On a fractional integro-differential inclusion

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Abstract. We study a Cauchy problem for a fractional integro-differential inclusion of order $\alpha \in (1, 2]$ involving a nonconvex set-valued map. Arcwise connectedness of the solution set is provided. Also we prove that the set of selections corresponding to the solutions of the problem considered is a retract of the space of integrable functions on a given interval.

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1   Introduction

Differential equations with fractional order have recently proved to be strong tools in the modelling of many physical phenomena. As a consequence there was an intensive development of the theory of differential equations and inclusions of fractional order ([12, 13, 14] etc.). Applied problems require definitions of fractional derivative allowing the utilization of physically interpretable initial conditions. Caputo's fractional derivative, originally introduced in [3] and afterwards adopted in the theory of linear visco elasticity, satisfies this demand. Very recently several qualitative results for fractional integro-differential equations were obtained in [6, 11, 17] etc.

In this paper we study fractional integro-differential inclusions of the form

$$D_c^\alpha x(t) \in F(t, x(t), V(x)(t)) \quad \text{a.e. in} \ ([0, T]), \quad x(0) = x_0, \quad x'(0) = x_1, \qquad (1.1)$$

where $\alpha \in (1, 2]$, D_c^α is the Caputo fractional derivative, $F: [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ is a set-valued map and $x_0, x_1 \in \mathbf{R}$, $x_0, x_1 \neq 0$. $V: C([0, T], \mathbf{R}) \to C([0, T], \mathbf{R})$ is a nonlinear Volterra integral operator defined by $V(x)(t) = \int_0^t k(t, s, x(s)) \, ds$ with $k(\cdot, \cdot, \cdot): [0, T] \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ a given function.

The aim of this paper is twofold. On one hand, we prove the arcwise connectedness of the solution set of problem (1.1) when the set-valued map is Lipschitz in the second and third variable. On the other hand, under such type of hypotheses on the set-valued map we establish a more general topological property of the solution set of problem (1.1). Namely, we prove

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that the set of selections of the set-valued map F that correspond to the solutions of problem (1.1) is a retract of $L^1([0,T], \mathbf{R})$. Both results are essentially based on the results of Bressan and Colombo [1] concerning the existence of continuous selections of lower semicontinuous multifunctions with decomposable values.

We note that in the classical case of differential inclusions similar results are obtained using various methods and tools ([2, 8] etc.). Our results may be interpreted as extensions of the results in [4, 15, 16] to fractional integro-differential inclusions.

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove our main results.

2   Preliminaries

Let $T > 0$, $I := [0, T]$ and denote by $\mathcal{L}(I)$ the σ-algebra of all Lebesgue measurable subsets of I. Let X be a real separable Banach space with the norm $|\cdot|$. Denote by $\mathcal{P}(X)$ the family of all nonempty subsets of X and by $\mathcal{B}(X)$ the family of all Borel subsets of X. If $A \subset I$ then $\chi_A(\cdot) \colon I \to \{0,1\}$ denotes the characteristic function of A. For any subset $A \subset X$ we denote by cl(A) the closure of A.

The distance between a point $x \in X$ and a subset $A \subset X$ is defined as usual by $\mathrm{d}(x, A) = \inf\{|x - a|; a \in A\}$. We recall that the Pompeiu–Hausdorff distance between the closed subsets $A, B \subset X$ is defined by $\mathrm{d}_H(A, B) = \max\{\mathrm{d}^*(A, B), \mathrm{d}^*(B, A)\}$, $\mathrm{d}^*(A, B) = \sup\{\mathrm{d}(a, B); a \in A\}$.

As usual, we denote by $C(I, X)$ the Banach space of all continuous functions $x \colon I \to X$ endowed with the norm $|x|_C = \sup_{t \in I} |x(t)|$ and by $L^1(I, X)$ the Banach space of all (Bochner) integrable functions $x \colon I \to X$ endowed with the norm $|x|_1 = \int_0^T |x(t)| \, \mathrm{d}t$.

We recall first several preliminary results we shall use in the sequel.

A subset $D \subset L^1(I, X)$ is said to be *decomposable* if for any $u, v \in D$ and any subset $A \in \mathcal{L}(I)$ one has $u\chi_A + v\chi_B \in D$, where $B = I \backslash A$.

We denote by $\mathcal{D}(I, X)$ the family of all decomposable closed subsets of $L^1(I, X)$.

