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On the existence of a component-wise positive radially symmetric solution for a superlinear system

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Abstract

The system under consideration is

$$-\Delta u + a_u u = u^3 - \beta u v^2, \quad u = u(x),$$

$$-\Delta v + a_v v = v^3 - \beta u^2 v, \quad v = v(x),\ x \in \mathbb{R}^3,$$

$$u\big|_{|x|\to\infty} = v\big|_{|x|\to\infty} = 0,$$

where a_u, a_v and β are positive constants. We prove the existence of a component-wise positive smooth radially symmetric solution of this system. This result is a part of the results presented in the recent paper [1]; in our opinion, our method allows one to treat the problem simpler and shorter.

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1 Introduction. Result

We look for solutions $(u, v) = (u(x), v(x)) \in C^2(\mathbb{R}^3) \times C^2(\mathbb{R}^3)$ of the argument $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ of the problem

$$-\Delta u + a_u u = u^3 - \beta u v^2, \tag{1}$$

$$-\Delta v + a_v v = v^3 - \beta u^2 v, \tag{2}$$

$$u\big|_{|x|\to\infty} = v\big|_{|x|\to\infty} = 0, \tag{3}$$

where all the quantities are real, a_u, a_v and β are positive constants and $\Delta = \frac{\partial^2}{\partial x_1} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator. In the following, the existence of a solution of (1)-(3), radially symmetric and component-wise positive in $\mathbb{R}^3$, is proved. System (1)-(3) is a model problem which naturally arises when one considers standing waves for a coupled system of nonlinear Schrödinger equations and which has various applications in different areas of physics, for instance, in the heat and diffusion theory, in the theory of nonlinear waves, for example, in plasma or in water, etc. The author's interest to this problem was mainly stimulated by the quite recent article [1] on one hand and by his publication [2] on the other hand. In fact, with the present note we improve the results in [2], where it is assumed that $\beta \in (0, 1]$, and obtain a result similar to one of those in [1] by another method in a simpler and shorter way; in fact, we proceed as in [2]. Readers may find a longer list of references on the subject in [1] and, also, in [3]. Here, our main result is the following.

Theorem *Let a_u, a_v and β be positive constants. Then, problem (1)-(3) has a C^2-solution radially symmetric and component-wise positive in $\mathbb{R}^3$.*

Remark 1 Of course, if $\beta \in (0, 1)$ and $a_u = a_v$, then the problem has a solution (u, v) satisfying $u \equiv v$ (see, for example, [3]). However, it seems to be surprising that the solution in the theorem above exists if $\beta \geq 1$. A similar statement was already presented in [1].

Now, we introduce some notation. Let $H^1 = H^1(0, \infty)$ be the standard Sobolev space of functions defined in $(0, \infty)$ and equal to 0 at the point 0, with the norm $\|w\| = \int_0^\infty [w^2(r) + (w'(r))^2]dr$ and let $\|w\|_u^2 = \int_0^\infty [a_u w^2(r) + (w'(r))^2]dr$ and $\|w\|_v^2 = \int_0^\infty [a_v w^2(r) + (w'(r))^2]dr$ be the equivalent norms in this space. Denote $X = H^1 \times H^1$ and, for $(y, z) \in X$,

$$s = s(y, z) = \beta \int_0^\infty \frac{y^2(r) z^2(r)}{r^2} dr, \quad p = p(y) = \int_0^\infty \frac{y^4(r)}{r^2} dr \quad \text{and} \quad q = q(z) = \int_0^\infty \frac{z^4(r)}{r^2} dr.$$

2 Proof of the theorem

In the class of radially symmetric solutions, system (1)-(3) reduces to the following:

$$u'' + \frac{2}{r}u' = a_u u - u^3 + \beta u v^2, \quad r = |x| > 0,\ u = u(r), \tag{4}$$

$$v'' + \frac{2}{r}v' = a_v v - v^3 + \beta u^2 v, \quad v = v(r), \tag{5}$$

$$u'(0) = v'(0) = u(+\infty) = v(+\infty) = 0, \tag{6}$$

where the prime denotes the differentiation in r. By the substitution $y(r) = ru(r),\ z(r) = rv(r)$ we reduce problem (4)-(6) to the following:

$$y'' = a_u y + \beta y \frac{z^2}{r^2} - \frac{y^3}{r^2}, \quad y = y(r), \tag{7}$$

$$z'' = a_v z + \beta \frac{y^2}{r^2} z - \frac{z^3}{r^2}, \quad z = z(r),\ r > 0, \tag{8}$$

$$y(0) = z(0) = y(+\infty) = z(+\infty) = 0. \tag{9}$$

System (7)-(9) is variational, and X-extremals of the functional

$$H = H(y,z) = \frac{1}{2}\|y\|_u^2 + \frac{1}{2}\|z\|_v^2 + \frac{1}{2}s(y,z) - \frac{1}{4}p(y) - \frac{1}{4}q(z)$$

are formally its solutions. In view of estimate (15) and the proof of lemma 2 (see below), it is well defined on X. In the following, we exploit a variant of the method of S.I. Pokhozhaev described, for example, in [3]. Let $S = \{(y,z) \in X :\ \|y\|_u^2 = 1 \text{ and } \|z\|_v^2 = 1\}$. Consider an arbitrary $(y_0, z_0) \in S$, $a, b > 0$ and a point $(y,z) = (ay_0, bz_0)$. For this point (y,z) to be an X-extremal of H, it is necessary that $\frac{\partial H(y,z)}{\partial a} = \frac{\partial H(y,z)}{\partial b} = 0$. This easily yields the following two conditions for X-extremals of H:

$$\|y\|_u^2 + s(y,z) = p(y), \tag{10}$$

$$\|v\|_v^2 + s(y,z) = q(z). \tag{11}$$

Lemma 1 *Let $(y_0, z_0) \in S$. Then, a point $(y,z) = (ay_0, bz_0)$, where $a, b > 0$, satisfying (10) and (11) exists if and only if $p_0 q_0 - s_0^2 > 0$ where $p_0 = p(y_0)$, $q_0 = q(z_0)$ and $s_0 = s(y_0, z_0)$.*

Proof Substitute $(y,z)=(ay_0,bz_0)$ in (10) and (11). Then, we obtain the system

$$1+s_0b^2=p_0a^2, \quad 1+s_0a^2=q_0b^2 \tag{12}$$

with the unknown quantities a and b. It is easily seen that (12) has a real solution (a,b), where $a,b>0$, if and only if $p_0q_0-s_0^2>0$ and this solution is given by

$$a^2=\frac{q_0+s_0}{p_0q_0-s_0^2} \quad \text{and} \quad b^2=\frac{p_0+s_0}{p_0q_0-s_0^2}. \tag{13}$$

Lemma 1 is proved.□

Consider the set

$$S_0=\{(y_0,z_0)\in S: \ p(y_0)\cdot q(z_0)-s^2(y_0,z_0)>0\}$$

and denote $T=\{(ay_0,bz_0): \ a,b>0$ are given by (13) and $(y_0,z_0)\in S_0\}$. By (10) and (11)

$$H=\frac{1}{4}[\|u\|_u^2+\|v\|_v^2]>0 \text{ on } T \tag{14}$$

so that the functional H is bounded from below on T.

Lemma 2 *The functionals p,q and s are weakly continuous in X.*

Proof Let a sequence $\{(y_n,z_n)\}_{n=1,2,3,...}$ be weakly converging in X. Then, it is bounded in X and, consequently, in $C(0,\infty)\times C(0,\infty)$. We have the estimate

$$|y_n(r_1)-y_n(r_2)|=\left|\int_{r_1}^{r_2} y_n'(r)dr\right|\le\int_{r_1}^{r_2}|y_n'(r)|dr\le|r_1-r_2|^{1/2}\cdot\|y_n\| \tag{15}$$

(and by analogy for z) which shows that for any $\epsilon>0$ there exists $\delta>0$ such that

$$\int_0^\delta \frac{y_n^4(r)}{r^2}dr<\epsilon, \ \beta\int_0^\delta\frac{y_n^2(r)z_n^2(r)}{r^2}dr<\epsilon \text{ and } \int_0^\delta\frac{z_n^4(r)}{r^2}dr<\epsilon.$$

In addition,

$$\int_R^\infty\frac{y_n^4(r)}{r^2}dr\le C\int_R^\infty\frac{dr}{r^2}=CR^{-1}\to+0 \text{ as } R\to+\infty$$

and by analogy for s and q. These estimates complete our proof of lemma 2.□

Remark 2 Using estimates of the type of estimate (15), one can easily prove that the functional $H = H(y,z)$ in X is continuously differentiable in each of its arguments when the other one is fixed.

