c] := N^{3/2}\mathrm{C}_\psi \mathrm{C}_\varphi^2 + \mu_{15}. \end{aligned}\right\} \tag{4.11}$$

Part 2.4 – determining $\mathrm{C}_{\Delta,v}$ *for* $v \in \{h, s, c\}$. As in Part 1.4 we apply Lemma 4.1 in a setting (3.22) extended with (4.8), (4.10) and (4.11). From (4.2) we obtain (4.4) in a case $V = \Delta, v \in \{h, s, c\}$ with

$$\begin{aligned} \mathrm{C}_{\Delta,h} >& \overline{\mathrm{C}}_{\Delta,h} := \frac{N\mathrm{C}_\varphi[\mathrm{C}_\psi \overline{\mathrm{C}}_\Delta(1 + \mathrm{C}_\varphi) + C_1]}{\mathrm{C}_{\min}}, \\ \mathrm{C}_{\Delta,s} >& \overline{\mathrm{C}}_{\Delta,s} := \frac{N\mathrm{C}_\varphi[\mathrm{C}_\psi \overline{\mathrm{C}}_\Delta(\mathrm{C}_\tau + \mathrm{C}_\varphi^2) + C_2 + C_4]}{\mathrm{C}_{\min}}, \\ \mathrm{C}_{\Delta,c} >& \overline{\mathrm{C}}_{\Delta,c} := \frac{N\mathrm{C}_\varphi[\mathrm{C}_\psi \overline{\mathrm{C}}_\Delta(\mathrm{C}_\tau + \sqrt{N}\mathrm{C}_\varphi) + C_3]}{\mathrm{C}_{\min}} \end{aligned}$$

for every m large, μ small. The proof is complete. □

Remark 3. Now as in the proof of Theorem 4.2 (see (4.9)) we get

$$\begin{aligned} D_v\mathcal{P}(s,c) - D_v\mathcal{P}(h,s,c) = (\bar{A}_1 - \bar{A}_2) D_v \bar{\Delta}_m + (\bar{A}_3 - \bar{A}_4) D_v \bar{x}^{m-1} \\ -(\bar{A}_5 + \bar{A}_2) D_v(\Delta_m - \bar{\Delta}_m) - (\bar{A}_6 + \bar{A}_4) D_v(X_m^{m-1} - \bar{x}^{m-1}) \end{aligned} \tag{4.12}$$

for $v \in \{h, s, c\}$, where

$$\begin{aligned} \bar{A}_1 :=& \varphi'_t(\bar{\Delta}_m, \bar{x}^{m-1}) - \varphi'_t(\Delta_m, x_m^{m-1}), \\ \bar{A}_2 :=& (p+1)\Delta_m^p \Upsilon(\Delta_m, x_m^{m-1}) + \Delta_m^{p+1}\Upsilon'_h(\Delta_m, x_m^{m-1}), \\ \bar{A}_3 :=& \varphi'_x(\bar{\Delta}_m, \bar{x}^{m-1}) - \varphi'_x(\Delta_m, x_m^{m-1}), \\ \bar{A}_4 :=& \Delta_m^{p+1}\Upsilon'_x(\Delta_m, x_m^{m-1}), \\ \bar{A}_5 :=& \varphi'_t(\Delta_m, x_m^{m-1}), \quad \bar{A}_6 := \varphi'_x(\Delta_m, x_m^{m-1}). \end{aligned}$$

From (3.4) we infer

$$|\Delta_m| \le |\bar{\Delta}_m| + |\Delta_m - \bar{\Delta}_m| \le \frac{1+\mu}{m} + \frac{\mathrm{C}_\Delta}{m^p} = \frac{1+\mu+\frac{\mathrm{C}_\Delta}{m^{p-1}}}{m}.$$

After a lengthy computation for $\mu_{16} > 0$ we get

$$|\bar{A}_1 - \bar{A}_2| \le \frac{C_5 + \mu_{16}}{m^p}, \quad |\bar{A}_3 - \bar{A}_4| \le \frac{\mathrm{C}_\varphi \overline{\mathrm{C}}_\Delta + \mu_{16}}{m^p},$$
$$|\bar{A}_5 + \bar{A}_2| \le \mathrm{C}_\varphi + \mu_{16}, \quad |\bar{A}_6 + \bar{A}_4| \le \mathrm{C}_\varphi + \mu_{16},$$

where $C_5 := \mathrm{C}_\varphi(\overline{\mathrm{C}}_\Delta + \overline{\mathrm{C}}_X) + (p+1)C_5^p \mathrm{C}_\Upsilon$, m is large μ is small enough. Using in addition (4.4), (4.5) and (4.8) for remaining terms in (4.12) we finally obtain

$$\left.\begin{aligned} |D_h[\mathcal{P}(s,c) - \mathcal{P}_m(h,s,c)]| \le& \kappa_h/m^{p-1}, \\ |D_v[\mathcal{P}(s,c) - \mathcal{P}_m(h,s,c)]| \le& \kappa_v/m^p, \quad v \in \{s,c\}, \end{aligned}\right\} \tag{4.13}$$

where

$$\begin{aligned} \kappa_h >& C_5 + \mathrm{C}_\varphi \overline{\mathrm{C}}_\Delta \mathrm{C}_\varphi + \mathrm{C}_\varphi(\mathrm{C}_{\Delta,h} + \mathrm{C}_{X,h}), \\ \kappa_s >& C_5 \mathrm{C}_\tau + \mathrm{C}_\varphi \overline{\mathrm{C}}_\Delta \mathrm{C}_\varphi^2 + \mathrm{C}_\varphi(\mathrm{C}_{\Delta,s} + \mathrm{C}_{X,s}), \\ \kappa_c >& C_5 \mathrm{C}_\tau + \mathrm{C}_\varphi \overline{\mathrm{C}}_\Delta \mathrm{C}_\varphi \sqrt{N} + \mathrm{C}_\varphi(\mathrm{C}_{\Delta,c} + \mathrm{C}_{X,c}). \end{aligned}$$