Next (S, d) is a separable metric space; we recall that a multifunction $G \colon S \to \mathcal{P}(X)$ is said to be lower semicontinuous (l.s.c.) if for any closed subset $C \subset X$, the subset $\{s \in S; G(s) \subset C\}$ is closed. The next lemmas may be found in [1].

Lemma 2.1. *If $F \colon I \to \mathcal{D}(I, X)$ is a lower semicontinuous multifunction with closed nonempty and decomposable values then there exists $f \colon I \to L^1(I, X)$ a continuous selection from F.*

Lemma 2.2. *Let $F^* \colon I \times S \to \mathcal{P}(X)$ be a closed-valued $\mathcal{L}(I) \otimes \mathcal{B}(S)$-measurable multifunction such that $F^*(t, .)$ is l.s.c. for any $t \in I$.*
Then the multifunction $G \colon S \to \mathcal{D}(I, X)$ defined by

$$G(s) = \{v \in L^1(I, X); \quad v(t) \in F^*(t, s) \quad \text{a.e. in } I\}$$

is l.s.c. with nonempty closed values if and only if there exists a continuous mapping $p \colon S \to L^1(I, X)$ such that

$$d(0, F^*(t, s)) \le p(s)(t) \quad \text{a.e. in } I, \ \forall s \in S.$$

Lemma 2.3. *Let $G \colon S \to \mathcal{D}(I, X)$ be a l.s.c. multifunction with closed decomposable values and let $\phi \colon S \to L^1(I, X)$, $\psi \colon S \to L^1(I, \mathbf{R})$ be continuous such that the multifunction $H \colon S \to \mathcal{D}(I, X)$ defined by*

$$H(s) = cl\{v(\cdot) \in G(s); \quad |v(t) - \phi(s)(t)| < \psi(s)(t) \quad \text{a.e. in } I\}$$

has nonempty values.

Then H has a continuous selection, i.e. there exists a continuous mapping $h: S \to L^1(I, X)$ such that $h(s) \in H(s) \quad \forall s \in S$.

Definition 2.4. a) The fractional integral of order $\alpha > 0$ of a Lebesgue integrable function $f : (0, \infty) \to \mathbf{R}$ is defined by

$$I^{\alpha} f(t) = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s) \, ds,$$

provided the right-hand side is pointwise defined on $(0, \infty)$ and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} \, dt$.

b) The Caputo fractional derivative of order $\alpha > 0$ of a function $f : [0, \infty) \to \mathbf{R}$ is defined by

$$D_c^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{-\alpha+n-1} f^{(n)}(s) \, ds,$$

where $n = [\alpha] + 1$. It is assumed implicitly that f is n times differentiable and its n-th derivative is absolutely continuous.

We recall (e.g., [12]) that if $\alpha > 0$ and $f \in C(I, \mathbf{R})$ or $f \in L^\infty(I, \mathbf{R})$ then $(D_c^{\alpha} I^{\alpha} f)(t) \equiv f(t)$.

Definition 2.5. A function $x \in C(I, \mathbf{R})$ is called a solution of problem (1.1) if there exists a function $f \in L^1(I, \mathbf{R})$ with $f(t) \in F(t, x(t), V(x)(t))$, a.e. in I such that $D_c^{\alpha} x(t) = f(t)$ a.e. in I and $x(0) = x_0, x'(0) = x_1$.

In this case $(x(\cdot), f(\cdot))$ is called a *trajectory-selection* pair of problem (1.1).

We shall use the following notations for the solution sets and for the selection sets of problem (1.1).

$$S(x_0, x_1) = \{x \in C(I, \mathbf{R}); \quad x \text{ is a solution of } (1.1)\}, \tag{2.1}$$

$$\tilde{f}(t) = x_0 + t x_1 + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} f(u) \, du,$$

$$\mathcal{T}(x_0, x_1) = \{f \in L^1(I, \mathbf{R}); \quad f(t) \in F(t, \tilde{f}(t), V(\tilde{f})(t)) \quad \text{a.e. in } I\}. \tag{2.2}$$

3   The main results

In order to prove our topological properties of the solution set of problem (1.1) we need the following hypotheses.

