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Oscillation of Second-Order Nonlinear Differential Equations with Damping Term

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Abstract

We present new oscillation criteria for the second order nonlinear differential equation with damping term of the form

$$(r(t)\psi(x)f(\dot{x}))^{\cdot} + p(t)\varphi\left(g(x), r(t)\psi(x)f(\dot{x})\right) + q(t)g(x) = 0,$$

where $p,\ q,\ r : [t_o, \infty) \to \mathbf{R}$ and $\psi,\ g,\ f : \mathbf{R} \to \mathbf{R}$ are continuous, $r(t) > 0$, $p(t) \geq 0$ and $\psi(x) > 0$, $xg(x) > 0$ for $x \neq 0$, $uf(u) > 0$ for $u \neq 0$. Our results generalize and extend some known oscillation criteria in the literature. The relevance of our results is illustrated with a number of examples.

1 Introduction

We are concerned with the oscillation of solutions of second order differential equations with damping terms of the following form

$$(r(t)\psi(x)f(\dot{x}))^{\cdot} + p(t)\varphi\left(g(x), r(t)\psi(x)f(\dot{x})\right) + q(t)g(x) = 0 \qquad \left(\cdot = \frac{d}{dt}\right) \qquad (E)$$

where $r \in C[[t_o, \infty), \mathbf{R}^+]$, $p \in C[[t_o, \infty), [0, \infty)]$, $q \in C[[t_o, \infty), \mathbf{R}]$, $\psi \in C[\mathbf{R}, \mathbf{R}^+]$ and $g \in C^1[\mathbf{R}, \mathbf{R}]$ such that $xg(x) > 0$ for $x \neq 0$ and $g'(x) > 0$ for $x \neq 0$. φ is defined and continuous on $\mathbf{R} \times \mathbf{R} - \{0\}$ with $u\varphi(u, v) > 0$ for $uv \neq 0$ and $\varphi(\lambda u, \lambda v) = \lambda\varphi(u, v)$ for $0 < \lambda < \infty$ and $(u, v) \in \mathbf{R} \times \mathbf{R} - \{0\}$.

We recall that a function $x : [t_o, t_1) \to \mathbf{R}$ is called a solution of equation (E) if $x(t)$ satisfies equation (E) for all $t \in [t_o, t_1)$. In the sequel it will be always assumed that solutions of equation (E) exist for any $t_o \geq 0$. Such a solution $x(t)$ is said to be oscillatory if it has arbitrarily large zeros, and otherwise it said to be nonoscillatory. Equation (E) is called oscillatory if all its solutions are oscillatory.

Oscillatory and nonoscillatory behavior of solutions for various classes of second-order differential equations have been widely discussed in the literature (see, for example, $[1-37]$ and the references quoted therein). There is a great number of papers dealing with particular cases of equation (E) such as the linear equations

$$\ddot{x}(t)+q(t)x(t)=0, \qquad (E_1)$$

$$\left(r(t)\dot{x}(t)\right)^{\cdot}+q(t)x(t)=0, \qquad (E_2)$$

the nonlinear equations

$$\ddot{x}(t)+q(t)g\left(x\right)=0, \qquad (E_3)$$

$$\left(r(t)\psi(x)\dot{x}(t)\right)^{\cdot}+q(t)g\left(x\right)=0, \qquad (E_4)$$

and the nonlinear equations with damping term

$$\left(r(t)\dot{x}(t)\right)^{\cdot}+p(t)\dot{x}(t)+q(t)g\left(x\right)=0, \qquad (E_5)$$

$$\left(r(t)\psi(x)\dot{x}(t)\right)^{\cdot}+p(t)\dot{x}(t)+q(t)g\left(x\right)=0, \qquad (E_6)$$

$$\left(r(t)f(\dot{x})\right)^{\cdot}+p(t)\varphi\left(g(x),r(t)f(\dot{x})\right)+q(t)g(x)=0. \qquad (E_7)$$

An important tool in the study of oscillatory behaviour of solutions for equations $(E_1)-(E_7)$ is the averaging technique. This goes back as far as the classical results of Wintner [32] which proved that (E_1) is oscillatory if

$$\lim_{t\to\infty}\frac{1}{t}\int_{t_o}^{t}\int_{t_o}^{s}q(u)du\ ds=\infty,$$

and Hartman [10] who showed the above limit cannot be replaced by the limit superior and proved the condition

$$\liminf_{t\to\infty}\frac{1}{t}\int_{t_o}^{t}\int_{t_o}^{s}q(u)du\ ds<\limsup_{t\to\infty}\frac{1}{t}\int_{t_o}^{t}\int_{t_o}^{s}q(u)du\ ds\leq\infty,$$

implies that equation (E_1) is oscillatory.

The result of Wintner was improved by Kamenev [11] who proved that the condition

$$\limsup_{t\to\infty}\frac{1}{t^{n-1}}\int_{t_o}^{t}(t-s)^{n-1}q(s)ds=\infty \quad \text{for some } n>2,$$

is sufficient for the oscillation of equation (E_1).

Some other results can be found in [21], [27], [34] and the references therein. Kong [12] and Li [15] employed the technique of Philos [27] and obtained several oscillation results for (E_2).

Butler [2], Philos [22], [23], [24], [25], [26], Philos and Purnaras [28], Wong and Yeh [33] and Yeh [37] obtained some sufficient conditions for the oscillation of equation (E_3) and Grace [7], Elabbasy [3], Manojlovic [18] and Tiryaki and Cakmak [31] for the oscillation of equation (E_4).

