Hybrid dynamical systems vs. ordinary differential equations: Examples of "pathological" behavior

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1 Introduction

We study the following scalar equation

$$\ddot{\xi} + \xi = u. \tag{1}$$

Here, the control u depends on the variable ξ. This is a controlled harmonic oscillator in which the external force u is allowed to depend only on the displacement ξ, but not on the velocity $\dot{\xi}$ of the pendulum.

Equation (1) with the constraint $u(\xi)$ is equivalent to the following controlled linear system

$$\begin{aligned} \dot{\xi} &= \eta, \\ \dot{\eta} &= -\xi + u, \\ y &= \xi \end{aligned} \tag{2}$$

with a control $u = u(y)$. Here u again depends only on the output $y = \xi$.

It can be shown (see e.g. [1]) that there is no output feedback control of the form $u = f(\xi) = f(\xi(t))$ that makes the system (2) asymptotically stable. Therefore, it was suggested in [1] to use *hybrid feedback controls* (abbr. HFC), which indeed can stabilize the system (2).

The idea used in [1] can be roughly described as follows. We incorporate a discrete device (*an automaton*) into the considered system (*a plant*). The device is able to switch on and off certain control functions at certain instances. The time interval between two consecutive switchings depends on the last observation of ξ. As it was demonstrated in [1], careful choice of design procedure and switching instances provides asymptotic stability of the system (2). The discrete nature of hybrid outputs makes their practical implementation simpler.

More results on stabilization of linear and nonlinear systems via HFC with a finite number of automata's locations are available (see e.g. [2], [3], [4], [5], [6], [8], [10]). In [7] it was proved that it is possible to stabilize an arbitrary linear system by using HFC with infinitely many locations.

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In this paper we show that the dynamics of solutions $x(t)$ of the system (2) which is controlled by the hybrid output designed in [1, Example 5.2], is quite erratic (see Figure 1). Trajectories' behavior indicates that the observed dynamics cannot be described by "classical" dynamical systems defined by ordinary differential equations. We suspect that this dynamics stems from differential equations with time lags, where the delay functions depend on solutions. We are planning to study this problem in the future.

The whole dynamics of hybrid dynamical systems is given by the *triplet* $(x(t), q(t), \tau(t))$, where $q(t)$ is the present location of the automaton, and $\tau(t)$ is the time remaining untill the next transition instance. We are interested here in dynamic properties of the first, most important, component, $x(t)$, which describes *the plant*. To be able to "track down" $x(t)$ we need however to study the dynamics of the whole triplet.

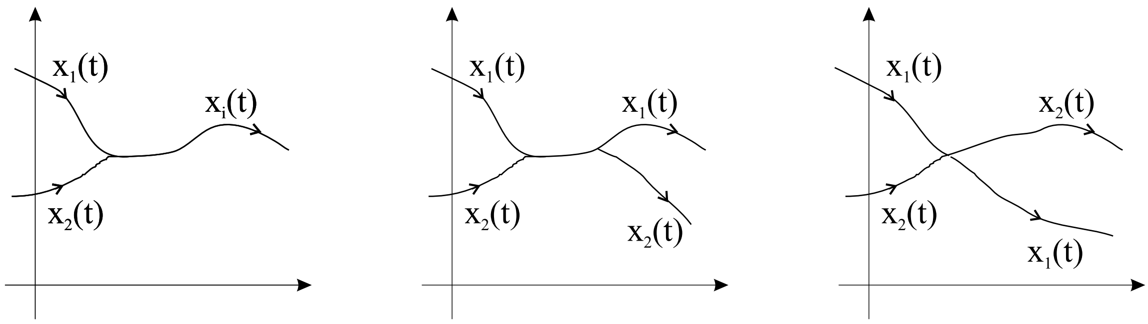

Figure 1

2 Main results

We consider the controlled harmonic oscillator (2) assuming that u is a specific HFC designed in [1, Example 5.2]. This control procedure provides asymptotic stability of the zero solution of the system. For the sake of brevity we, as in [8], denote this HFC by $u = \mathcal{A}(\delta)$, where δ is to be specified.