Hypothesis. i) $F(\cdot, \cdot): I \times \mathbf{R} \times \mathbf{R} \to \mathcal{P}(\mathbf{R})$ *has nonempty closed values and is* $\mathcal{L}(I) \otimes \mathcal{B}(\mathbf{R} \times \mathbf{R})$ *measurable.*

ii) *There exists* $L(\cdot) \in L^1(I, (0, \infty))$ *such that, for almost all* $t \in I, F(t, \cdot, \cdot)$ *is* $L(t)$-*Lipschitz in the sense that*

$$d_H(F(t, x_1, y_1), F(t, x_2, y_2)) \le L(t)(|x_1 - x_2| + |y_1 - y_2|) \quad \forall \ x_1, x_2, y_1, y_2 \in \mathbf{R}.$$

iii) *There exists* $p \in L^1(I, \mathbf{R})$ *such that*

$$d_H(\{0\}, F(t, 0, V(0)(t))) \le p(t) \quad a.e. \text{ in } I.$$

iv) $k(\cdot, \cdot, \cdot): I \times \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *is a function such that* $\forall x \in \mathbf{R}, \ (t, s) \to k(t, s, x)$ *is measurable.*

v) $|k(t, s, x) - k(t, s, y)| \leq L(t)|x - y|$ *a.e.* $(t, s) \in I \times I$, $\forall x, y \in \mathbf{R}$.

We use next the following notations

$$M(t) := L(t) \left(1 + \int_0^t L(u) \, \mathrm{d}u\right), \quad t \in I, \quad |I^\alpha M| := \sup_{t \in I} |I^\alpha M(t)|.$$

Theorem 3.1. *Assume that Hypothesis is satisfied and $|I^\alpha M| < 1$.*
Then for any $\xi_0, \xi_1 \in \mathbf{R}$ the solution set $\mathcal{S}(\xi_0, \xi_1)$ is arcwise connected in the space $C(I, \mathbf{R})$.

Proof. Let $\xi_0, \xi_1 \in \mathbf{R}$ and $x_0, x_1 \in \mathcal{S}(\xi_0, \xi_1)$. Therefore there exist $f_0, f_1 \in L^1(I, \mathbf{R})$ such that
$x_0(t) = \xi_0 + t\xi_1 + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} f_0(u) \, \mathrm{d}u$ and $x_1(t) = \xi_0 + t\xi_1 + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} f_1(u) \, \mathrm{d}u$, $t \in I$.
    For $\lambda \in [0, 1]$ define

$$x^0(\lambda) = (1 - \lambda)x_0 + \lambda x_1 \quad \text{and} \quad g^0(\lambda) = (1 - \lambda)f_0 + \lambda f_1.$$

Obviously, the mapping $\lambda \mapsto x^0(\lambda)$ is continuous from $[0, 1]$ into $C(I, \mathbf{R})$ and since
$|g^0(\lambda) - g^0(\lambda_0)|_1 = |\lambda - \lambda_0| \cdot |f_0 - f_1|_1$ it follows that $\lambda \mapsto g^0(\lambda)$ is continuous from $[0, 1]$
into $L^1(I, \mathbf{R})$.
    Define the set-valued maps

$$\Psi^1(\lambda) = \left\{v \in L^1(I, \mathbf{R}); \; v(t) \in F\big(t, x^0(\lambda)(t), V(x^0(\lambda))(t)\big) \; \text{a.e. in } I\right\},$$

$$\Phi^1(\lambda) = \begin{cases} \{f_0\} & \text{if } \lambda = 0, \\ \Psi^1(\lambda) & \text{if } 0 < \lambda < 1, \\ \{f_1\} & \text{if } \lambda = 1 \end{cases}$$

and note that $\Phi^1 \colon [0, 1] \to \mathcal{D}(I, \mathbf{R})$ is lower semicontinuous. Indeed, let $C \subset L^1(I, \mathbf{R})$ be
a closed subset, let $\{\lambda_m\}_{m \in \mathbf{N}}$ converge to some λ_0 and $\Phi^1(\lambda_m) \subset C$ for any $m \in \mathbf{N}$. Let
$v_0 \in \Phi^1(\lambda_0)$. Since the multifunction $t \mapsto F(t, x^0(\lambda_m)(t), V(x^0(\lambda_m))(t))$ is measurable, it
admits a measurable selection $v_m(\cdot)$ such that

$$|v_m(t) - v_0(t)| = \mathrm{d}\big(v_0(t), F\big(t, x^0(\lambda_m)(t), V(x^0(\lambda_m))(t)\big)\big) \quad \text{a.e. in } I.$$