One may wish to continue in this direction developing bounds for

$$D^2_{v_1 v_2}[\mathcal{P}_m(h,s,c) - \mathcal{P}(s,c)], \quad v_1, v_2 \in \{h,s,c\}.$$

This is quite technical (computations rather for computer), therefore we only show the key equipment namely the natural extension of Lemma 2.2 to the next level in the spirit of Lemma 4.1.

Lemma 4.3. *Suppose all the assumptions of Lemma 2.2 with $X = X_1 \times X_2 \times X_3$ ($X_i, i \in \{1,2,3\}$ are Banach spaces, and $|\cdot|_X := \max_{i\in\{1,2,3\}} |\cdot|_{X_i}$). Let us have $F \in C^r(U \times V, Z)$ for $r \ge 2$ and also $\bar{y} \in C^2(U,V)$. Suppose (like in (4.1)) that*

$$\left.\begin{aligned} |D_{x_i}\bar{y}| \le \alpha_{1,i}, \quad |D_{x_i}\vartheta| \le \alpha_{2,i}, \\ |F'_{x_i}(x,y_1) - F'_{x_i}(x,y_2)| \le l_{1,i}|y_1 - y_2|, \quad x \in U, y_1, y_2 \in \overline{B(\bar{y}(x),\varrho)} \end{aligned}\right\} \tag{4.14}$$

for $i \in \{1,2,3\}$. *Introduce also* $\varrho_{1,i} := \frac{\beta}{1-\beta l \varrho}\left(l\varrho\alpha_{1,i} + l_{1,i}\varrho + \alpha_{2,i}\right)$ *accordingly to* (4.2). *Further let us have*

$$\left.\begin{aligned} |D^2_{x_i x_j}\bar{y}| \le \alpha_{3,i,j}, \quad |D^2_{x_i x_j}\vartheta| \le \alpha_{4,i,j}, \\ |F''_{x_i y}(x,\bar{y}(x))| \le \alpha_{5,i}, \quad |F''_{yy}(x,\bar{y}(x))| \le \alpha_6, \\ |F''_{x_i x_j}(x,y_1) - F''_{x_i x_j}(x,y_2)| \le l_{2,i,j}|y_1-y_2|, \\ |F''_{x_i y}(x,y_1) - F''_{x_i y}(x,y_2)| \le l_{3,i}|y_1-y_2|, \\ |F''_{yy}(x,y_1) - F''_{yy}(x,y_2)| \le l_4|y_1-y_2| \end{aligned}\right\} \tag{4.15}$$

for $i,j \in \{1,2,3\}, i \le j$ *and for all* $x \in U$ *and* $y_1, y_2 \in \overline{B(\bar{y}(x),\varrho)}$. *Then*

$$|D_{x_i x_j}\mathrm{y}(x) - D_{x_i x_j}\bar{y}(x)| \le \varrho_{2,i,j}, \quad x \in U, \quad i,j \in \{1,2,3\}, i \le j, \tag{4.16}$$

where

$$\begin{aligned} \varrho_{2,i,j} :=& \frac{\beta}{1-\beta l\varrho}\Big(l\varrho\alpha_{3,i,j} + \alpha_{4,i,j} + \varrho l_{2,i,j} + \varrho_{1,j}\alpha_{5,i} + \varrho l_{3,i}(\alpha_{1,j} + \varrho_{1,j}) \\ &+ \varrho_{1,i}\alpha_{5,j} + \varrho l_{3,j}(\alpha_{1,i} + \varrho_{1,i}) + \varrho_{1,i}\alpha_6\alpha_{1,j} + \varrho_{1,j}\alpha_6(\alpha_{1,i} + \varrho_{1,i}) \\ &+ \varrho l_4(\alpha_{1,i} + \varrho_{1,i})(\alpha_{1,j} + \varrho_{1,j})\Big). \end{aligned}$$

Proof. Partial derivations with respect to x_i into the direction $\delta v \in X_i$ of the equations $F(x,\mathrm{y}(x)) = 0$ and $F(x,\bar{y}(x)) = \vartheta(x)$ gives (we use notation $\bar{F}$ from the proof of Lemma 4.1)

$$F'_{x_i}\delta v + F'_y \mathrm{y}'_{x_i}\delta v = 0, \quad \bar{F}'_{x_i}\delta v + \bar{F}'_y\bar{y}'_{x_i}\delta v = \vartheta'_{x_i}\delta v.$$