In the presence of damping, a number of oscillation criteria have been obtained by Li and Agarwal [16], Grace and Lalli [9], Rogovchenko [29], Kirane and Rogovchenko [14], Li et al. [17], Yang [36] Nagabuchi and Minora Yamamoto [20], Yan [35], Elabbasy, Hassan and Saker [5] for equation (E_5), Ayanlar and Tiryaki [1], Tiryaki and Zafer [30], Kirane and Rogovchenko [13], Grace [6], [7], [8] and Manojlovic [19] for equation (E_6) and Elabbasy and Elsharabasy [4] for equation (E_7).

In this paper we extend the results of Wintner [32], Yan [35], Elabbasy [3], Elabbasy and Elsharabasy [4], and Nagabuchi and Yamamoto [20] for a broad class of second order nonlinear equation of the type (E).

2 Main Results

Theorem 1. Suppose, in addition to condition

$$\varphi(1,z) \geq z \quad \text{for all } z \neq 0, \tag{1}$$

$$\frac{g^{'}(x)}{\psi(x)} \geq K > 0 \quad \text{for all} \quad x \neq 0, \tag{2}$$

$$\int_{\pm\varepsilon}^{\pm\infty} \frac{\psi(y)}{g(y)} dy < \infty \quad \text{for all} \quad \varepsilon > 0, \tag{3}$$

and

$$0 < L_1 \leq \frac{f(y)}{y} \leq L_2 \text{ for all } y \neq 0 \text{ and } \lim_{y\to\infty} \frac{f(y)}{y} \text{ exist,} \tag{4}$$

that there exist a positive function $\rho \in C^1[t_o,\infty)$ such that $(\rho(t)r(t))^{\cdot} \leq 0$ for all $t \geq t_o$. Then equation (E) is oscillatory if

$$\lim_{t\to\infty} \sup \frac{1}{t} \int_{t_o}^{t} \int_{t_o}^{s} \left[\rho(u)q(u) - \frac{r(u)\rho(u)}{4M}\left(\frac{\dot{\rho}(u)}{\rho(u)} - p(u)\right)^2\right] duds = \infty, \tag{5}$$

where $M = \frac{K}{L_2}$.

Proof. On the contrary we assume that (E) has a nonoscillatory solution $x(t)$. We suppose without loss of generality that $x(t) > 0$ for all $t \in [t_o,\infty)$. We define the function $\omega(t)$ as

$$\omega(t) = \rho(t)\frac{r(t)\psi(x)f(\dot{x})}{g(x(t))} \quad \text{for all } t \geq t_o.$$

This and equation (E) imply

$$\dot{\omega}(t) = \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \rho(t)[q(t) + p(t)\varphi(1, \frac{\omega(t)}{\rho(t)})] - \rho(t)\frac{r(t)\psi(x)f(\dot{x})g^{'}(x)\dot{x}}{g^2(x(t))}.$$

From (1), (2) and (4) we obtain

$$\dot{\omega}(t) \leq -\rho(t)q(t) - \left(\frac{\dot{\rho}(t)}{\rho(t)} - p(t)\right)\omega(t) - \frac{M}{\rho(t)r(t)}\omega^2(t).$$

Integrating from t_o to t we obtain

$$\int_{t_o}^{t} \rho(s)q(s)ds \le \omega(t_o) - \omega(t) - \int_{t_o}^{t} \left[\frac{M\omega^2(s)}{\rho(s)r(s)} - \left(\frac{\dot\rho(s)}{\rho(s)} - p(s)\right)\omega(s)\right] ds,$$

Thus, for every $t \ge t_o$ we have

$$\int_{t_o}^{t} \left[\rho(s)q(s) - \frac{r(s)\rho(s)}{4M}\left(\frac{\dot\rho(s)}{\rho(s)} - p(s)\right)^2\right] ds \le \omega(t_o) - \omega(t)$$

$$- \int_{t_o}^{t} \left[\sqrt{\frac{M}{\rho(s)r(s)}}\omega(s) - \frac{\sqrt{r(s)\rho(s)}}{2\sqrt{M}}\left(\frac{\dot\rho(s)}{\rho(s)} - p(s)\right)\right]^2 ds.$$

Hence, for all $t \ge t_o$ we have

$$\int_{t_o}^{t} \left[\rho(s)q(s) - \frac{r(s)\rho(s)}{4M}\left(\frac{\dot\rho(s)}{\rho(s)} - p(s)\right)^2\right] ds \le \omega(t_o) - \omega(t),$$

or

$$\int_{t_o}^{t} \left[\rho(s)q(s) - \frac{r(s)\rho(s)}{4M}\left(\frac{\dot\rho(s)}{\rho(s)} - p(s)\right)^2\right] ds \le \omega(t_o) - \rho(t)\frac{r(t)\psi(x)f(\dot x)}{g(x(t))}.$$

From (4) we obtain

$$\int_{t_o}^{t} \left[\rho(s)q(s) - \frac{r(s)\rho(s)}{4M}\left(\frac{\dot\rho(s)}{\rho(s)} - p(s)\right)^2\right] ds \le \omega(t_o) - L_1\rho(t)\frac{r(t)\psi(x)\dot x}{g(x(t))}.$$

Integrate again from t_0 to t we obtain

$$\int_{t_o}^{t}\int_{t_o}^{s} \left[\rho(u)q(u) - \frac{r(u)\rho(u)}{4M}\left(\frac{\dot\rho(u)}{\rho(u)} - p(u)\right)^2\right] duds$$

$$\le \omega(t_o)(t - t_o) - L_1\int_{t_o}^{t} \rho(s)r(s)\frac{\psi(x)\dot x}{g(x)}ds. \qquad (6)$$