Taking into account Hypothesis one may write

$$\begin{aligned} |v_m(t) - v_0(t)| &\leq \mathrm{d}_H\big(F(t, x^0(\lambda_m)(t), V(x^0(\lambda_m))(t)), F(t, x^0(\lambda_0)(t), V(x^0(\lambda_0))(t)) \\ &\leq L(t) \left[\left|x^0(\lambda_m)(t) - x^0(\lambda_0)(t)\right| + \int_0^t L(s) \left|x^0(\lambda_m)(s) - x^0(\lambda_0)(s)\right| \, \mathrm{d}s\right] \\ &= L(t)|\lambda_m - \lambda_0| \left[|x_0(t) - x_1(t)| + \int_0^t L(s)|x_0(s) - x_1(s)| \, \mathrm{d}s\right], \end{aligned}$$

hence

$$|v_m - v_0|_1 \leq |\lambda_m - \lambda_0| \int_0^T L(t) \left[|x_0(t) - x_1(t)| + \int_0^t L(s)|x_0(s) - x_1(s)| \, \mathrm{d}s\right] \mathrm{d}t,$$

which implies that the sequence v_m converges to v_0 in $L^1(I, \mathbf{R})$. Since C is closed we infer that
$v_0 \in C$; hence $\Phi^1(\lambda_0) \subset C$ and $\Phi^1(\cdot)$ is lower semicontinuous.

Next we use the following notation

$$p_0(\lambda)(t) = |g^0(\lambda)(t)| + p(t) + L(t)\left(|x^0(\lambda)(t)| + \int_0^t L(s)\,|x^0(\lambda)(s)|\,\mathrm{d}s\right),$$

$t \in I$, $\lambda \in [0,1]$.
   Since

$$|p_0(\lambda)(t) - p_0(\lambda_0)(t)|$$
$$\leq |\lambda - \lambda_0|\left[|f_1(t) - f_0(t)| + L(t)(|x_0(t) - x_1(t)| + \int_0^t L(s)|x_0(s) - x_1(s)|\,\mathrm{d}s)\right],$$

we deduce that $p_0(\cdot)$ is continuous from $[0,1]$ to $L^1(I,\mathbf{R})$.
   At the same time, from Hypothesis it follows that

$$\mathrm{d}\left(g^0(\lambda)(t), F(t, x^0(\lambda)(t), V(x^0(\lambda))(t))\right) \leq p_0(\lambda)(t) \quad \text{a.e. in } I. \tag{3.1}$$

   Fix $\delta > 0$ and for $m \in \mathbf{N}$ we set $\delta_m = \frac{m+1}{m+2}\delta$.
   We shall prove next that there exists a continuous mapping $g^1\colon [0,1] \to L^1(I,\mathbf{R})$ with the following properties

a) $g^1(\lambda)(t) \in F\left(t, x^0(\lambda)(t), V(x^0(\lambda))(t)\right)$ a.e. in I,

b) $g^1(0) = f_0$, $g^1(1) = f_1$,

c) $|g^1(\lambda)(t) - g^0(\lambda)(t)| \leq p_0(\lambda)(t) + \delta_0\frac{\Gamma(\alpha+1)}{T^\alpha}$ a.e. in I.

   Define

$$G^1(\lambda) = \mathrm{cl}\left\{v \in \Phi^1(\lambda); \quad |v(t) - g^0(\lambda)(t)| < p_0(\lambda)(t) + \delta_0\frac{\Gamma(\alpha+1)}{T^\alpha}, \quad \text{a.e. in } I\right\}$$

and, by (3.1), we find that $G^1(\lambda)$ is nonempty for any $\lambda \in [0,1]$. Moreover, since the mapping $\lambda \mapsto p_0(\lambda)$ is continuous, we apply Lemma 2.3 and we obtain the existence of a continuous mapping $g^1\colon [0,1] \to L^1(I,\mathbf{R})$ such that $g^1(\lambda) \in G^1(\lambda)\ \forall \lambda \in [0,1]$, hence with properties a)–c).
   Define now

$$x^1(\lambda)(t) = \xi_0 + t\xi_1 + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)}g^1(\lambda)(u)\,\mathrm{d}u, \quad t \in I$$