Now differentiating once more with respect to x_j into the direction $\delta w \in X_j$ we get

$$\begin{gathered} F''_{x_i x_j}[\delta v,\delta w] + F''_{x_i y}[\delta v, \mathrm{y}'_{x_j}\delta w] + F''_{y x_j}[\mathrm{y}'_{x_i}\delta v, \delta w] + F''_{yy}[\mathrm{y}'_{x_i}\delta v, \mathrm{y}'_{x_j}\delta w] \\ +F'_y \mathrm{y}''_{x_i x_j}[\delta v,\delta w] = 0, \\ \bar{F}''_{x_i x_j}[\delta v,\delta w] + \bar{F}''_{x_i y}[\delta v, \bar{y}'_{x_j}\delta w] + \bar{F}''_{y x_j}[\bar{y}'_{x_i}\delta v, \delta w] + \bar{F}''_{yy}[\bar{y}'_{x_i}\delta v, \bar{y}'_{x_j}\delta w] \\ +\bar{F}'_y\bar{y}''_{x_i x_j}[\delta v,\delta w] = \vartheta''_{x_i x_j}[\delta v,\delta w]. \end{gathered}$$

Therefore as in the proof of Lemma 4.1 we infer

$$\begin{aligned}
&(\mathrm{y}''_{x_ix_j} - \bar{y}''_{x_ix_j})[\delta v, \delta w] \\
=&(F'_y)^{-1}\Big\{(\bar{F}'_y - F'_y)\bar{y}''_{x_ix_j} + F'_y\mathrm{y}''_{x_ix_j} - \bar{F}'_y\bar{y}''_{x_ix_j}\Big\}[\delta v, \delta w] \\
=&(F'_y)^{-1}\Big\{\big[(\bar{F}'_y - F'_y)\bar{y}''_{x_ix_j} - \vartheta''_{x_ix_j} + (\bar{F}''_{x_ix_j} - F''_{x_ix_j})\big][\delta v, \delta w] \\
&+ \bar{F}''_{x_iy}[\delta v, (\bar{y}'_{x_j} - \mathrm{y}'_{x_j})\delta w] + (\bar{F}''_{x_iy} - F''_{x_iy})[\delta v, \mathrm{y}'_{x_j}\delta w] \\
&+ \bar{F}''_{yx_j}[(\bar{y}'_{x_i} - \mathrm{y}'_{x_i})\delta v, \delta w] + (\bar{F}''_{yx_j} - F''_{yx_j})[\mathrm{y}'_{x_i}\delta v, \delta w] \\
&+ \bar{F}''_{yy}[(\bar{y}'_{x_i} - \mathrm{y}'_{x_i})\delta v, \bar{y}'_{x_j}\delta w] + \bar{F}''_{yy}[\mathrm{y}'_{x_i}\delta v, (\bar{y}'_{x_j} - \mathrm{y}'_{x_j})\delta w] \\
&+ (\bar{F}''_{yy} - F''_{yy})[\bar{y}'_{x_i}\delta v, \bar{y}'_{x_j}\delta w]\Big\}.
\end{aligned}$$

Now using the symmetry of the second derivatives, switching to the norms and employing the assumptions of the theorem the final statement (4.16) follows and the proof is finished. □

Now we show a sketch of one possible application of Lemma 4.3. Let the equation $G_m(h, s, c, X) = 0$ be in the role of $F(x_1, x_2, x_3, y) = 0$ with a basic framework given in (3.16). We only deal with the case $i = j = 3$, when we are looking for a bound of $|D^2_{cc}X_m - D^2_{cc}\bar{X}_m|$. The proof of Theorem 4.2 – namely (4.5), (4.6) and (4.7) – using notations of (4.14) implies

$$\alpha_1 = \sqrt{N}\mathrm{C}_\varphi, \quad \alpha_2 = \frac{\sqrt{N}\mathrm{C}_\varphi\mathrm{C}_\Upsilon + \mu_{10}}{m^{p+1}}, \quad l_1 = 0$$

needed in (4.14). Remaining constants in (4.15), skipping the details of the lengthy computation, are

$$\begin{gathered}
\alpha_3 = N\mathrm{C}_\varphi, \quad \alpha_4 = \frac{N(1+\mu)^{p+1}\mathrm{C}_\varphi\mathrm{C}_\Upsilon(1+\mathrm{C}_\varphi)}{m^{p+1}}, \quad \alpha_5 = 0, \\
\alpha_6 := \frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m}, \quad l_2 = l_3 = 0, \quad l_4 = \frac{(1+\mu)\mathrm{C}_\varphi + \mu_2}{m}.
\end{gathered}$$

Now application of Lemma 4.3 yields that for $\mathrm{C}_{X,cc} > \overline{\mathrm{C}}_{X,cc}$, m large and μ small enough we have

$$|D^2_{cc}X_m - D^2_{cc}\bar{X}_m| \le \mathrm{C}_{X,cc}/m^p, \tag{4.17}$$

where $\overline{\mathrm{C}}_{X,cc} := \mathrm{C}^2_\varphi[N\mathrm{C}_\varphi\overline{\mathrm{C}}_X + N\mathrm{C}_\Upsilon(1+\mathrm{C}_\varphi) + 2\sqrt{N}\mathrm{C}_\varphi\overline{C}_{X,c} + N\mathrm{C}^2_\varphi\overline{\mathrm{C}}_X]$.