The HFC $\mathcal{A}(\delta)$ is given by the following diagram

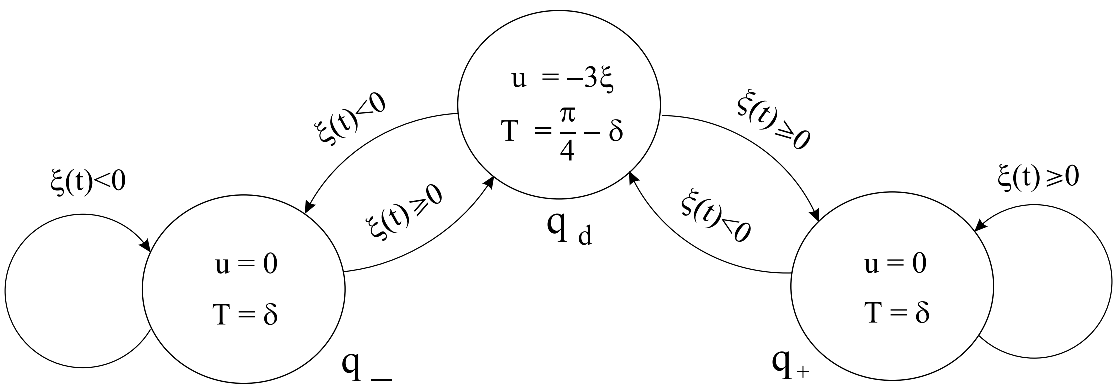

Figure 2

The automaton has 3 locations called q_+, q_- and q_d, and the values of T indicate the time of staying in the respective locations.

Remark 1 *We have slightly modified the definition of $\mathcal{A}(\delta)$ suggested in [1], where $T(q_d) = \pi/4-$*

2δ. *Our alteration is technical and does not influence the main results.*

As was already mentioned the dynamics of the system (1) governed by the HFC, $u = \mathcal{A}(\delta)$ is a triplet $(x(t), q(t), \tau(t))$. However it is clear that the value $\tau(t)$ is uniquely determined by the value $q(s)$, where $s \le t$ is the moment of the last observation. In particular, $\tau(0)$ is a function of $q(0)$. In what follows we fix an arbitrary initial location $q(0)$ (as we will show, all the results below are independent of the choice of $q(0)$). Then, given an initial value $x(0)$, the trajectory $x(t)$ is uniquely defined, so that we, at least formally, can set up a single functional-differential equation

$$\dot{x} = Fx = F(q(0))x \tag{3}$$

for all $x(t)$ (see details in [7] and [9]). We are interested in the dynamics of this equation.

We start with some technical remarks. Consider a solution $x(t) = (\xi(t), \eta(t))$ of the equation (3), i.e. of the system (2) governed by the HFC $u = \mathcal{A}(\delta)$ with $q(0)$ fixed. The trajectory $x(t)$ is assumed to start at $x(0) \neq 0$.

We will use polar coordinates in the plane, so that any solution $x(t) = (\xi(t), \eta(t))$ of (3) is described by the (uniquely defined) pair of functions $r : [0, \infty) \to [0, \infty)$, $\varphi : [0, \infty) \to \mathbf{R}/(2\pi\mathbf{Z})$, where $\xi(t) = r(t)\cos\varphi(t)$, $\eta(t) = r(t)\sin\varphi(t)$.

In what follows we assume that the function φ takes on values from the interval $(-\pi, \pi]$.

Within any interval $S = (s_1, s_2) \subset [0, \infty)$, where no change of locations occurs, the solution $x(t)$ satisfies one of the following systems of differential equations:

$$(\ q(t) = q_+,\ t \in S\ ) \ \vee\ (\ q(t) = q_-,\ t \in S\ ) \implies \begin{cases} \frac{\dot{r}}{r} = 0 \\ \dot{\varphi} = -1, \end{cases} \tag{4}$$

$$(\ q(t) = q_d,\ t \in S\ ) \implies \begin{cases} \frac{\dot{r}}{r} = -\frac{3}{2}\sin 2\varphi \\ \dot{\varphi} = -1 - 3\cos^2\varphi. \end{cases} \tag{5}$$

Theorem 1 *There exist $\delta > 0$, $t^* > 0$ and two distinct initial states $x_1(0)$, $x_2(0)$, for which the corresponding solutions $x_1(t)$ and $x_2(t)$ to (2) governed by the HFC $u = \mathcal{A}(\delta)$ coincide for $t \ge t^*$, i.e. $x_1(t) = x_2(t)$, $t \ge t^*$. Moreover, in this case the "true" hybrid trajectories $H_1(t) = (x_1(t), q_1(t), \tau_1(t))$ and $H_2(t) = (x_2(t), q_2(t), \tau_2(t))$ coincide for $t \ge t^*$, too.*

Proof. We use equations (4) and (5) to derive estimates for solutions of (3) (i. e. of the system (2) with $u = \mathcal{A}(\delta)$). Assume that $s_1, s_2 \in [0, \infty)$, $s_1 \le s_2$.