and note that, since $|x^1(\lambda) - x^1(\lambda_0)|_C \leq \frac{T^{\alpha-1}}{\Gamma(\alpha)}|g^1(\lambda) - g^1(\lambda_0)|_1$, $x^1(\cdot)$ is continuous from $[0,1]$ into $C(I,\mathbf{R})$.
   Set $p_m(\lambda) := |I^\alpha M|^{m-1}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)}|p_0(\lambda)|_1 + \delta_m\right)$.
   We shall prove that for all $m \geq 1$ and $\lambda \in [0,1]$ there exist $x^m(\lambda) \in C(I,\mathbf{R})$ and $g^m(\lambda) \in L^1(I,\mathbf{R})$ with the following properties:

i) $g^m(0) = f_0$, $g^m(1) = f_1$,

ii) $g^m(\lambda)(t) \in F\left(t, x^{m-1}(\lambda)(t), V(x^{m-1}(\lambda))(t)\right)$ a.e. in I,

iii) $g^m\colon [0,1] \to L^1(I,\mathbf{R})$ is continuous,

iv) $|g^1(\lambda)(t) - g^0(\lambda)(t)| \leq p_0(\lambda)(t) + \delta_0\frac{\Gamma(\alpha+1)}{T^\alpha}$,

v) $|g^m(\lambda)(t) - g^{m-1}(\lambda)(t)| \le M(t)p_m(\lambda), \quad m \ge 2,$

vi) $x^m(\lambda)(t) = \xi_0 + t\xi_1 + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} g^m(\lambda)(u)\, du, \quad t \in I.$

Assume that we have already constructed $g^m(\cdot)$ and $x^m(\cdot)$ with i)–vi) and define

$$\Psi^{m+1}(\lambda) = \left\{ v \in L^1(I, \mathbf{R}); \quad v(t) \in F\big(t, x^m(\lambda)(t), V(x^m(\lambda))(t)\big) \quad \text{a.e. in } I \right\},$$

$$\Phi^{m+1}(\lambda) = \begin{cases} \{f_0\} & \text{if } \lambda = 0, \\ \Psi^{m+1}(\lambda) & \text{if } 0 < \lambda < 1, \\ \{f_1\} & \text{if } \lambda = 1. \end{cases}$$

As in the case $m = 1$ we obtain that $\Phi^{m+1} \colon [0,1] \to \mathcal{D}(I, \mathbf{R})$ is lower semicontinuous. From ii), v) and Hypothesis, for almost all $t \in I$, we have

$$\left| x^m(\lambda)(t) - x^{m-1}(\lambda)(t) \right|$$
$$\le \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} \left| g^m(\lambda)(u) - g^{m-1}(\lambda)(u) \right|\, du$$
$$\le \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} M(u) p_m(\lambda)\, du$$
$$= I^\alpha M(t) p_m(\lambda)$$
$$\le |I^\alpha M| p_m(\lambda)$$
$$< p_{m+1}(\lambda).$$

For $\lambda \in [0,1]$ consider the set

$$G^{m+1}(\lambda) = \text{cl}\left\{ v \in \Phi^{m+1}(\lambda); \quad |v(t) - g^m(\lambda)(t)| < M(t) p_{m+1}(\lambda) \quad \text{a.e. in } I \right\}.$$

To prove that $G^{m+1}(\lambda)$ is not empty we note first that $r_m := |I^\alpha M|^m (\delta_{m+1} - \delta_m) > 0$ and by Hypothesis and ii) one has

$$\text{d}\big(g^m(t), F\big(t, x^m(\lambda)(t), V(x^m(\lambda))(t)\big)\big)$$
$$\le L(t) \left(\left| x^m(\lambda)(t) - x^{m-1}(\lambda)(t) \right| + \int_0^t L(s) \left| x^m(\lambda)(s) - x^{m-1}(\lambda)(s) \right|\, ds \right)$$
$$\le L(t) \left(1 + \int_0^t L(s)\, ds \right) |I^\alpha M(t)| p_m(\lambda)$$
$$= M(t)(p_{m+1}(\lambda) - r_m) < M(t) p_{m+1}(\lambda).$$

Moreover, since $\Phi^{m+1} \colon [0,1] \to \mathcal{D}(I, \mathbf{R})$ is lower semicontinuous and the maps $\lambda \to p_{m+1}(\lambda)$, $\lambda \to h^m(\lambda)$ are continuous, we apply Lemma 2.3. and we obtain the existence of a continuous selection g^{m+1} of G^{m+1}.