Since $r(s)\rho(s)$ is nonincreasing, then by the Bonnet's theorem there exists a $\eta \in [t_o, t]$ such that

$$\begin{aligned} -L_1\int_{t_o}^{t} r(s)\rho(s)\frac{\psi(x)\dot x}{g(x(s))}ds &= -L_1 r(t_o)\rho(t_o)\int_{t_o}^{\eta}\frac{\psi(x)\dot x}{g(x(s))}ds \\ &= L_1 r(t_o)\rho(t_o)\int_{x(\eta)}^{x(t_o)}\frac{\psi(y)}{g(y)}dy \\ &< \begin{cases} 0 & \text{if } x(t_o) < x(\eta), \\ L_1 r(t_o)\rho(t_o)\int_{\varepsilon}^{\infty}\frac{\psi(y)}{g(y)}dy & \text{if } x(t_o) > x(\eta), \end{cases} \end{aligned}$$

hence

$$-\infty < -L_1 \int_{t_o}^{t} r(s)\rho(s)\frac{\psi(x)\dot{x}}{g(x(s))}ds < k_1,$$

where

$$k_1 = L_1 r(t_o)\rho(t_o)\int_{\varepsilon}^{\infty}\frac{\psi(y)}{g(y)}dy.$$

Hence (6) becomes

$$\int_{t_o}^{t}\int_{t_o}^{s}\left[\rho(u)q(u) - \frac{r(u)\rho(u)}{4M}\left(\frac{\dot{\rho}(u)}{\rho(u)} - p(u)\right)^2\right]duds \leq \omega(t_o)(t-t_o) + k_1. \quad (7)$$

Divide (7) by t and take the upper limit as $t \to \infty$ which contradicts the assumption (5). This completes the proof.

Corollary 1. If the condition (5) in the above theorem is replaced by

$$\lim_{t\to\infty}\sup\frac{1}{t}\int_{t_o}^{t}\int_{t_o}^{s} r(u)\rho(u)\left(\frac{\dot{\rho}(u)}{\rho(u)} - p(u)\right)^2 duds < \infty,$$

$$\lim_{t\to\infty}\sup\frac{1}{t}\int_{t_o}^{t}\int_{t_o}^{s}\rho(u)q(u)duds = \infty,$$

then the conclusion of theorem 1 still true.

Remark 1. If $p(t) \equiv 0$, $r(t) \equiv 1$ and $\rho(t) \equiv 1$, then Theorem 1 reduce to Wintner theorem in [32].

Theorem 2. Suppose, in addition to conditions (1) , (2), (3) and (4), that there exist a positive function $\rho \in C^1[t_o,\infty)$ such that

$$(r(t)\rho(t))^{\cdot} \geq 0, \ \ ((r(t)\rho(t))^{\cdot\cdot} \leq 0,$$

$$\gamma(t) = (r(t)p(t)\rho(t) - \dot{\rho}(t)r(t)) \geq 0 \ \text{ and } \ \dot{\gamma}(t) \leq 0 \ \text{ for all } t \geq t_o, \quad (8)$$

and

$$\lim_{t\to\infty}\inf\int_{t_o}^{t}\rho(s)q(s)ds > -\infty \quad (9)$$

hold. Then equation (E) is oscillatory if

$$\lim_{t\to\infty}\sup\frac{1}{t}\int_{t_o}^{t}\left[\int_{t_o}^{s}\rho(u)q(u)du\right]^2 ds = \infty. \quad (10)$$

Proof. On the contrary we assume that (E) has a nonoscillatory solution $x(t)$. We suppose without loss of generality that $x(t) > 0$ for all $t \in [t_o,\infty)$. We define the function $\omega(t)$ as

$$\omega(t) = \rho(t)\frac{r(t)\psi(x(t))f(\dot{x}(t))}{g(x(t))} \ \text{ for all } \ t \geq t_o.$$

This and equation (E) imply

$$\dot{\omega}(t) \le \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \rho(t)[q(t) + p(t)\varphi(1, \frac{\omega(t)}{\rho(t)})] - \rho(t)\frac{r(t)\psi(x(t))f(\dot{x})g^{'}(x(t))\dot{x}(t)}{g^2(x(t))}.$$

From (1), (2) and (4) we obtain

$$\rho(t)q(t) \le -\dot{\omega}(t) + \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - p(t)\omega(t) - M\frac{1}{\rho(t)r(t)}\omega^2(t), \tag{11}$$

or

$$\rho(t)q(t) \le -\dot{\omega}(t) - \gamma(t)\frac{\psi(x)f(\dot{x})}{g(x(t))} - M\frac{1}{\rho(t)r(t)}\omega^2(t).$$

From (4) we have

$$\rho(t)q(t) \le -\dot{\omega}(t) - L_1\gamma(t)\frac{\psi(x)\dot{x}}{g(x(t))} - M\frac{1}{\rho(t)r(t)}\omega^2(t). \tag{12}$$

Integrating from T to t we obtain

$$\int_T^t \rho(s)q(s)ds \le \omega(T) - \omega(t) - L_1\int_T^t \gamma(s)\frac{\psi(x)\dot{x}}{g(x(s))}ds - M\int_T^t \frac{1}{\rho(s)r(s)}\omega^2(s)ds. \tag{13}$$