Similarly it is possible to handle the equation $z(h, s, c, \Delta) = 0$ in a setting (3.22). From (4.8), (4.10) and (4.11) we get

$$\alpha_1 = \mathrm{C}_\tau, \quad \alpha_2 = \frac{N\mathrm{C}_\varphi C_3 + \mu_{14}}{m^p}, \quad l_1 = N^{3/2}\mathrm{C}^2_\varphi\mathrm{C}_\psi + \mu_{15}.$$

Omitting again the details we get for $\mu_{17} > 0$, m large and μ small enough that

$$\alpha_3 = \mathrm{C}_\tau, \quad \alpha_4 = \frac{N(p+1)p\mathrm{C}_\varphi \mathrm{C}_\Upsilon + \mu_{17}}{m^{p-1}},$$
$$\alpha_5 = N^{3/2}\mathrm{C}_\varphi^2 \mathrm{C}_\psi + \mu_{17}, \quad \alpha_6 = N\mathrm{C}_\varphi \mathrm{C}_\psi,$$
$$l_2 = N^2 \mathrm{C}_\varphi^2 \mathrm{C}_\psi(\mathrm{C}_\varphi + 1) + \mu_{17}. \quad l_3 = \alpha_5, \quad l_4 = \alpha_6.$$

So Lemma 4.3 gives

$$|D^2_{cc}\Delta_m - D^2_{cc}\bar{\Delta}_m| \le \mathrm{C}_{\Delta,cc}/m^{p-1} \tag{4.18}$$

for m large enough where $\mathrm{C}_{\Delta,cc} > \overline{\mathrm{C}}_{\Delta,cc} := \frac{N(p+1)p\mathrm{C}_\varphi \mathrm{C}_\Upsilon}{\mathrm{C}_{\min}}$.

Now as in the Remark 1 it would be possible to derive

$$|D^2_{cc}\mathcal{P}_m(h,s,c) - D^2_{cc}\mathcal{P}(s,c)| \le C/m^{p-1}$$

for some constant C. Instead of this we show a weaker result, namely that $|D^2_{cc}\mathcal{P}_m(h,s,c)|$ is uniformly bounded for every m large enough (uniformity is related to m-s).

Differentiation yields

$$\begin{aligned}
&D^2_{cc}\mathcal{P}_m(h,s,c)[\delta v,\delta w] = \psi''_{hh}(\Delta_m, x_m^{m-1})[D_c\Delta_m\delta v, D_c\Delta_m\delta w]\\
&+ \psi''_{hx}(\Delta_m, x_m^{m-1})[D_c\Delta_m\delta v, D_c x_m^{m-1}\delta w] + \psi'_h(\Delta_m, x_m^{m-1})D^2_{cc}\Delta_m[\delta v,\delta w]\\
&+ \psi''_{xh}(\Delta_m, x_m^{m-1})[D_c x_m^{m-1}\delta v, D_c\Delta_m\delta w]\\
&+ \psi''_{xx}(\Delta_m, x_m^{m-1})[D_c x_m^{m-1}\delta v, D_c x_m^{m-1}\delta w] + \psi'_x(\Delta_m, x_m^{m-1})D^2_{cc}x_m^{m-1}[\delta v,\delta w].
\end{aligned}$$

Switching to the norms, using (3.2), (4.4), (4.17) and (4.18) after some computations we obtain

$$|D^2_{cc}\mathcal{P}_m(h,s,c)| \le C_6 \tag{4.19}$$

for $C_6 > \overline{C}_6 := \mathrm{C}_\psi\big[(\mathrm{C}_\tau + \sqrt{N}\mathrm{C}_\varphi)^2 + \mathrm{C}_\tau + N\mathrm{C}_\varphi\big]$, large m and small μ.

5 A closed curve for a discrete dynamics

The *nondegeneracy* condition of γ

$$1 \text{ is a simple eigenvalue of } \varphi'_x(1,\xi_0) \tag{5.1}$$

is in the central role in this section.

The word *simple* means that the algebraic multiplicity of the eigenvalue 1 is one, in other words $\lambda = 1$ is a simple root of the characteristic polynomial

$$\det\big(\lambda I - \varphi_x'(1,\xi_0)\big).$$

Noting

$$\varphi_x'(1,\gamma(s)) = Q\varphi_x'(1,\xi_0)Q^{-1}, \quad Q := \varphi_x'(s,\xi_0), \quad s \in \mathbb{R}$$

we have that (5.1) is equivalent to

$$1 \text{ is a simple eigenvalue of } \varphi_x'(1,\gamma(s)) \tag{5.2}$$

for any $s \in \mathbb{R}$.

Introduce $\mathbb{A}_s := E(s)^T\varphi_x'(1,\gamma(s))E(s) - I_{N-1}$ where I_{N-1} is an $(N-1)\times(N-1)$ identity matrix. Condition (5.1) implies that $\mathbb{A}_s$ is invertible. Indeed, suppose on the contrary that $\mathbb{A}_s v = 0$ for $v \in \mathbb{R}^{N-1}$, $v \neq 0$. Then for $w := E(s)v \neq 0$ we infer

$$\varphi_x'(1,\gamma(s))w = \alpha f(\gamma(s)) + w, \qquad \text{for some } \alpha \in \mathbb{R}.$$

Using also that $\varphi_x'(1,\gamma(s))f(\gamma(s)) = f(\gamma(s))$ we get $(I - \varphi_x'(1,\gamma(s)))^2 w = 0$. Therefore the geometric multiplicity of the eigenvalue 1 is at least 2 (w and $f(\gamma(s))$ are linearly independent vectors from the generalized eigenspace). This is a contradiction with (5.2) (geometric multiplicity is always less than or equal to algebraic multiplicity – for more details see [12, Chapter 6 and Appendix III]).