Therefore,

$$|x^m(\lambda) - x^{m-1}(\lambda)|_C \le |I^\alpha M| p_m(\lambda) \le |I^\alpha M|^m \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} |p_0(\lambda)|_1 + \delta \right)$$

and thus $\{x^m(\lambda)\}_{m \in \mathbf{N}}$ is a Cauchy sequence in the Banach space $C(I, \mathbf{R})$, hence it converges to some function $x(\lambda) \in C(I, \mathbf{R})$.

Let $g(\lambda) \in L^1(I, \mathbf{R})$ be such that

$$x(\lambda)(t) = \xi_0 + t\xi_1 + \int_0^t \frac{(t-u)^{\alpha-1}}{\Gamma(\alpha)} g(\lambda)(u) \, du, \quad t \in I.$$

The function $\lambda \mapsto \frac{T^{\alpha-1}}{\Gamma(\alpha)}|p_0(\lambda)|_1 + \delta$ is continuous, so it is locally bounded. Therefore the Cauchy condition is satisfied by $\{x^m(\lambda)\}_{m \in \mathbf{N}}$ locally uniformly with respect to λ and this implies that the mapping $\lambda \to x(\lambda)$ is continuous from $[0,1]$ into $C(I, \mathbf{R})$. Obviously, the convergence of the sequence $\{x^m(\lambda)\}$ to $x(\lambda)$ in $C(I, \mathbf{R})$ implies that $g^m(\lambda)$ converges to $g(\lambda)$ in $L^1(I, \mathbf{R})$.

Finally, from ii), Hypothesis and from the fact that the values of F are closed we obtain that $x(\lambda) \in \mathcal{S}(\xi_0, \xi_1)$. From i) and v) we have $x(0) = x_0, x(1) = x_1$ and the proof is complete. $\square$

In what follows we use the notations

$$\tilde{u}(t) = x_0 + tx_1 + \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} u(s) \, ds, \quad u \in L^1(I, \mathbf{R}) \tag{3.2}$$

and

$$p_0(u)(t) = |u(t)| + p(t) + L(t)\left(|\tilde{u}(t)| + \int_0^t L(s)|\tilde{u}(s)| \, ds\right), \quad t \in I \tag{3.3}$$

Let us note that

$$\mathrm{d}(u(t), F(t, \tilde{u}(t), V(\tilde{u})(t)) \le p_0(u)(t) \quad \text{a.e. in } I \tag{3.4}$$

and, since for any $u_1, u_2 \in L^1(I, \mathbf{R})$

$$|p_0(u_1) - p_0(u_2)|_1 \le (1 + |I^\alpha M(T)|)|u_1 - u_2|_1$$

the mapping $p_0 \colon L^1(I, \mathbf{R}) \to L^1(I, \mathbf{R})$ is continuous.

Proposition 3.2. *Assume that Hypothesis is satisfied and let $\phi \colon L^1(I, \mathbf{R}) \to L^1(I, \mathbf{R})$ be a continuous map such that $\phi(u) = u$ for all $u \in \mathcal{T}(x_0, x_1)$. For $u \in L^1(I, \mathbf{R})$, we define*

$$\Psi(u) = \left\{ u \in L^1(I, \mathbf{R}); \quad u(t) \in F(t, \widetilde{\phi(u)}(t), V(\widetilde{\phi(u)})(t)) \quad \text{a.e. in } I \right\},$$

$$\Phi(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}(x_0, x_1), \\ \Psi(u) & \text{otherwise.} \end{cases}$$

Then the multifunction $\Phi \colon L^1(I, \mathbf{R}) \to \mathcal{P}(L^1(I, \mathbf{R}))$ is lower semicontinuous with closed decomposable and nonempty values.

Proof. According to (3.4), Lemma 2.2 and the continuity of p_0 we obtain that Ψ has closed decomposable and nonempty values and the same holds for the set-valued map Φ.