Now evaluate the integral

$$-\int_T^t \gamma(s)\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds.$$

Since $\gamma(t)$ is nonincreasing, then by the Bonnet's theorem there exists a $\eta \in [T, t]$ such that

$$\begin{aligned}
-\int_T^t \gamma(s)\frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds &= -\gamma(T)\int_T^\eta \frac{\psi(x(s))\dot{x}(s)}{g(x(s))}ds \\
&= -\gamma(T)\int_{x(T)}^{x(\eta)} \frac{\psi(y)}{g(y)}dy \\
&= \gamma(T)\int_{x(\eta)}^{x(T)} \frac{\psi(y)}{g(y)}dy \\
&< \begin{cases} 0 & \text{if } x(\eta) > x(T), \\ \gamma(T)\int_\varepsilon^\infty \frac{\psi(y)}{g(y)}dy & \text{if } x(\eta) < x(T), \end{cases}
\end{aligned}$$

hence

$$-\infty < -\int_T^t \gamma(s)\frac{\psi(x)\dot{x}}{g(x(s))}ds \le k_2, \tag{14}$$

where

$$k_2 = \gamma(T)\int_\varepsilon^\infty \frac{\psi(y)}{g(y)}dy.$$

From (14) in (13), we have

$$\int_T^t \rho(s)q(s)ds \leq \omega(T) - \omega(t) + L_1k_2 - M\int_T^t \frac{1}{\rho(s)r(s)}\omega^2(s)ds. \qquad (15)$$

Thus, we obtain for $t \geq T \geq t_o$

$$\begin{aligned} \int_{t_o}^t \rho(s)q(s)ds &= \int_{t_o}^T \rho(s)q(s)ds + \int_T^t \rho(s)q(s)ds \\ &\leq C_1 - \omega(t) - M\int_T^t \frac{\omega^2(s)}{\rho(s)r(s)}ds, \qquad (16) \end{aligned}$$

where

$$C_1 = \omega(T) + L_1k_2 + \int_{t_o}^T \rho(s)q(s)ds.$$

We consider the following two cases:
Case 1. The integral

$$\int_T^\infty \frac{\omega^2(s)}{\rho(s)r(s)}ds \quad \text{is finite.}$$

Thus, there exists a positive constant N such that

$$\int_T^t \frac{\omega^2(s)}{\rho(s)r(s)}ds \leq N \quad \text{for all } t \geq T. \qquad (17)$$

Thus, we obtain for $t \geq T$

$$\begin{aligned} \left[\int_{t_o}^t \rho(s)q(s)ds\right]^2 &\leq \left\{C_1 - \omega(t) - M\int_T^t \frac{\omega^2(s)}{\rho(s)r(s)}ds\right\}^2 \\ &\leq 4C_1^2 + 4\omega^2(t) + 4M^2\left[\int_T^t \frac{\omega^2(s)}{\rho(s)r(s)}ds\right]^2. \end{aligned}$$

Therefore, by taking into account (17), we conclude that

$$\left[\int_{t_o}^t \rho(s)q(s)ds\right]^2 \leq C_2 + 4\omega^2(t),$$

where

$$C_2 = 4C_1^2 + 4M^2\left[\int_T^t \frac{\omega^2(s)}{\rho(s)r(s)}ds\right]^2.$$

Thus, we obtain for every $t \geq T$

$$\begin{aligned}\int_{t_o}^{t}\left[\int_{t_o}^{s}\rho(u)q(u)du\right]^2 ds &= \int_{t_o}^{T}\left[\int_{t_o}^{s}\rho(u)q(u)du\right]^2 ds \\ &\quad + \int_{T}^{t}\left[\int_{t_o}^{s}\rho(u)q(u)du\right]^2 ds \\ &= C_3 + \int_{T}^{t}\left[\int_{t_o}^{s}\rho(u)q(u)du\right]^2 ds \\ &\leq C_3 + C_2(t-T) + 4\int_{T}^{t}\omega^2(s)ds \\ &= C_3 + C_2(t-T) + 4\int_{T}^{t}\rho(s)r(s)\frac{\omega^2(s)}{\rho(s)r(s)}ds.\end{aligned}$$

Since $\rho(t)r(t)$ is positive and nondecreasing for $t \in [t_o, \infty)$, the Bonnet's theorem would ensures the existence of $T_1 \in [T, t]$ such that

$$\int_{T}^{t}\rho(s)r(s)\frac{\omega^2(s)}{\rho(s)r(s)}ds = \rho(t)r(t)\int_{T_1}^{t}\frac{\omega^2(s)}{\rho(s)r(s)}ds.$$

Also, since $\rho(t)r(t)$ is positive on $[t_o, \infty)$ and $(\rho(t)r(t))^{\cdot}$ is nonnegative and bounded above, it follow that $\rho(t)r(t) \leq \beta t$ for all large t where $\beta > 0$ is a constant. Which ensures

$$\int_{t_o}^{\infty}\frac{ds}{\rho(s)r(s)} = \infty.$$

Thus, we conclude that

$$\int_{t_o}^{t}\left[\int_{t_o}^{s}\rho(u)q(u)du\right]^2 ds \leq C_3 + C_2(t-T) + 4\beta t\int_{T}^{t}\frac{\omega^2(s)}{\rho(s)r(s)}ds.$$

So, we have

$$\int_{t_o}^{t}\left[\int_{t_o}^{s}\rho(u)q(u)du\right]^2 ds \leq C_3 + C_2(t-T) + 4N\beta t.$$

This implies

$$\lim_{t\to\infty}\sup\frac{1}{t}\int_{t_o}^{t}\left[\int_{t_o}^{s}\rho(u)q(u)du\right]^2 \leq C_2 + 4N\beta < \infty,$$

which contradicts (10).
Case 2. The integral

$$\int_{T}^{\infty}\frac{K\omega^2(s)}{\rho(s)r(s)}ds \quad \text{is infinite.} \tag{18}$$

By condition (2) , we see that

$$\int_T^{\infty} \frac{g'(x)}{\rho(s)r(s)\psi(x)}\omega^2(s)ds = \infty.$$

By virtue of condition (9), it follow from (16) for constant B

$$-\omega(t) \geq B + \frac{1}{L_2}\int_T^t \frac{g'(x(s))}{\rho(s)r(s)\psi(x(s))}\omega^2(s)ds \quad \text{for every } t \geq T. \tag{19}$$