1) If either $q(\cdot) \equiv q_+$, or $q(\cdot) \equiv q_+$ on $[s_1, s_2]$, then

$$\varphi(s_1) - \varphi(s_2) = s_2 - s_1, \tag{6}$$

$$r(s_1) = r(s_2). \tag{7}$$

2) If $q(\cdot) \equiv q_d$ on $[s_1, s_2]$ and $\varphi([s_1, s_2]) \subset (-\pi/2, \pi/2)$, then

$$s_2 - s_1 = \frac{1}{2}\left(\arctan\frac{\tan\varphi(s_1)}{2} - \arctan\frac{\tan\varphi(s_2)}{2}\right) \tag{8}$$

$$\frac{r(s_2)}{r(s_1)} = \sqrt{\frac{1 + 3\cos^2\varphi(s_1)}{1 + 3\cos^2\varphi(s_2)}}. \tag{9}$$

Put $T_d = \pi/4 - \delta$ as the time of stay in the location q_d. Let $\bar{t}$ be the moment of switching to the location q_d. Then $\bar{t} + T_d$ is the moment of switching from q_d to another location. We define a function $\theta : [\pi/2 - \delta, \pi/2] \to \mathbf{R}$ by $\theta(\varphi) = \varphi(\bar{t} + T_d)$, if $\varphi = \varphi(\bar{t})$. Due to (8),

$$\theta(\varphi) = -\arctan\left(2 \cdot \cot\left(\arctan\frac{\tan\varphi}{2} + 2\delta\right)\right), \quad \varphi \in \left(\frac{\pi}{2} - \delta, \frac{\pi}{2}\right), \tag{10}$$

so that $\theta(\varphi)$ is well-defined for sufficiently small δ, namely for those satisfying

$$\theta\left(\frac{\pi}{2}\right) - \theta\left(\frac{\pi}{2} - \delta\right) > \delta. \tag{11}$$

We also define $\beta : [\pi/2 - \delta, \pi/2] \to \mathbf{R}$ by

$$\beta(\varphi) = \sqrt{\frac{1 + 3\cos^2\varphi}{1 + 3\cos^2\theta(\varphi)}}. \tag{12}$$

From (11) and continuity of the function θ, it is easy to derive the existence of $\psi_1, \psi_2 \in \mathbf{R}$, for which

$$\frac{\pi}{2} - \delta < \psi_1 < \psi_2 < \frac{\pi}{2}, \qquad \theta(\psi_2) - \theta(\psi_1) = \delta. \tag{13}$$

From now on we fix a positive and sufficiently small δ as well as two constants ψ_i satisfying (13).

We pick two different trajectories $x_1(t)$, $x_2(t)$ being the "shadows" of the "true" hybrid trajectories

$$H_i(t) = (x_i(t), q_i(t), \tau_i(t)), \quad i = 1, 2,$$

We assume that at $t = t_0 \geq 0$ the automaton either switches from q_- to q_d, or keeps staying in q_-. In polar coordinates one has

$$\varphi_1(t_0) = \psi_1 + \delta, \quad r_1(t_0) = r_0 > 0, \quad \varphi_2(t_0) = \psi_2, \quad r_2(t_0) = r_0\,\frac{\beta(\psi_1)}{\beta(\psi_2)}. \tag{14}$$

An example of such a situation is given by $q(0) = q_-$ and $t_0 = n\delta$, where n is a nonnegative integer satisfying $n\delta < \frac{\pi}{2} - \delta$.

Clearly, $r_2(t_0) < r_1(t_0)$ (the function β is strictly increasing).

The two observations below can easily be derived from (13). See also Figure 3.

1) In the case of the trajectory $x_1(t)$, the automaton keeps staying in the location q_- near $t = t_0$; i.e., $H_1(t) = (x_1(t), q_-, \delta)$ for $t_0 < t < t_0 + \delta$. The first transition to the location q_d occurs at $t = t_0 + \delta$; i.e., $H_1(t_0 + \delta) = (x_1(t_0 + \delta), q_d, T_d)$.

2) In the case of the trajectory $x_2(t)$, the automaton switches from q_- to q_d at $t = t_0$; i.e., $H_2(t_0) = (x_2(t_0), q_d, T_d)$. At the moment $t = t_0 + T_d$ the automaton switches from q_d to q_+; i.e., $H_2(t_0 + T_d) = (x_1(t_0 + T_d), q_+, \delta)$.

These observations imply that

$$q_i(t^*) = q_+, \quad \tau_i(t^*) = \delta, \quad i = 1, 2, \tag{15}$$

where $t^* = t_0 + \frac{\pi}{4} = t_0 + T_d + \delta$.