Let $\{(y_n, z_n)\}_{n=1,2,3,...}$ be an arbitrary minimizing sequence for the functional H on T. By (14), it is bounded in X and therefore, weakly compact. Without the loss of generality we accept that it is weakly converging in X to a point $(\overline{y}, \overline{z})$ and that there exist limits of $\|y_n\|_u$ and $\|z_n\|_v$ as $n \to \infty$.

Lemma 3 *There exists $c > 0$ such that $\|y_n\|_u \geq c$ and $\|z_n\|_v \geq c$ for all n.*

Proof Consider an arbitrary $(y_0, z_0) \in S_0$, the corresponding $(y,z) = (ay_0, bz_0) \in T$, where $a, b > 0$, and the condition

$$\|y_1\|_u^2 = \int_0^\infty \frac{y_1^4(r)}{r^2} dr,$$

where $y_1 = a_1 y$, $a_1 > 0$. By (15),

$$\int_0^\infty \frac{y_1^4(r)}{r^2} dr \leq C_1 \|y_1\|_u^4$$

for a constant $C_1 > 0$. This immediately implies that there exists $c_1 > 0$ such that $a_1 \geq c_1$ for any $(y_0, z_0) \in S_0$. Observe now that $a \geq a_1$. To estimate b, one can proceed by analogy.$\square$

By lemma 2, (10) and (11), $\overline{y} \neq 0$ and $\overline{z} \neq 0$ in H^1.

Lemma 4 *The sequence $\{(y_n, z_n)\}$ converges to $(\overline{y}, \overline{z})$ strongly in X.*

Proof By lemma 3, (10) and (11), $p(y_n) \geq c$ and $q(z_n) \geq c$ where $c > 0$ is the constant from lemma 3. By (14), $\|y_n\|_u \leq c_1$ and $\|z_n\|_v \leq c_1$ for a constant $c_1 > 0$ independent of n. Hence, $p\left(\frac{y_n}{\|y_n\|_u}\right) \geq c_2$ and $q\left(\frac{z_n}{\|z_n\|_v}\right) \geq c_2$ where $c_2 > 0$ does not depend on n. Therefore, in formulas (13), written for $\frac{y_n}{\|y_n\|_u}$ and $\frac{z_n}{\|z_n\|_v}$, the denominator can be estimated as follows:

$$p\left(\frac{y_n}{\|y_n\|_u}\right) \cdot q\left(\frac{z_n}{\|z_n\|_v}\right) - s^2\left(\frac{y_n}{\|y_n\|_u}, \frac{z_n}{\|z_n\|_v}\right) \geq c_0 \tag{16}$$

for a constant $c_0 > 0$ independent of n. But

$$p\left(\frac{y_n}{\|y_n\|_u}\right)\cdot q\left(\frac{z_n}{\|z_n\|_v}\right)-s^2\left(\frac{y_n}{\|y_n\|_u},\frac{z_n}{\|z_n\|_v}\right)=\frac{1}{\|y_n\|_u^4\cdot\|z_n\|_v^4}[p(y_n)\cdot q(z_n)-s^2(y_n,z_n)]$$

and thus, by lemma 2 and (16),

$$p\left(\frac{\overline{y}}{\|\overline{y}\|_u}\right)\cdot q\left(\frac{\overline{z}}{\|\overline{z}\|_v}\right)-s^2\left(\frac{\overline{y}}{\|\overline{y}\|_u},\frac{\overline{z}}{\|\overline{z}\|_v}\right)=\frac{1}{\|\overline{y}\|_u^4\cdot\|\overline{z}\|_v^4}[p(\overline{y})\cdot q(\overline{z})-s^2(\overline{y},\overline{z})]>0. \quad (17)$$

Therefore, according to lemma 1, there exist $\overline{a},\overline{b}>0$ such that $\left(\overline{a}\frac{\overline{y}}{\|\overline{y}\|_u},\overline{b}\frac{\overline{z}}{\|\overline{z}\|_v}\right)\in T$.