We can consider a $T_2 \geq T$ such that

$$C = B + \frac{1}{L_2}\int_T^{T_2} \frac{g'(x(s))}{\rho(s)r(s)\psi(x(s))}\omega^2(s)ds > 0.$$

Then (19)ensures that $w(t)$ is negative on $[T_2, \infty)$. Now, (19) gives

$$\begin{aligned}\frac{1}{L_2}\frac{g'(x)\omega^2(t)}{\rho(t)r(t)\psi(x)}\left\{B + \frac{1}{L_2}\int_T^t \frac{g'(x(s))}{\rho(s)r(s)\psi(x)}\omega^2(s)ds\right\}^{-1} &\geq \frac{-g'(x(t))f(\dot{x}(t))}{L_2 g(x(t))} \\ &\geq \frac{-g'(x(t))\dot{x}(t)}{g(x(t))},\end{aligned}$$

and consequently for all $t \geq T_2$

$$\log \frac{B + \frac{1}{L_2}\int_T^t \frac{g'(x)}{\rho(s)r(s)\psi(x)}\omega^2(s)ds}{C} \geq \log \frac{g(x(T_2))}{g(x(t))} \quad \text{for } t \geq T_2.$$

Hence

$$B + \frac{1}{L_2}\int_T^t \frac{g'(x)}{\rho(s)r(s)\psi(x)}\omega^2(s)ds \geq C\ \frac{g(x(T))}{g(x(t))}.$$

So, (19) yields

$$-\omega(t) \geq \frac{C'}{g(x(t))},$$

where $C' = Cg(x(T))$. Thus, we have

$$\psi(x)f(\dot{x}) \leq -C'\frac{1}{\rho(t)r(t)}.$$

From (4) we have

$$\psi(x)\dot{x} \leq \frac{-C'}{L_1}\frac{1}{\rho(t)r(t)}.$$

Integrate from T_2 to t, we have

$$\int_{x(T)}^{x(t)} \psi(y)dy \leq \frac{-C'}{L_1}\int_T^t \frac{1}{\rho(s)r(s)}ds \to -\infty \quad \text{as} \quad t \to \infty,$$

a contradiction to the fact $x(t) > 0$ for $t \geq t_o$, and hence (18) fails. This completes the proof.

Remark 2. If $p(t) \equiv 0$ and $f(\dot{x}) = \dot{x}$ then Theorem 2 reduce to Theorem 1 of Elabbasy [3].

Theorem 3. Suppose, in addition to (1) , (2) and

$$f^2(y) \leq Lyf(y), \tag{20}$$

that there exist a positive function $\rho \in C^1[t_o, \infty)$. Moreover, assume that there exist continuous functions $H,\ h : D \equiv \{(t,s), t \geq s \geq t_o\} \to \mathbf{R}$ and H has a continuous and nonpositive partial derivative on D with respect to the second variable such that

$$\begin{aligned} H(t,t) &= 0 \text{ for all } t \geq t_o, \quad H(t,s) > 0 \text{ for all } t > s \geq t_o, \\ -\frac{\partial H(t,s)}{\partial s} &= h(t,s)\sqrt{H(t,s)} \quad \text{for all} \quad (t,s) \in D. \end{aligned}$$

Then equation (E) is oscillatory if

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_o)} \int_{t_o}^{t} \left\{ H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4M} Q^2(t,s) \right\} ds = \infty. \tag{21}$$

Where

$$Q(t,s) = \left[h(t,s) - \left(\frac{\dot{\rho}(s)}{\rho(s)} - p(s) \right) \sqrt{H(t,s)} \right] \quad \text{and} \quad M = \frac{K}{L}.$$

Proof. On the contrary we assume that (E) has a nonoscillatory solution $x(t)$. We suppose without loss of generality that $x(t) > 0$ for all $t \in [t_o, \infty)$. We define the function $\omega(t)$ as

$$\omega(t) = \rho(t) \frac{r(t)\psi(x(t))f(\dot{x}(t))}{g(x(t))} \qquad \text{for all } t \geq t_o.$$

This and equation (E) imply

$$\dot{\omega}(t) \leq \frac{\dot{\rho}(t)}{\rho(t)}\omega(t) - \rho(t)[q(t) + p(t)\varphi(1, \frac{\omega(t)}{\rho(t)})] - \rho(t)\frac{r(t)\psi(x(t))f(\dot{x}(t))g'(x(t))\dot{x}(t)}{g^2(x(t))}.$$

From (1), (2) and (20) we obtain

$$\dot{\omega}(t) \leq \left(\frac{\dot{\rho}(t)}{\rho(t)} - p(s) \right) \omega(t) - \rho(t)q(t) - \frac{M}{\rho(t)r(t)}\omega^2(t).$$

Thus, for every $t \geq T$, we have

$$\int_T^t H(t,s)\rho(s)q(s)ds \leq \int_T^t \left(\frac{\dot{\rho}(t)}{\rho(t)} - p(s) \right) H(t,s)\omega(s)ds$$

$$-\int_T^t H(t,s)\dot{\omega}(s)ds - M\int_T^t \frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds.$$

Since

$$\begin{aligned} -\int_T^t H(t,s)\dot{\omega}(s)ds &= H(t,T)\omega(T) + \int_T^t \frac{\partial H(t,s)}{\partial s}\omega(s)ds \\ &= H(t,T)\omega(T) - \int_T^t h(t,s)\sqrt{H(t,s)}\omega(s)ds. \end{aligned}$$