At the same time, from (6), (14) and the observations above it follows that

$$\begin{aligned} &\varphi_1(t_0 + \delta) = (\psi_1 + \delta) - \delta = \psi_1, \quad \varphi_1(t^*) = \theta(\psi_1), \\ &\varphi_2(t_0 + T_d) = \theta(\varphi_2(t_0)) = \theta(\psi_2), \quad \varphi_2(t^*) = \theta(\psi_2) - \delta = \theta(\psi_1). \end{aligned} \tag{16}$$

Thus, it is shown that

$$\varphi_1(t^*) = \varphi_2(t^*). \tag{17}$$

Since φ_i is strictly monotone on $[t_0, t^*]$, there exist functions ρ_i $(i = 1, 2)$, defined on the set $D_i = \varphi_i^{-1}([t_0, t^*])$ and satisfying

$$r_i(t) = \rho_i(\varphi_i(t)). \tag{18}$$

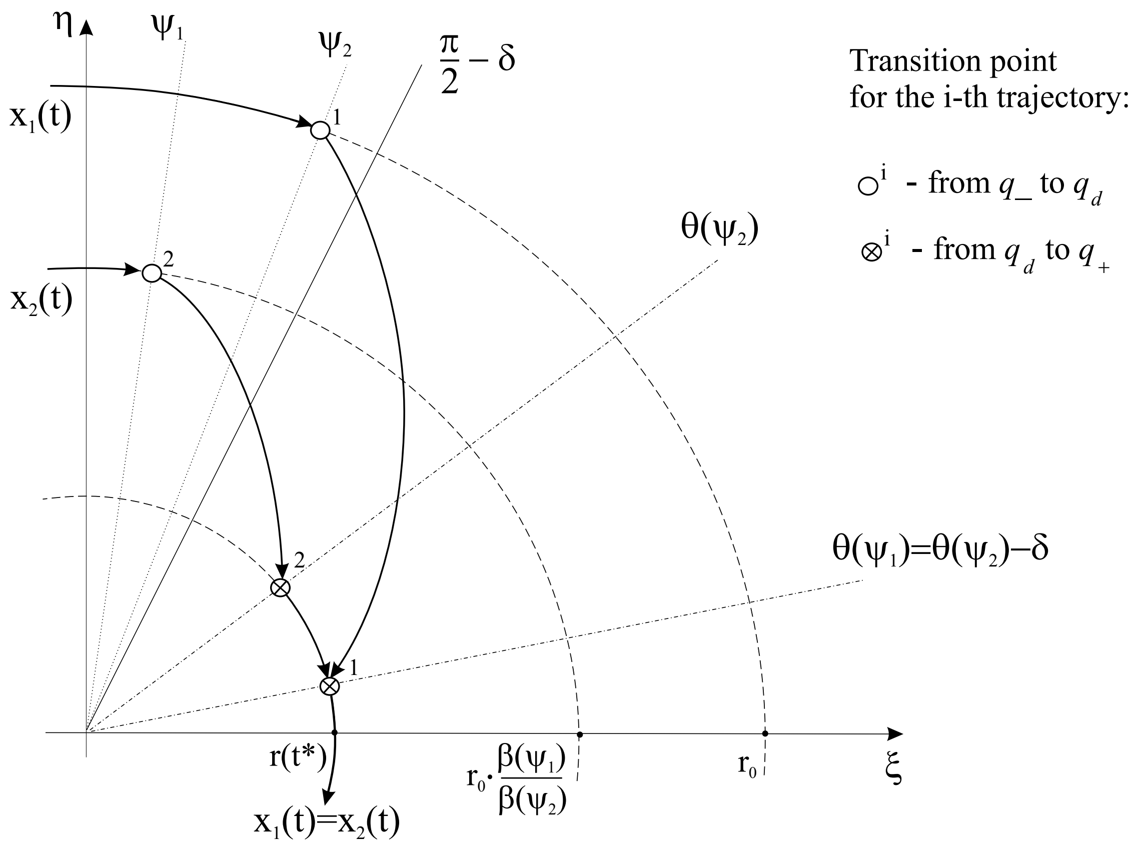

Figure 3

Then (7), (9), (12), (16) imply

$$r_1(t^*) = \rho_1(\varphi_1(t^*)) = \rho_1(\theta(\psi_1)) = \beta(\psi_1)\rho_1(\psi_1) = \beta(\psi_1)r_0,$$

$$r_2(t^*) = r_2(t_0 + T_d) = \rho_2(\theta(\psi_2)) = \beta(\psi_2)\rho_2(\psi_2) = \beta(\psi_2)r_2(t_0).$$

From this and (14) one gets

$$r_1(t^*) = r_2(t^*). \tag{19}$$

Now, (17), (19) imply $x_1(t^*) = x_2(t^*)$. Taking this and (15) into account one observes that $H_1(t) = H_2(t)$ for $t \geq t^*$ and, in particular, $x_1(t) = x_2(t)$ for $t \geq t^*$. $\square$