Let $C \subset L^1(I, \mathbf{R})$ be a closed subset, let $\{u_m\}_{m \in \mathbf{N}}$ converge to some $u_0 \in L^1(I, \mathbf{R})$ and $\Phi(u_m) \subset C$, for any $m \in \mathbf{N}$. Let $v_0 \in \Phi(u_0)$ and for every $m \in \mathbf{N}$ consider a measurable selection v_m from the set-valued map $t \to F(t, \widetilde{\phi(u_m)}(t), V(\widetilde{\phi(u_m)})(t))$ such that $v_m = u_m$ if $u_m \in \mathcal{T}(x_0, x_1)$ and

$$|v_m(t) - v_0(t)| = \mathrm{d}\left(v_0(t), F\left(t, \widetilde{\phi(u_m)}(t), V(\widetilde{\phi(u_m)})(t)\right)\right) \quad \text{a.e. in } I$$

otherwise. One has

$$|v_m(t) - v_0(t)| \leq \mathrm{d}_H\big(F\big(t, \widetilde{\phi(u_m)}(t), V(\widetilde{\phi(u_m)})(t)\big), F\big(t, \widetilde{\phi(u_0)}(t), V(\widetilde{\phi(u_0)})(t)\big)\big)$$
$$\leq L(t)\big(|\widetilde{\phi(u_m)}(t) - \widetilde{\phi(u_0)}(t)| + \int_0^t L(s)|\widetilde{\phi(u_m)}(s) - \widetilde{\phi(u_0)}(s)|\,\mathrm{d}s\big),$$

hence

$$|v_m - v_0|_1 \leq |I^\alpha M(T)| \cdot |\phi(u_m) - \phi(u_0)|_1.$$

Since $\phi\colon L^1(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ is continuous, it follows that v_m converges to v_0 in $L^1(I,\mathbf{R})$. On the other hand, $v_m \in \Phi(u_m) \subset C \ \forall m \in \mathbf{N}$ and since C is closed we infer that $v_0 \in C$. Hence $\Phi(u_0) \subset C$ and Φ is lower semicontinuous. $\qquad\square$

Theorem 3.3. *Assume that Hypothesis is satisfied, consider $x_0, x_1 \in \mathbf{R}$ and assume $|I^\alpha M| < 1$. Then there exists a continuous mapping $g : L^1(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ such that*

i) $g(u) \in \mathcal{T}(x_0, x_1), \quad \forall u \in L^1(I,\mathbf{R})$,

ii) $g(u) = u, \quad \forall u \in \mathcal{T}(x_0, x_1)$.

Proof. Fix $\delta > 0$ and for $m \geq 0$ set $\delta_m = \frac{m+1}{m+2}\delta$ and define

$$p_m(u) := |I^\alpha M|^{m-1}\left(\frac{T^{\alpha-1}}{\Gamma(\alpha)}|p_0(u)|_1 + \delta_m\right),$$

where $\tilde{u}$ and $p_0(\cdot)$ are defined in (3.2) and (3.3). By the continuity of the map $p_0(\cdot)$, already proved, we obtain that $p_m\colon L^1(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ is continuous.

We define $g_0(u) = u$ and we shall prove that for any $m \geq 1$ there exists a continuous map $g_m\colon L^1(I,\mathbf{R}) \to L^1(I,\mathbf{R})$ that satisfies

a) $g_m(u) = u, \quad \forall u \in \mathcal{T}(x_0, x_1)$,

b) $g_m(u)(t) \in F\big(t, \widetilde{g_{m-1}(u)}(t), V(\widetilde{g_{m-1}(u)})(t)\big) \quad$ a.e. in I,

c) $|g_1(u)(t) - g_0(u)(t)| \leq p_0(u)(t) + \delta_0\frac{\Gamma(\alpha+1)}{T^\alpha} \quad$ a.e. in I,

d) $|g_m(u)(t) - g_{m-1}(t)| \leq M(t)p_m(u) \quad$ a.e. in I, $m \geq 2$.

For $u \in L^1(I,\mathbf{R})$, we define

$$\Psi_1(u) = \Big\{v \in L^1(I,\mathbf{R}); \quad v(t) \in F(t, \tilde{u}(t), V(\tilde{u})(t)) \quad \text{a.e. in } I\Big\},$$

$$\Phi_1(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}(x_0, x_1), \\ \Psi_1(u) & \text{otherwise} \end{cases}$$

and by Proposition 3.2 (with $\phi(u) = u$) we obtain that $\Phi_1\colon L^1(I,\mathbf{R}) \to \mathcal{D}(I,\mathbf{R})$ is lower semicontinuous. Moreover, due to (3.4) the set

$$G_1(u) = \mathrm{cl}\left\{v \in \Phi_1(u); \quad |v(t) - u(t)| < p_0(u)(t) + \delta_0\frac{\Gamma(\alpha+1)}{T^\alpha} \quad \text{a.e. in } I\right\}$$

is not empty for any $u \in L^1(I,\mathbf{R})$. So applying Lemma 2.3, we find a continuous selection $g_1(\cdot)$ of $G_1(\cdot)$ that satisfies a)–c).