The previous inequality becomes

$$\begin{aligned} \int_T^t H(t,s)\rho(s)q(s)ds &\leq H(t,T)\omega(T) - \int_T^t Q(t,s)\sqrt{H(t,s)}\omega(s)ds \\ &\quad -M\int_T^t \frac{H(t,s)}{\rho(s)r(s)}\omega^2(s)ds. \end{aligned}$$

Hence we have

$$\int_T^t \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4M}Q^2(t,s)\right] ds \leq H(t,T)\omega(T)$$

$$-\int_T^t \left(\frac{\sqrt{M}\sqrt{H(t,s)}}{\sqrt{r(s)\rho(s)}}\omega(s) + \frac{\sqrt{r(s)\rho(s)}}{2\sqrt{M}}Q(t,s)\right)^2 ds. \quad (22)$$

By (22) we have for every $t \geq T \geq t_o$

$$\begin{aligned} \int_T^t \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4M}Q^2(t,s)\right] ds &\leq H(t,T)\omega(T) \quad (23) \\ &\leq H(t,T)\left|\omega(T)\right| \\ &\leq H(t,t_o)\left|\omega(T)\right|. \end{aligned}$$

We use the above inequality for $T = T_o$ to obtain

$$\int_{T_o}^t \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4M}Q^2(t,s)\right] ds \leq H(t,t_o)\left|\omega(T_o)\right|.$$

Therefore,

$$\begin{aligned} &\int_{t_o}^t \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4M}Q^2(t,s)\right] ds \\ &= \int_{t_o}^{T_o} \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4M}Q^2(t,s)\right] ds \\ &\quad + \int_{T_o}^t \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4M}Q^2(t,s)\right] ds. \end{aligned}$$

Hence for every $t \geq t_o$ we have

$$\int_{t_o}^{t} \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4M} Q^2(t,s) \right] ds$$
$$\leq H(t,t_o) \left\{ \int_{t_o}^{T_o} \rho(s) |q(s)| \, ds + |\omega(T_o)| \right\}. \tag{24}$$

Dividing (24) by $H(t,t_o)$ and take the upper limit as $t \to \infty$ we get

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_o)} \int_{t_o}^{t} \left[H(t,s)\rho(s)q(s) - \frac{r(s)\rho(s)}{4M} Q^2(t,s) \right] ds < \infty,$$

which contradicts (21). This completes the proof.
Corollary 2. If condition (21) in Theorem 3 is replaced by

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_o)} \int_{t_o}^{t} H(t,s)\rho(s)q(s) ds = \infty,$$

$$\lim_{t\to\infty} \sup \frac{1}{H(t,t_o)} \int_{t_o}^{t} r(s)\rho(s)Q^2(t,s) ds < \infty,$$

then the conclusion of Theorem 3 remains valid.
Corollary 3. If we take $H(t,s) = (t-s)^\alpha$ for $\alpha \geq 2$, then the condition (21) in the above theorem becomes

$$\lim_{t\to\infty} \sup \frac{1}{t^\alpha} \int_{t_o}^{t} (t-s)^\alpha \rho(s)q(s) - \frac{r(s)\rho(s)}{4M}$$
$$\times \left[\alpha - (t-s) \left(\frac{\rho^{\cdot}(s)}{\rho(s)} - p(s) \right) \right]^2 (t-s)^{\alpha-2} ds = \infty.$$

Remark 3. Theorem 3 extended and improved Theorem 2 in [4].
Remark 4. If $f(\dot{x}) = \dot{x}$, $\psi(x) = 1$ and $\varphi\left(g(x), r(t)\psi(x)f(\dot{x})\right) = \dot{x}$, then Corollary 2 extend Nagabuchi and Yamamoto theorem in [20] and Yan Theorem with $g(x) = x$ in [35].
Example 1. Consider the differential equation

$$\left(t\frac{4t^2 + 2\cos^2(\ln t)}{t^2 + 2\cos^2(\ln t)} \dot{x}(t) \left(\frac{1 + 2\dot{x}^2(t)}{4 + 2\dot{x}^2(t)} \right) \right)^{\cdot} + \frac{\sin^2(\ln t)}{t^2}$$
$$\times \left[\frac{\left(t\frac{4t^2+2\cos^2(\ln t)}{t^2+2\cos^2(\ln t)} \dot{x}(t) \left(\frac{1+2\dot{x}^2(t)}{4+2\dot{x}^2(t)} \right) \right)^2 + x^2(t)}{x(t)} \right] + \frac{1}{2t} x(t) = 0, \ t \geq t_o = 1 \,.$$

If we take $\rho(t) = 1$ and $H(t,s) = (t-s)^2$, then we see that all hypotheses of

Theorem 3 are satisfied where

$$\begin{aligned}
&\lim_{t\to\infty}\sup\frac{1}{H(t,t_o)}\int_{t_o}^{t}\left\{H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4M}Q^2(t,s)\right\}ds\\
=&\lim_{t\to\infty}\sup\frac{1}{(t-1)^2}\int_{1}^{t}\left[\frac{(t-s)^2}{2s}-\frac{s}{4}\frac{4s^2+2\cos^2(\ln s)}{s^2+2\cos^2(\ln s)}\right.\\
&\qquad\left.\times\left(2+\frac{(t-s)\sin^2(\ln s)}{2s}\right)^2\right]ds\\
\geq&\lim_{t\to\infty}\sup\frac{1}{(t-1)^2}\int_{1}^{t}\left[\frac{(t-s)^2}{2s}-\frac{s}{4}\left[2+\frac{(t-s)}{2s}\right]^2\right]ds\\
=&\lim_{t\to\infty}\sup\frac{1}{(t-1)^2}\left(\frac{11}{8}t-\frac{45}{32}t^2+\frac{7}{16}t^2\ln t+\frac{1}{32}\right)=\infty
\end{aligned}$$

Hence, this equation is oscillatory by theorem 3. One such solution of this equation is $x(t)=\sin(\ln t)$.