Theorem 2 *There exist positive* δ, ε, t^*, t^{**} $(t^* < t^{**})$ *and distinct initial states* $x_1(0), x_2(0)$, *for which the corresponding solutions* $x_1(t)$ *and* $x_2(t)$ *to (2) governed by the HFC* $u = \mathcal{A}(\delta)$ *satisfy the following properties*
1) $x_1(t) = x_2(t)$ *for* $t^* \leq t \leq t^{**}$,
2) $x_1(t) \neq x_2(t)$ *for* $t^* - \varepsilon < t < t^*$ *and* $t^{**} < t < t^{**} + \varepsilon$,
while for the "true" hybrid trajectories $H_1(t) = (x_1(t), q_1(t), \tau_1(t))$ *and* $H_2(t) = (x_2(t), q_2(t), \tau_2(t))$ *one has*
3) $H_1(t) \neq H_2(t)$ *for* $t^* \leq t \leq t^{**}$.

Proof. Putting again $T_d = \frac{\pi}{4}$ we will use the same functions θ and β as in the course of the proof of Theorem 1. Similar to (13) we fix sufficiently small $\delta > 0$, $\mu > 0$ and two constants ψ_1 and ψ_2, giving an increasing function β and relations

$$\frac{\pi}{2} - \delta < \psi_1 < \psi_2 < \frac{\pi}{2}, \qquad \theta(\psi_2) - \theta(\psi_1) = \delta + \mu, \qquad 0 < \mu < \psi_1 + \delta - \frac{\pi}{2}. \tag{20}$$

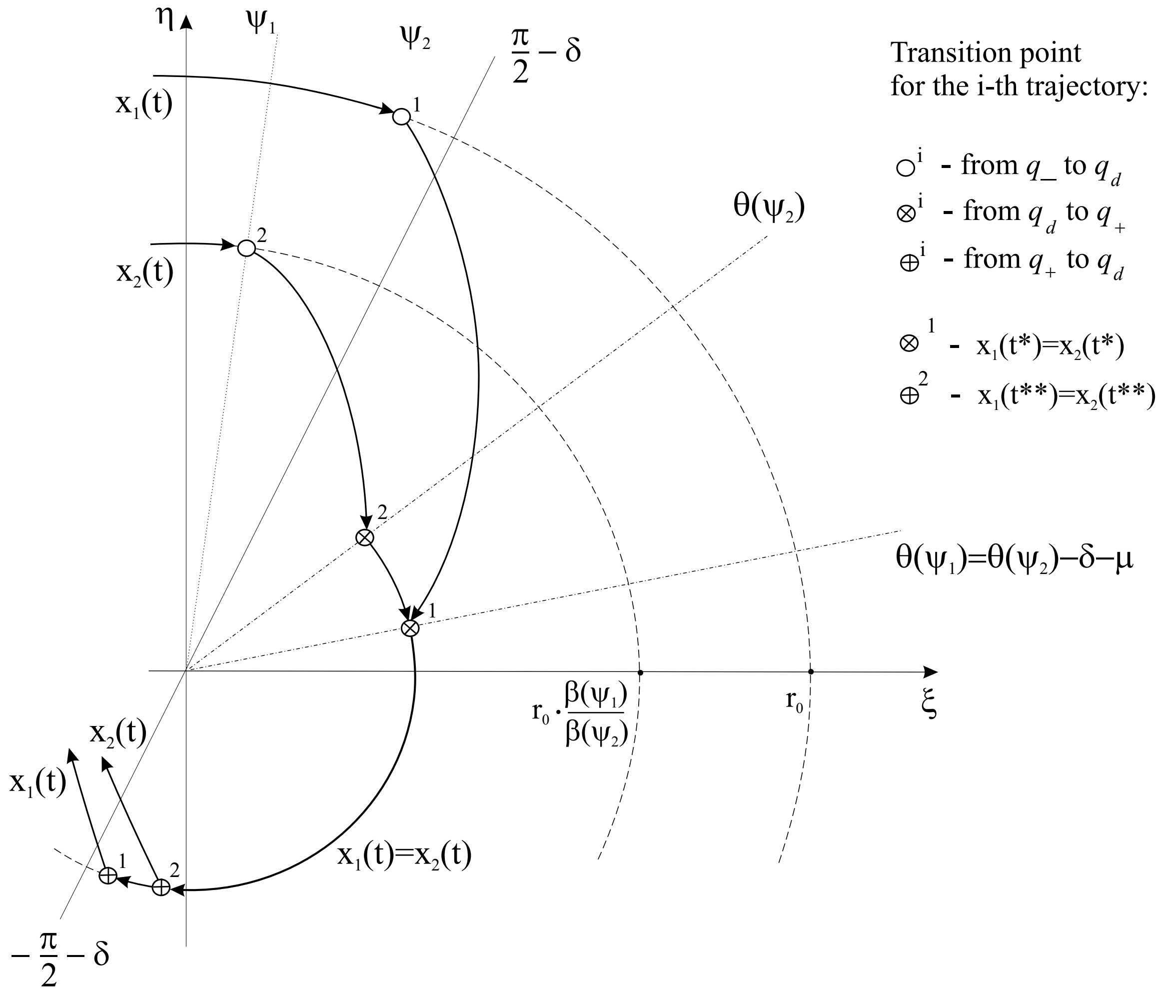

Figure 4

Consider two different trajectories $x_1(t), x_2(t),\ t \geq t_0 > 0$ of the system (2) governed by the HFC $u = \mathcal{A}(\delta)$, for which (14) hold true. Assume that