Theorem 5.1. *Suppose that* (5.1) *holds and we have* $\delta > \sqrt{N}\overline{\kappa}a$ *where* $a := \max_{s\in[0,1]}\left|\mathbb{A}_s^{-1}\right|$. *Then for every* m *large enough and* μ *sufficiently small there is a unique function*

$$\zeta_m : \mathcal{I}_m(p,\delta,\mu)\times\mathbb{R} \to \mathcal{B}_m(p,\delta)$$

such that

$$\mathcal{P}_m(h,s,\zeta_m(h,s)) = \xi(s,\zeta_m(h,s)), \quad (h,s) \in \mathcal{I}_m\times\mathbb{R}. \tag{5.3}$$

In addition ζ_m *is* C^3*-smooth in its arguments and* $\zeta_m(h,s+1) = \zeta_m(h,s)$ *for all* $(h,s) \in \mathcal{I}_m\times\mathbb{R}$.

Proof. Introduce $g(h,s,c) := E(s)^T (\mathcal{P}_m(h,s,c) - \gamma(s)) - c$ for $(h,s,c) \in \mathcal{H}_m$. Then it is easy to see that (5.3) is equivalent to $g(h,s,\zeta_m(h,s)) = 0$. To settle this we apply again Lemma 2.2 in the framework

$$U = \mathcal{I}_m \times \mathbb{R}, \quad V = \mathbb{R}^{N-1}, \quad x = (h,s), \quad \bar{y}(x) = 0 \in \mathbb{R}^{N-1}.$$

From (3.24) we get

$$|g(h,s,0)| = |E(s)^T||\mathcal{P}_m(h,s,0) - \mathcal{P}(s,0)| \leq \frac{\sqrt{N}\kappa}{m^p}.$$

Further using $\mathcal{P}_c'(s,0) = f(\gamma(s))\tau_c'(s,0) + \varphi_x'(1,\gamma(s))E(s)$ and (2.2) it is straightforward to verify that

$$g_c'(h,s,0) = \mathbb{A}_s + W, \quad W := E(s)^T \left[(\mathcal{P}_m)_c' (h,s,0) - \mathcal{P}_c'(s,0) \right].$$

From (4.13) we have $|W| \leq \frac{\sqrt{N}\kappa_c}{m^p}$. Picking up any $\mu_{18} \in [0,1)$ for every m large enough we obtain

$$|\mathbb{A}_s^{-1} W| \leq \frac{a\sqrt{N}\kappa_c}{m^p} \leq \mu_{18} < 1.$$

So from Lemma 2.1 we infer that $g_c'(h,s,0)$ is invertible with

$$|g_c'(h,s,0)^{-1}| \leq \frac{a}{1-\mu_{18}}.$$

Next (4.19) easily gives for $c_1, c_2 \in \mathcal{B}_m$ that

$$\begin{aligned} & |g_c'(h,s,c_1) - g_c'(h,s,c_2)| \\ = & \left|E(s)^T\right| \int_0^1 \left|(\mathcal{P}_m)_{cc}''(h,s,c_2 + \vartheta(c_1 - c_2))\right| \mathrm{d}\vartheta |c_1 - c_2| \\ \leq & \sqrt{N} C_6 |c_1 - c_2|. \end{aligned}$$

In the context of the setting of Lemma (2.2) we have derived

$$\alpha = \frac{\sqrt{N}\kappa}{m^p}, \quad \beta = \frac{a}{1-\mu_{18}}, \quad l = \sqrt{N} C_6$$

and we have also $\varrho = \frac{\delta}{m^p}$. Now for any $\mu_{19} \in (0,1)$ we get

$$\beta l \varrho = \frac{a\sqrt{N} C_6 \delta}{(1-\mu_{18}) m^p} \leq \mu_{19} < 1$$

for m large enough. So (2.5) holds. Further

$$\frac{\alpha\beta}{\varrho(1-\beta l\varrho)} = \frac{\sqrt{N}\kappa a}{(1-\mu_{18})(1-\mu_{19})\delta}$$

yields that (2.6) is valid if and only if

$$\frac{\sqrt{N}\kappa a}{(1-\mu_{18})(1-\mu_{19})} < \delta.$$

This is satisfied for m large and $\kappa - \overline{\kappa}, \mu, \mu_{18}, \mu_{19}$ small enough because of (3.24) and $\delta > \sqrt{N}\overline{\kappa}a$. Application of Lemma 2.2 gives ζ_m with the desired properties and the proof is finished. □

Remark 4. Introducing

$$\mathcal{N}_m := \mathcal{N}_m(p,\delta) := \left\{\xi(s,c) \in \mathbb{R}^N \,:\, s \in \mathbb{R},\, c \in \mathcal{B}_m(p,\delta)\right\}$$

according to Theorem 5.1 we can state for the appropriate values of parameters that

$$\left\{x \in \mathcal{N}_m(p,\delta) \,:\, \mathcal{P}_m\left(h, \xi^{-1}(x)\right) = x\right\} = \left\{\xi(s, \zeta_m(h,s)) \,:\, s \in \mathbb{R}\right\}.$$

Thorough study of the set of m-periodic points for discretized dynamics was done in [4]. Our approach implies some results also to this direction.