Theorem 4. Suppose, in addition to condition (1), (2) and (20), that there exist a positive function $\rho\in C^1[t_o,\infty)$. Moreover, assume that the function H and h be as in Theorem 3, and let

$$0<\inf_{s\,\geq\,t_o}\left[\lim_{t\to\infty}\inf\frac{H(t,s)}{H(t,t_o)}\right]\leq\infty. \tag{25}$$

Suppose that there exists a function $\Omega\in C([t_o,\infty),\mathbf{R})$ such that

$$\lim_{t\to\infty}\sup\frac{1}{H(t,t_o)}\int_{t_o}^{t}\rho(s)r(s)Q^2(t,s)ds<\infty, \tag{26}$$

$$\lim_{t\to\infty}\sup\frac{1}{H(t,T)}\int_{T}^{t}\{H(t,s)\rho(s)q(s)-\frac{\rho(s)r(s)Q^2(t,s)}{4M}\}ds\geq\Omega(T), \tag{27}$$

for every $T\geq t_o$. Then equation (E) is oscillatory if

$$\int_{t_o}^{\infty}\frac{\Omega_+^2(s)}{\rho(s)r(s)}=\infty, \tag{28}$$

where $\Omega_+(t)=\max\{\Omega(t),0\}$ for $t\geq t_o$.

Proof. On the contrary we assume that (E) has a nonoscillatory solution $x(t)$. We suppose without loss of generality that $x(t)>0$ for all $t\in[t_o,\infty)$. Defining $\omega(t)$ as in the proof of Theorem 3, we obtain (22) and hence for $t>T\geq t_o$, we get

$$\begin{aligned}
&\lim_{t\to\infty}\sup\frac{1}{H(t,T)}\int_{T}^{t}\left[H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4M}Q^2(t,s)\right]ds\\
&\leq\omega(T)-\lim_{t\to\infty}\inf\frac{1}{H(t,T)}\int_{T}^{t}\left[\frac{\sqrt{M}\sqrt{H(t,s)}}{\sqrt{r(s)\rho(s)}}\omega(s)+\frac{\sqrt{r(s)\rho(s)}}{2\sqrt{M}}Q(t,s)\right]^2ds.
\end{aligned}$$

Thus, by condition (27) we have for $T \geq t_o$

$$\omega(T) \geq \Omega(T) + \lim_{t\to\infty} \inf \frac{1}{H(t,T)} \int_T^t \left[\frac{\sqrt{M}\sqrt{H(t,s)}}{\sqrt{r(s)\rho(s)}} \omega(s) + \frac{\sqrt{r(s)\rho(s)}}{2\sqrt{M}} Q(t,s) \right]^2 ds.$$

This shows that

$$\omega(T) \geq \Omega(T), \tag{29}$$

and

$$\lim_{t\to\infty} \inf \frac{1}{H(t,T)} \int_T^t \left[\frac{\sqrt{M}\sqrt{H(t,s)}}{\sqrt{r(s)\rho(s)}} \omega(s) + \frac{\sqrt{r(s)\rho(s)}}{2\sqrt{M}} Q(t,s) \right]^2 ds < \infty.$$

Hence

$$\infty > \lim_{t\to\infty} \inf \frac{1}{H(t,t_o)} \int_{t_o}^t \left[\frac{\sqrt{M}\sqrt{H(t,s)}}{\sqrt{r(s)\rho(s)}} \omega(s) + \frac{\sqrt{r(s)\rho(s)}}{2\sqrt{M}} Q(t,s) \right]^2 ds$$

$$\geq \lim_{t\to\infty} \inf \frac{1}{H(t,t_o)} \int_{t_o}^t \left[\frac{M\ H(t,s)}{r(s)\rho(s)} \omega^2(s) + Q(t,s)\sqrt{H(t,s)}\ \omega(s) \right] ds. \tag{30}$$

Define the function $\alpha(t)$ and $\beta(t)$ as follows

$$\alpha(t) = \frac{1}{H(t,t_o)} \int_{t_o}^t \frac{M\ H(t,s)}{r(s)\rho(s)} \omega^2(s) ds,$$

$$\beta(t) = \frac{1}{H(t,t_o)} \int_{t_o}^t Q(t,s)\sqrt{H(t,s)}\ \omega(s) ds.$$

Then (30) may be written as

$$\lim_{t\to\infty} \inf \left[\alpha(t) + \beta(t)\right] < \infty. \tag{31}$$

In order to show that

$$\int_{t_o}^{\infty} \frac{\omega^2(s)}{\rho(s)r(s)} < \infty. \tag{32}$$

Now, suppose that

$$\int_{t_o}^{\infty} \frac{\omega^2(s)}{\rho(s)r(s)} = \infty. \tag{33}$$

By (25) we can easily see that

$$\lim_{t\to\infty} \alpha(t) = \infty. \tag{34}$$

Next, let us consider a sequence $\{T_n\}_{n=1,2,3,\ldots.}$ in (t_o, ∞) with $\lim_{n\to\infty} T_n = \infty$ and such that

$$\lim_{n\to\infty} \left[\alpha(T_n) + \beta(T_n)\right] = \lim_{t\to\infty} \inf \left[\alpha(t) + \beta(t)\right].$$