$$q_1(t_0) = q_-, \qquad \tau_1(t_0) = \mu, \qquad q_2(t_0) = q_d, \qquad \tau_2(t_0) = T_d. \tag{21}$$

Clearly, (20) and (21) imply

$$q_1(t_0 + \mu + \delta) = q_d, \qquad \tau_1(t_0 + \mu + \delta) = T_d, \qquad q_2(t_0 + T_d) = q_+, \qquad \tau_2(t_0 + T_d) = \delta, \tag{22}$$

and hence

$$q_1(t^*) = q_+, \qquad \tau_1(t^*) = \delta, \qquad q_2(t^*) = q_+, \qquad \tau(t^*) = \delta - \mu, \tag{23}$$

where $t^* = t_0 + T_d + \delta + \mu$ (see Figure 4).

From (8), (14), (20), (23) it follows that

$$
\begin{array}{rl}
\varphi_1(t_0+\delta+\mu)=\psi_1+\delta-\delta=\psi_1, & \varphi_1(t^*)=\theta(\psi_1), \\
\varphi_2(t_0+T_d)=\theta(\varphi_2(t_0))=\theta(\psi_2), & \varphi_2(t^*)=\theta(\psi_2)-\delta-\mu=\theta(\psi_1).
\end{array} \tag{24}
$$

Thus, the condition (17) is verified.

On the other hand, an argument similar to that used in the proof of Theorem 1 shows that

$$
\begin{array}{c}
r_1(t^*)=\beta(\psi_1)r_1(t_0) \\
r_2(t^*)=r_2(t_0+T_d)=\beta(\psi_2)r_2(t_0)
\end{array}
$$

due to (7), (9), (12), (23), (24).

From this and (14) one easily derives (19).

Due to (22), $q_1(t^*-\varepsilon)=q_d$ and $q_2(t^*-\varepsilon)=q_+$ for sufficiently small ε. This and (7) together with (9) imply

$$
r_1(t)\neq r_2(t), \quad t^*-\varepsilon<t<t^*. \tag{25}
$$

By (6), (7), (17), we have that (19) and (23) imply the existence of $t^{**}>t^*$, for which

$$
x_1(t)=x_2(t) \text{ for } t^*\leq t\leq t^{**}, \tag{26}
$$

and

$$
\begin{array}{ll}
\varphi_1(t^{**})\in[-\frac{\pi}{2}-\delta,-\frac{\pi}{2}), & \tau_1(t)=\tau_2(t)+\mu \text{ for } t^*\leq t\leq t^{**}, \\
q_1(t^{**})=q_+, \quad \tau_1(t^{**})=\mu, & q_2(t^{**})=q_d, \quad \tau_2(t^{**})=T_d.
\end{array}
$$

The last four equalities say that in the case of the trajectory $x_2(t)$ the automaton switches from q_+ to q_d at time $t=t^{**}$, while in the case of the trajectory $x_1(t)$ switching occurs at time $t=t^{**}+\mu$.

From (9) and (26) one obtains

$$
\begin{array}{c}
H_1(t)\neq H_2(t) \text{ for } t^*\leq t\leq t^{**} \\
r_1(t)\neq r_2(t) \text{ for } t^{**}<t<t^{**}+\varepsilon,
\end{array} \tag{27}
$$

for sufficiently small $\varepsilon>0$.

The relations (25) - (27) prove the theorem. □

Theorem 3 . *There exist $\delta>0$ and two distinct initial states $x_1(0)$, $x_2(0)$, for which the corresponding solutions to (2) governed by the HFC $u=\mathcal{A}(\delta)$ meet transversely at some time $t^*>0$. In other words, $x_1(t^*)=x_2(t^*)$, $x_1(t)\neq x_1(t)$ for small $|t-t^*|\neq 0$, and the vectors $\dot{x}_1(t^*)$, $\dot{x}_2(t^*)$ are linearly independent.*

Proof. As in the proof of theorem 2 let us fix sufficiently small $\delta>0$, $\mu>0$ and some constants ψ_1, ψ_2, so that the function β defined by (12) is increasing and (20) holds.