Theorem 5.2. *Suppose all the assumptions of Theorem 5.1 and fix any $\eta \in (0,1)$. Then for m large enough we have for every $s \in \mathbb{R}$ a unique element $h^\star(s) \in \mathcal{I}_m$ such that*

$$\Delta_m(h^\star(s), s, \zeta_m(h^\star(s), s)) = h^\star(s).$$

Further $h^\star(s+1) = h^\star(s)$ and $h^\star \in C^3(\mathbb{R}, \mathcal{I}^\star_m)$ where

$$\mathcal{I}^\star_m := \left(\frac{1}{m} - d^\star_m, \frac{1}{m} + d^\star_m\right), \quad d^\star_m := \frac{\mathrm{C}_\Delta + \mathrm{C}_\tau\delta}{m^p(m-\eta)} < d_m.$$

Therefore

$$\left\{x \in \mathcal{N}_m(p,\delta) \,:\, x = \psi^m(h,x)\right\} = \left\{\xi(s, \zeta_m(h^\star(s), s)) \,:\, s \in \mathbb{R}\right\}.$$

Proof. It is an elementary fact that for

$$\mathrm{g}(h,s) := \Delta_m(h, s\zeta_m(h,s)) - h, \quad h \in \mathcal{I}_m, s \in \mathbb{R}$$

we have

$$\mathrm{g}(h,s) = \mathrm{g}(1/m,s) + \int_{1/m}^{h} \mathrm{g}'_h(\vartheta,s)\mathrm{d}\vartheta. \tag{5.4}$$

Now (3.2) and (4.4) yields

$$\begin{aligned}|\mathrm{g}(1/m,s)| =& |\Delta_m(1/m,s,\zeta_m(1/m,s)) - \bar{\Delta}_m(1/m,s,0)|\\ \leq& |\Delta_m(1/m,s,\zeta_m(1/m,s)) - \bar{\Delta}_m(1/m,s,\zeta_m(1/m,s))|\\ &- |\bar{\Delta}_m(1/m,s,\zeta_m(1/m,s)) - \bar{\Delta}_m(1/m,s,0)|\\ \leq& \frac{\mathrm{C}_\Delta + \mathrm{C}_\tau\delta}{m^p}.\end{aligned}$$

Further

$$\mathrm{g}'_h(\vartheta,s) = (\Delta_m)'_h(\vartheta,s,\zeta_m(\vartheta,s)) + (\Delta_m)'_c(\vartheta,s,\zeta_m(\vartheta,s))(\zeta_m)'_h(\vartheta,s) - 1.$$

From Theorem 5.1 (using notation for g from its proof) we infer

$$(\zeta_m)'_h(\vartheta,s) = -g'_h(\vartheta,s,\zeta_m(\vartheta,s))\left[g'_c(\vartheta,s,\zeta_m(\vartheta,s))\right]^{-1}$$

hence (cf. Lemma 2.2 and bound (4.13))

$$|(\zeta_m)'_h(\vartheta,s)| \leq \frac{a\sqrt{N}\kappa_h}{(1-\mu_{18})(1-\mu_{19})m^{p-1}}.$$

In addition using (4.4) elementary computation shows

$$\begin{gathered}(\Delta_m)'_h(\vartheta,s,\zeta_m(\vartheta,s)) = (\bar{\Delta}_m)'_h(\vartheta,s,\zeta_m(\vartheta,s)) + w_m = -m + 1 + v_m,\\ v_m := (\Delta_m)'_h(\vartheta,s,\zeta_m(\vartheta,s)) - (\bar{\Delta}_m)'_h(\vartheta,s,\zeta_m(\vartheta,s)), \quad |v_m| \leq \frac{\mathrm{C}_{\Delta,h}}{m^{p-1}},\\ |(\Delta_m)'_c(\vartheta,s,\zeta_m(\vartheta,s))| \leq \mathrm{C}_\tau + \frac{\mathrm{C}_{\Delta,c}}{m^p}.\end{gathered}$$

Combining these facts we get

$$\mathrm{g}'_h(\vartheta,s) = -m + w_m, \qquad |w_m| \leq \eta \tag{5.5}$$

for every m large enough.

The relation $d_m^\star < d_m$ holds evidently for m sufficiently large. Now (5.4) implies after easy computations that

$$\left.\begin{aligned} &g(h,s) < 0, \text{ for } h \in \left[\frac{1}{m} + d_m^\star, \frac{1}{m} + d_m\right), \\ &g(h,s) > 0, \text{ for } h \in \left(\frac{1}{m} - d_m, \frac{1}{m} - d_m^\star\right]. \end{aligned}\right\} \tag{5.6}$$

Because of $g(\cdot, s) : \mathcal{I}_m \to \mathbb{R}$ is a C^1-function with properties (5.6) and (5.5) we get a unique element $h^\star(s) \in \mathcal{I}_m$ such that $g(h^\star(s), s) = 0$ moreover $h^\star(s) \in \mathcal{I}_m^\star$. Application of the Implicit Function Theorem for the equation $g(h, s) = 0$ in the neighbourhood of the solution $(h^\star(s'), s')$ for any $s' \in \mathbb{R}$ yields also the C^3-smoothness of $h^\star : \mathbb{R} \to \mathcal{I}_m^\star$ and the proof is completed (the periodicity of $h^\star$ is straightforward). □