Now, by (31), there exists a constant N such that

$$\alpha(T_n) + \beta(T_n) \le N \qquad (n = 1, 2, ...). \tag{35}$$

Furthermore (34) guarantees that

$$\lim_{n\to\infty} \alpha(T_n) = \infty, \tag{36}$$

and hence (35) gives

$$\lim_{n\to\infty} \beta(T_n) = -\infty. \tag{37}$$

By taking into account (36), from (35), we derive

$$1 + \frac{\beta(T_n)}{\alpha(T_n)} \le \frac{N}{\alpha(T_n)} < \frac{1}{2},$$

provided that n is sufficiently large. Thus,

$$\frac{\beta(T_n)}{\alpha(T_n)} < \frac{-1}{2} \quad \text{for all large } n,$$

which by (37), ensures that

$$\lim_{n\to\infty} \frac{\beta^2(T_n)}{\alpha(T_n)} = \infty. \tag{38}$$

On the other hand, by Schawrz inequality, we have for any positive integer n,

$$\beta^2(T_n) = \frac{1}{H^2(T_n, t_o)} \left[\int_{t_o}^{T_n} Q(T_n, s)\sqrt{H(T_n, s)}\ \omega(s)ds\right]^2$$

$$\le \left[\frac{1}{H(T_n, t_o)} \int_{t_o}^{T_n} \frac{M\ H(T_n, s)}{r(s)\rho(s)}\omega^2(s)ds\right]$$
$$\times \left[\frac{1}{H(T_n, t_o)} \int_{t_o}^{T_n} \frac{r(s)\rho(s)}{M} Q^2(T_n, s)ds\right]$$

$$= \alpha(T_n)\left[\frac{1}{H(T_n, t_o)} \int_{t_o}^{T_n} \frac{r(s)\rho(s)}{M} Q^2(T_n, s)ds\right],$$

or

$$\frac{\beta^2(T_n)}{\alpha(T_n)} \le \frac{1}{H(T_n, t_o)} \int_{t_o}^{T_n} \frac{r(s)\rho(s)}{M} Q^2(T_n, s)ds.$$

It follow from (38) that

$$\lim_{n\to\infty} \frac{1}{H(T_n, t_o)} \int_{t_o}^{T_n} r(s)\rho(s)Q^2(T_n, s)ds = \infty. \tag{39}$$

Consequently,

$$\lim_{t\to\infty}\sup \frac{1}{H(t,t_o)}\int_{t_o}^{t}\rho(s)r(s)Q^2(t,s)ds=\infty,$$

but the latter contradicts the assumption (26) . Hence, (33) fails to hold. Consequently, we have proved that inequality (32) holds. Finally, by (29) we obtain

$$\int_{t_o}^{\infty}\frac{\Omega_+^2(s)}{\rho(s)r(s)}\le\int_{t_o}^{\infty}\frac{\omega^2(s)}{\rho(s)r(s)}<\infty,$$

which contradicts the assumption (28). Therefore, equation (E) is oscillatory.
Example 2. Consider the differential equation

$$\left(\frac{1+\sin^2 t}{t^2}\frac{1}{1+x^2}\dot{x}\right)^{\cdot}+\frac{2}{t^3}\dot{x} \\ +\frac{1+\sin^2 t}{t^2(3+2\sin^2 t)}x\left[2+\frac{1}{1+x^2}\right]=0,\quad t\ge t_o=1. \tag{40}$$

We note that

$$\frac{g^{'}(x)}{\psi(x)}=3+\frac{2x^4}{(1+x^2)}\ge 3=K.$$

If we take $\rho(t)=1$ and $H(t,s)=(t-s)^2$, then we have

$$\begin{aligned}
&\lim_{t\to\infty}\sup\frac{1}{H(t,t_o)}\int_1^t r(s)\rho(s)Q^2(t,s)ds\\
=\ &\lim_{t\to\infty}\sup\frac{1}{(t-1)^2}\int_1^t\frac{\left(1+\sin^2 s\right)}{s^2}\left(2+\frac{2t}{s^3}-\frac{2}{s^2}\right)^2 ds\\
\le\ &\lim_{t\to\infty}\sup\frac{1}{(t-1)^2}\int_1^t\frac{2}{s^2}\left(2+\frac{2t}{s^3}-\frac{2}{s^2}\right)^2 ds=\frac{8}{7}<\infty,
\end{aligned}$$

$$\begin{aligned}
&\lim_{t\to\infty}\sup\frac{1}{H(t,T)}\int_T^t\left(H(t,s)\rho(s)q(s)-\frac{r(s)\rho(s)}{4M}Q^2(t,s)\right)ds\\
&=\lim_{t\to\infty}\sup\frac{1}{(t-T)^2}\int_T^t\frac{(t-s)^2\left(1+\sin^2 s\right)}{s^2(3+2\sin^2 s)}-\frac{\left(1+\sin^2 s\right)}{12s^2}\left(2+\frac{2t}{s^3}-\frac{2}{s^2}\right)^2 ds\\
\ge\ &\lim_{t\to\infty}\sup\frac{1}{(t-T)^2}\int_T^t\left((t-s)^2(\frac{1}{3s^2})-\frac{1}{6s^2}\left(2+\frac{2t}{s^3}-\frac{2}{s^2}\right)^2\right)ds\\
=\ &\frac{1}{21}\frac{7T^6-2}{T^7}\overset{def}{=}\Omega(T),
\end{aligned}$$

and, finally,

$$\begin{aligned}\int_1^\infty \frac{\Omega_+^2(s)}{\rho(s)r(s)}ds &= \int_1^\infty \left(\frac{1}{21}\frac{7s^6-2}{s^7}\right)^2\left(\frac{s^2}{1+\sin^2 s}\right)ds \\ &\geq \int_1^t \left(\frac{1}{21}\frac{7s^6-2}{s^7}\right)^2\left(\frac{s^2}{2}\right)ds = \infty.\end{aligned}$$

Thus we conclude that (40) is oscillatory by Theorem 4. As a matter of fact, $x(t) = \sin t$ is an oscillatory solution of this equation.

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