Suppose we have already constructed $g_i(\cdot)$, $i = 1, \ldots, m$ satisfying a)–d). For $u \in L^1(I, \mathbf{R})$ we define

$$\Psi_{m+1}(u) = \left\{ v \in L^1(I, \mathbf{R}); \quad v(t) \in F(t, \widetilde{g_m(u)}(t), V(\widetilde{g_m(u)}))(t)) \quad \text{a.e. in } I \right\},$$

$$\Phi_{m+1}(u) = \begin{cases} \{u\} & \text{if } u \in \mathcal{T}(x_0, x_1), \\ \Psi_{m+1}(u) & \text{otherwise.} \end{cases}$$

We apply Proposition 3.2 (with $\phi(u) = g_m(u)$) and obtain that $\Phi_{m+1}(\cdot)$ is a lower semicontinuous multifunction with closed decomposable and nonempty values. Define the set

$$G_{m+1}(u) = \mathrm{cl}\{ v \in \Phi_{m+1}(u); \quad |v(t) - g_{m+1}(u)(t)| < M(t) p_{m+1}(u) \quad \text{a.e. in } I \}.$$

To prove that $G_{m+1}(u)$ is not empty we note first that $r_m := |I^\alpha L|^m (\delta_{m+1} - \delta_m) > 0$ and by Hypothesis and b) one has

$$\mathrm{d}\big(g_m(t), F\big(t, \widetilde{g_m(u)}(t), V(\widetilde{g_m(u)})(t)\big)\big)$$
$$\leq L(t) \left(|\widetilde{g_m(u)}(t) - \widetilde{g_{m-1}(u)}(t)| + \int_0^t L(s) |\widetilde{g_m(u)}(s) - \widetilde{g_{m-1}(u)}(s)| \, \mathrm{d}s \right)$$
$$\leq M(t) |I^\alpha M| p_m(u) = M(t)(p_{m+1}(u) - r_m)$$
$$< M(t) p_{m+1}(u).$$

Thus $G_{m+1}(u)$ is not empty for any $u \in L^1(I, \mathbf{R})$. With Lemma 2.3, we find a continuous selection g_{m+1} of G_{m+1}, satisfying a)–d).

Therefore, we obtain that

$$|g_{m+1}(u) - g_m(u)|_1 \leq |I^\alpha M|^m \left(\frac{T^{\alpha-1}}{\Gamma(\alpha)} |p_0(u)|_1 + \delta \right)$$

and this implies that the sequence $\{g_m(u)\}_{m \in \mathbf{N}}$ is a Cauchy sequence in the Banach space $L^1(I, \mathbf{R})$. Let $g(u) \in L^1(I, \mathbf{R})$ be its limit. The function $u \to |p_0(u)|_1$ is continuous, hence it is locally bounded and the Cauchy condition is satisfied by $\{g_m(u)\}_{m \in \mathbf{N}}$ locally uniformly with respect to u. Hence the mapping $g(\cdot) \colon L^1(I, \mathbf{R}) \to L^1(I, \mathbf{R})$ is continuous.

From a) it follows that $g(u) = u$, $\forall u \in \mathcal{T}(x_0, x_1)$ and from b) and the fact that F has closed values we obtain that

$$g(u)(t) \in F(t, \widetilde{g(u)}(t), V(\widetilde{g(u)}))(t)) \quad \text{a.e. in } I \quad \forall u \in L^1(I, \mathbf{R}).$$

and the proof is complete. □

Remark 3.4. We recall that if Y is a Hausdorff topological space, a subspace X of Y is called a retract of Y if there is a continuous map $h \colon Y \to X$ such that $h(x) = x$, $\forall x \in X$.

Therefore, by Theorem 3.3, for any $x_0, x_1 \in \mathbf{R}$, the set $\mathcal{T}(x_0, x_1)$ of selections of solutions of (1.1) is a retract of the Banach space $L^1(I, \mathbf{R})$.

Acknowledgements

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