Remark 5. Usual arguments yield that for any $A_0 \in \mathcal{L}(\mathbb{R}^{N-1})$ and $r > 0$ we have that the following minimum is attained and

$$\min_{\substack{\lambda \in \mathbb{C}\setminus B_r \\ z \in \mathbb{R}^{N-1}, |z|=1}} |(\lambda I - A_0)z| := c(r) > 0,$$

where $B_r := \bigcup_{\mu \in \sigma(A_0)} B(\mu, r)$ and $\sigma(A_0) \subset \mathbb{C}$ is the spectrum of A_0. Therefore for any $A \in \mathcal{L}(\mathbb{R}^{N-1})$ such that $|A - A_0| < c(r)$ we have $\sigma(A) \subset B_r$ (for more general statement see [3, Corollary 2.6, pp. 470]). Indeed, for $\lambda \in \mathbb{C} \setminus B_r$ we have

$$\lambda I - A = (\lambda I - A_0)(I + (\lambda I - A_0)^{-1}(A_0 - A))$$

and

$$\left|(\lambda I - A_0)^{-1}(A_0 - A)\right| \le \left|(\lambda I - A_0)^{-1}\right| |A_0 - A| < \frac{1}{c(r)} \cdot c(r) = 1.$$

From Lemma 2.1 we get $\lambda \in \mathbb{C} \setminus \sigma(A)$ which gives $\mathbb{C} \setminus B_r \subset \mathbb{C} \setminus \sigma(A)$ and we are done. Now set

$$A_0 := E(s)^T \mathcal{P}_c'(s, 0), \qquad A := E(s)^T (\mathcal{P}_m)_c'(h, s, c)$$

for any $(h, s, c) \in \mathcal{H}_m$. Then after careful computations using primarily (4.13) we get $|A - A_0| < c(r)$ for m large enough. This yields

$$\sigma(A) \subset \bigcup_{\mu \in \sigma(A_0)} B(\mu, r).$$

Hence with the substitution $c = \zeta_m(h, s)$, or $h = h^\star(s)$ and $c = \zeta_m(h^\star(s), s)$, we get for above detected curves the corresponding closeness statements about their (h, s) and s dependent spectrums.

Appendix

For the sake of completeness we collect here some proofs.

Proof of Lemma 2.2. We transform the task to the fixed point problem of the mapping

$$G(x,y) := y - [D_yF(x,\bar{y}(x))]^{-1}F(x,y).$$

Choose an arbitrary $x \in U$. For $y_1, y_2 \in \overline{B(\bar{y}(x),\varrho)}$ we have

$$\begin{aligned}|G(x,y_1) - G(x,y_2)| &= \left|y_1 - y_2 - D_yF(x,\bar{y}(x))^{-1}(F(x,y_1) - F(x,y_2))\right| \\ &\leq \beta \left|\int_0^1 \left[D_yF(x,\bar{y}(x)) - D_yF(x,y_2 + s(y_1-y_2))\right](y_1-y_2)\mathrm{d}s\right| \\ &\leq \beta l \int_0^1 |y_2 + s(y_1-y_2)|\,|y_1-y_2|\mathrm{d}s \leq \beta l\varrho|y_1-y_2|,\end{aligned}$$

where we used $y_2 + s(y_1 - y_2) \in \overline{B(\bar{y}(x),\varrho)}$ which is caused by the convexity of the closed ball $\overline{B(\bar{y}(x),\varrho)}$.

On the other hand $|G(x,\bar{y}(x)) - \bar{y}(x)| \leq \beta\alpha < \varrho(1-\beta l\varrho)$ (cf. (2.6)) implies for any $y \in \overline{B(\bar{y}(x),\varrho)}$ that

$$\begin{aligned}|G(x,y) - \bar{y}(x)| \leq& |G(x,y) - G(x,\bar{y}(x))| + |G(x,\bar{y}(x)) - \bar{y}(x)| \\ <& \beta l\varrho|y - \bar{y}(x)| + \varrho(1-\beta l\varrho) \leq \beta l\varrho^2 + \varrho(1-\beta l\varrho) = \varrho\end{aligned}$$

therefore $G(x,\cdot) : \overline{B(\bar{y}(x),\varrho)} \to B(\bar{y}(x),\varrho) \subset \overline{B(\bar{y}(x),\varrho)}$ and it is a contraction (from (2.5)). Banach's theorem yields a unique fixed point $\mathrm{y}(x) \in \overline{B(\bar{y}(x),\varrho)}$ of this mapping which lies in $B(\bar{y}(x),\varrho)$.

Now because of

$$\begin{aligned}&D_yF(x,\mathrm{y}(x)) = D_yF(x,\bar{y}(x)) + D_yF(x,\mathrm{y}(x)) - D_yF(x,\bar{y}(x)) \\ &= D_yF(x,\bar{y}(x))\left[I + D_yF(x,\bar{y}(x))^{-1}\left(D_yF(x,\mathrm{y}(x)) - D_yF(x,\bar{y}(x))\right)\right]\end{aligned}$$

and

$$\left|D_yF(x,\bar{y}(x))^{-1}\left(D_yF(x,\mathrm{y}(x)) - D_yF(x,\bar{y}(x))\right)\right| \leq \beta l\varrho < 1$$

we get from Lemma 2.1 that $D_yF(x,\mathrm{y}(x))$ is invertible with

$$\begin{aligned}\left|D_yF(x,\mathrm{y}(x))^{-1}\right| \leq& \frac{\left|D_yF(x,\bar{y}(x))^{-1}\right|}{1 - |D_yF(x,\bar{y}(x))^{-1}(D_yF(x,\mathrm{y}(x)) - D_yF(x,\bar{y}(x))|} \\ \leq& \frac{\beta}{1-\beta l\varrho}.\end{aligned}$$