Consider two solutions $x_1(t)$, $x_2(t)$ of the system (2) governed by the HFC $u=\mathcal{A}(\delta)$. The solutions are assumed to satisfy

$$
\varphi_1(t_0)=\psi_1+\delta, \quad r_1(t_0)=r_{01}, \quad \varphi_2(t_0)=\psi_2, \quad r_2(t_0)=r_{02}<r_{01} \tag{28}
$$

at some time $t_0\geq 0$. We also assume that there occurs switching to a different location at $t=t_0$. According to (6) and (20) switching from q_- to q_d occurs at $t=t_0+\delta$ in the case of the trajectory $x_1(t)$, and at $t=t_0$ in the case of the trajectory $x_2(t)$ (see Figure 5).

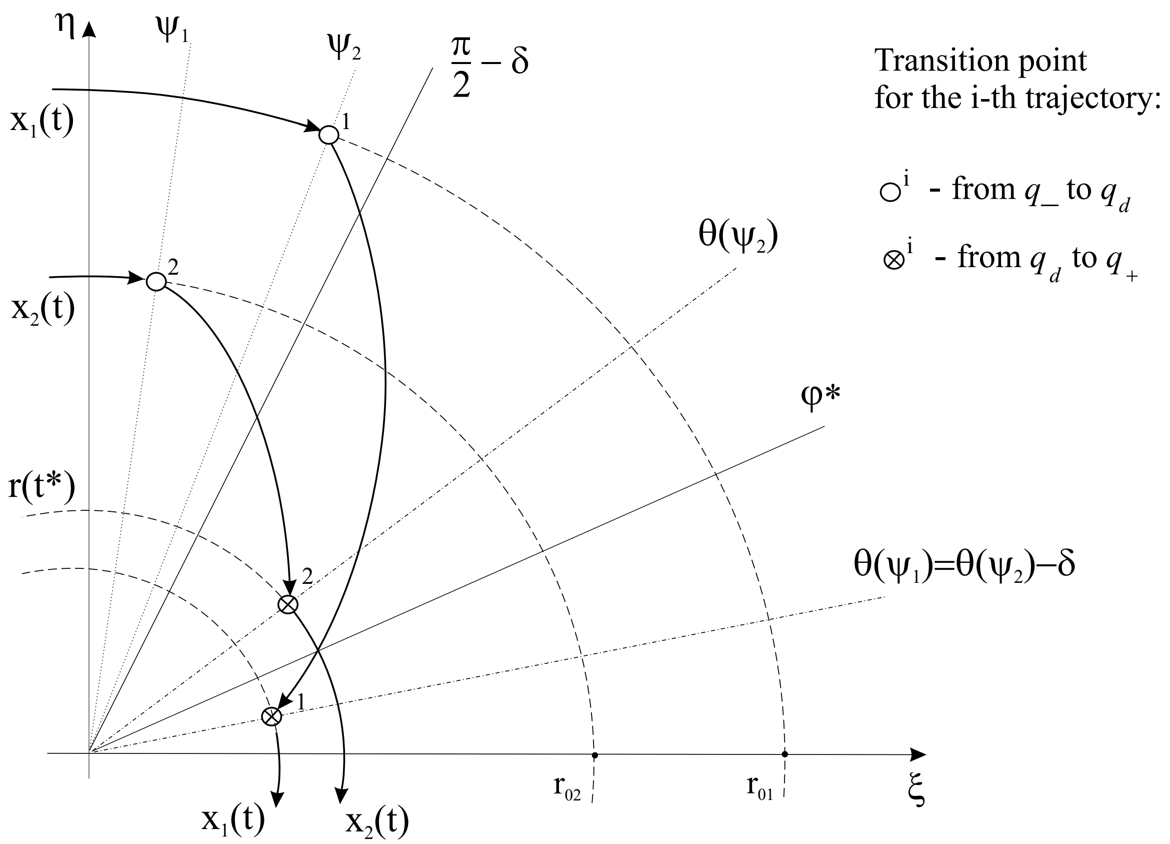

Figure 5

As in the proof of Theorem 1, the relations (28) imply that

$$\varphi_1(t_0+\delta+T_d)=\theta(\psi_1), \qquad \varphi_2(t_0+\delta+T_d)=\theta(\psi_2)-\delta=\theta(\psi_1)+\mu.$$

Moreover, using the second inequality in (5), the mean value theorem and (20), (28) one can easily show that

$$\varphi_1(t_0+T_d)>\varphi_2(t_0+T_d), \qquad \varphi_1(t_0+T_d+\delta)<\varphi_2(t_0+T_d+\delta)$$

for sufficiently small $\mu>0$.