Suppose that the statement of our lemma is wrong. Then, simple calculations similar to those performed to obtain (13) show that $\frac{\overline{a}}{\|\overline{y}\|_u}\leq 1$ and $\frac{\overline{b}}{\|\overline{z}\|_v}\leq 1$ and, in addition, at least one of these two inequalities is strict. Therefore,

$$H\left(\frac{\overline{a}}{\|\overline{y}\|_u}\overline{y},\frac{\overline{b}}{\|\overline{z}\|_v}\overline{z}\right)<\frac{1}{4}[\|\overline{y}\|_u^2+\|\overline{z}\|_v^2]<\liminf_{n\to\infty}H(y_n,z_n)=\inf_{(y,z)\in T}H(y,z),$$

which is a contradiction. So, lemma 4 is proved.□

By lemma 4, $(\overline{y},\overline{z})\in T$ is a point of minimum of the functional H on the set T. According to lemmas 1 and 4 and (17), in S, there exists a neighborhood of the point $\left(\frac{\overline{y}}{\|\overline{y}\|_u},\frac{\overline{z}}{\|\overline{z}\|_v}\right)$ belonging to S_0. Therefore, since the pair (a,b) is a smooth function of $(y,z)\in S_0$, according to the Pokhozhaev theorem (see, for example, theorem II.2.2 in [3]) $\overline{y}$ is a critical point of the functional $H(y,\overline{z})$ taken with the fixed second argument and $\overline{z}$ is a critical point of the functional $H(\overline{y},z)$ taken with the fixed first argument. Therefore, by standard arguments, the pair $(\overline{y},\overline{z})$ belongs to $(C^2(0,\infty)\times C^2(0,\infty))\cap(C([0,\infty))\times C([0,\infty)))$, and it is a solution of problem (7)-(9).

Lemma 5 *One has* $\overline{y}(r)\neq 0$ *and* $\overline{z}(r)\neq 0$ *in* $(0,\infty)$.

Proof First, observe that it cannot be that $\overline{y}(r_0)=\overline{y}'(r_0)=0$ at some $r_0\in(0,\infty)$ because otherwise $\overline{y}(r)\equiv 0$ by the uniqueness theorem (and by analogy for $\overline{z}$). Further, suppose that the function $\overline{y}$ changes sign at some $r_0\in(0,\infty)$. Observe that the pair $(|\overline{y}|,|\overline{z}|)$ is still a point of minimum of H on the set T, hence, a smooth solution of problem (7)-(9). But by our supposition, the function $|\overline{y}|'$ is discontinuous at r_0. This contradiction completes our proof (for the function $\overline{z}$ one can proceed by the complete analogy).$\square$

Now, we can accept that $\overline{y}(r) > 0$ and $\overline{z}(r) > 0$ for all $r > 0$. As is well known, there exist $C > 0$ and $\kappa > 0$ such that $|\overline{y}(r)| + |\overline{z}(r)| \leq Ce^{-\kappa r}$ for all $r \geq 1$. Then, it is a component-wise positive smooth solution of problem (7)-(9). According to (15) and equations (7) and (8), one has: $|\overline{y}''(r)| + |\overline{z}''(r)| \leq Cr^{-1/2}$ for all $r \in (0,1]$. Therefore, there exist limits $\overline{y}'(+0) = \overline{y}'(1) - \int_0^1 \overline{y}''(r)dr$ and $\overline{z}'(+0) = \overline{z}'(1) - \int_0^1 \overline{z}''(r)dr$. Let $(u(r), v(r)) = \frac{1}{r}(\overline{y}(r), \overline{z}(r))$. Then, we have $(u(+0), v(+0)) = (\overline{y}'(+0), \overline{z}'(+0))$ so that the functions u and v are continuous and bounded in $(0,\infty)$ and $u(+\infty) = v(+\infty) = 0$. Now, it can be proved similarly to theorem II.2.1 in [3] that the pair $(u(|x|), v(|x|))$ is a smooth solution of system (1)-(3). So, our theorem is proved.

References

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