Now we show C^r-smoothness. Choose any $x_0 \in U$ and let $y_0 := \mathrm{y}(x_0)$. From the results above we have $F(x_0, y_0) = 0$ and that $D_y F(x_0, y_0)$ is continuously invertible. The Implicit Function Theorem yields a unique function $y^\star \in C^r(U', V')$ such that $F(x, y) = 0$ holds for $(x, y) \in U' \times V'$ if and only if $y = y^\star(x)$. Here U', V' are sufficiently small open sets with properties

$$x_0 \in U' \subset U, \quad y_0 \in V' \subset B(\bar{y}(x_0), \varrho).$$

Next

$$|y^\star(x) - \bar{y}(x)| \leq |y^\star(x) - y_0| + |y_0 - \bar{y}(x_0)| + |\bar{y}(x_0) - \bar{y}(x)|.$$

Here the second term is smaller than ϱ and other two terms are arbitrarily small if x is sufficiently close to x_0 (because of the continuity of $y^\star$ and $\bar{y}$ at x_0). Therefore we have an open set U'' such that $x_0 \in U'' \subset U'$ and for which $|y^\star(x) - \bar{y}(x)| < \varrho$ for every $x \in U''$. The uniqueness of the first part of this proof ensures that $\mathrm{y} = y^\star$ on U'', so $\mathrm{y}|_{U''} \in C^r(U'', V)$. Because x_0 was chosen arbitrarily in U we also get $\mathrm{y} \in C^r(U, V)$ and the proof is finished. □

Proof of Lemma 3.1. The C^3-smoothness of $z : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^{N-1} \to \mathbb{R}$ is straightforward. It is easy to see that

$$z(1, s, 0) = 0 \text{ and } D_t z(t, s, c)|_{t=1, c=0} = |f(\gamma(s))|_2^2 \neq 0.$$

From the Implicit Function Theorem we get for all $s' \in [0, 1]$ numbers $\delta(s') > 0, \eta(s') > 0, \varepsilon(s') \in (0, 1/2)$ and C^3-smooth implicit functions

$$\tau^{s'} : (s' - \eta(s'), s' + \eta(s')) \times B_{N-1}^{\delta(s')} \to (1 - \varepsilon(s'), 1 + \varepsilon(s'))$$

determined uniquely by the equation (3.1) for

$$(t, s, c) \in (1 - \varepsilon(s'), 1 + \varepsilon(s')) \times (s' - \eta(s'), s' + \eta(s')) \times B_{N-1}^{\delta(s')}.$$

Now $\bigcup_{s' \in [0,1]} \big(s' - \eta(s')/2, s' + \eta(s')/2\big) \supset [0, 1]$ so we can choose a finite number of elements $0 \leq s_1 \leq \cdots \leq s_k \leq 1$ such that $\bigcup_{i=1}^k \big(s_i - \eta(s_i)/2, s_i + \eta(s_i)/2\big) \supset [0, 1]$. Introduce

$$\delta := \min\{\delta_{\mathrm{tr}}, \min_{i=1,\ldots,k} \{\delta(s_i)\}\} \text{ and } \varepsilon^\star := \min_{i=1,\ldots,k} \{\varepsilon(s_i)\}.$$

Now $\tau^{s_i}(s, 0) = 1$ together with uniform continuity of τ^{s_i} on $[s_i - \eta(s_i)/2, s_i + \eta(s_i)/2] \times \overline{B_{N-1}^{\delta/2}}$ implies that for every $\varepsilon \in (0, \varepsilon^\star]$ there is a $\delta_{\mathrm{re}} = \delta_{\mathrm{re}}(\varepsilon) \in (0, \delta/2]$ such that

$$\tau^{s_i}(s, c) \in (1 - \varepsilon, 1 + \varepsilon)$$

for all $i = 1, \ldots, k$, $s \in \big(s_i - \eta(s_i)/2, s_i + \eta(s_i)/2\big)$ and $c \in B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon)}$. Therefore $\tau : [0,1] \times B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon)} \to (1-\varepsilon, 1+\varepsilon)$ can be defined by functions τ^{s_i} naturally as follows, for $s \in [0,1]$ choose any s_i such that $s \in \big(s_i - \eta(s_i)/2, s_i + \eta(s_i)/2\big)$ then set $\tau(s,\cdot) := \tau^{s_i}(s,\cdot)$ (equality of τ^{s_i} and τ^{s_j} on the intersection of their domains comes from $\tau^{s_i}(s,0) = 1 = \tau^{s_j}(s,0)$ and the Implicit Function Theorem – so τ is well-defined).

Because the determining equation (3.1) is 1-periodic in s we can easily extend τ to become a function $\mathbb{R} \times B_{N-1}^{\delta_{\mathrm{re}}(\varepsilon)} \to (1-\varepsilon, 1+\varepsilon)$ which is 1-periodic in the first variable by the identity $\tau(s+k, c) := \tau(s,c), k \in \mathbb{Z}, s \in [0,1]$. The proof is complete. □

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