Due to the continuity of $\varphi_i(t)$ there exists $t^*\in(t_0+T_d, t_0+T_d+\delta)$, for which (17) holds true. We also put $\varphi^*=\varphi_i(t^*)$.

Let $\omega:[\frac{\pi}{2}-\delta,\frac{\pi}{2}]\to R$ be a function defined by

$$\omega(\psi_1,\psi_2)=\sqrt{\frac{1+3\cos^2\psi_1}{1+3\cos^2\psi_2}}.$$

Putting $\psi_i=\varphi(t_i)$ and comparing the definition of ω with (9) we, as in Theorem 1, obtain

$$\frac{\rho_i(s_2)}{\rho_i(s_1)}=\omega(s_1,s_2), \qquad s_1,s_2\in\varphi_i(I) \tag{29}$$

being valid for any time interval $I\subset[t_0,t_1]$, during which the automaton keeps staying in the location q_d. This applies to both of solutions $x_1(t)$ and $x_2(t)$, so that we may put $\rho_1=\rho_2=\rho$.

According to our calculations, neither t^*, nor φ^* depends on r_{0i}. This means that we can always find a pair r_{01}, r_{02}, for which the following additional assumption holds:

$$\omega(\psi_2,\theta(\psi_2))\,r_{02}=\omega(\psi_1,\varphi^*)\,r_{01}. \tag{30}$$

According to (17) and (29),

$$r_1(t^*) = \rho_1(\varphi^*) = \omega(\psi_1, \varphi^*)\, r_{01},$$
$$r_2(t^*) = r_2(t_0 + T_d) = \rho_2(\theta(\psi_2)) = \omega(\psi_2, \theta(\psi_2))\, r_{02},$$

so that (30) implies (19). From (17) and (19) it immediately follows that $x_1(t^*) = x_2(t^*)$. Since $r_1(t)$ is strictly monotone and $\rho_2(t)$ is a constant in some neighbourhood O_{t^*} of the point t^*, we see that $x_1(t) \neq x_2(t)$ $(\forall t \in O_{t^*} \setminus \{t^*\})$.

Finally, we observe that

$$q_1(t^*) = q_d, \quad q_2(t^*) = q_+. \tag{31}$$

Put $(\xi, \eta)^T = x_1(t^*) = x_2(t^*)$. Evidently, $\xi\eta \neq 0$. Hence,

$$\alpha\dot{x}_1 + \beta\dot{x}_2 = \begin{pmatrix} 0 & \alpha+\beta \\ -\alpha-4\beta & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} \neq 0$$

for $|\alpha| + |\beta| \neq 0$. Thus, the velocity vectors are linearly independent, and the theorem is proved. $\square$

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References

[1] Z. Artstein, *Example of stabilization with hybrid feedback*, in Hybrid Systems III. Verification and Control. Springer-Verlag, Berlin, Lecture Notes in Computer Science., 1066 (1996), pp. 173–185.

[2] C. Benassi, A. Gavioli, *Hybrid stabilization of planar linear systems with one-dimensional outputs*, preprint, Dipartamenta di Matematica Pura e Applicata, Universita di Modena, Italy, August 1999.

[3] M. S. Branicky, *Stability of Switched and Hybrid Systems*, in Proc. 33rd IEEE Conf. Decision Control, Lake Buena Vista, FL (1994), pp. 3498-3503.

[4] M. S. Branicky, *Studies in hybrid systems: modeling, analysis and control*, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, 1995.

[5] D. Liberzon, *Stabilizing a linear system with finite-state hybrid output feedback*, The 7th IEEE Mediterranean Conference on Control and Automation (1999).

[6] D. Liberzon, A.S.Morse, *Benchmark problems in stability and design of switched systems*, preprint, March 1999.

[7] E. Litsyn, Yu.V. Nepomnyaschchikh, and A. Ponosov, *Stabilization of linear differential systems via hybrid feedback controls,* accepted for publication in SIAM J. Control and Optimization (1999).

[8] E. Litsyn, Yu.V. Nepomnyaschchikh, and A. Ponosov, *Stabilization of Quasilinear Systems and Hybrid Feedback Controls,* The 7-th Mediterranean Conference on Control and Automation, Haifa (1999), pp. 37 - 52.

[9] E. Litsyn, Yu.V. Nepomnyaschchikh, and A. Ponosov, *Uniform exponential stability of controlled quasilinear systems and functional differential equations,* "Topics on Functional Differential and Difference Equations" T. Faria and P. Freitas Eds, *Fields Institute Communications* (submitted for publication in 2000).

[10] A. Nerode and W. Kohn, *Models for hybrid systems: automata, topologies, controllability, observability*, in Hybrid Systems, R. L. Grossman et al eds., Springer-Verlag, Berlin, Lecture Notes in Computer Scient., 736 (1993), pp. 317–356.