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Attraction property of local center-unstable manifolds for differential equations with state-dependent delay

Dedicated to Professor Hans-Otto Walther on the occasion of his 65th birthday

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Abstract. In the present paper we consider local center-unstable manifolds at a stationary point for a class of functional differential equations of the form $\dot{x}(t) = f(x_t)$ under assumptions that are designed for application to differential equations with state-dependent delay. Here, we show an attraction property of these manifolds. More precisely, we prove that, after fixing some local center-unstable manifold W_{cu} of $\dot{x}(t) = f(x_t)$ at some stationary point φ, each solution of $\dot{x}(t) = f(x_t)$ which exists and remains sufficiently close to φ for all $t \geq 0$ and which does not belong to W_{cu} converges exponentially for $t \to \infty$ to a solution on W_{cu}.

Keywords: attraction, center-unstable manifold, functional differential equation, state-dependent delay.

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1 Introduction

In the last decade the theory of differential equations with state-dependent delay made significant progress. Apart from other results, the framework developed by Walther in [7, 8, 9] had a remarkable impact. This series of works is concerned with a class of abstract functional differential equations and contains a proof that under certain mild conditions the solutions of the associated Cauchy problems define a continuous semiflow on a smooth submanifold of a function space. In particular, the resulting semiflow has continuously differentiable solution operators and the linearization of the semiflow along a solution is described by linear variational equations. The vital point of that framework with respect to delay differential equations is the fact that it seems to be typically applicable in cases where the functional differential equation represents an autonomous differential equation with state-dependent delay. Consequently, under the assumption of applicability, one obtains a general setting of smooth dynamical systems for the study of differential equations with state-dependent delay.

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Nowadays, the semiflow mentioned above is analyzed in various articles and many of its dynamical aspects are well-understood. For instance, a general survey of basic properties together with the linearization process at stationary points as well as the principle of linearized stability is presented in [1]. In addition, [1] contains a proof of the existence of local stable and local center manifolds at stationary points. The counterpart of the principle of linearized stability, that is, the principle of linearized instability is discussed in [4]. For the existence of continuously differentiable local unstable manifolds at stationary points we refer the reader to [2]. The construction of C^1-smooth local center-unstable manifolds is carried out in [5], whereas the authors of [3] show the existence and smoothness of local center-stable manifolds.

In the present article we address another feature from the dynamical systems theory of semiflows as laid out in the framework [7, 8, 9]; namely, an attraction property of local center-unstable manifolds obtained in [5]. We show that each solution which starts and stays close enough to a stationary point converges exponentially for $t \to \infty$ to a solution on a local center-unstable manifold of the semiflow. In particular, this property provides an asymptotic description of the dynamics of such solutions: for all sufficiently large t they behave like solutions on the considered local center-unstable manifold. However, in order to formulate our main result in detail we have to recall some relevant material. This is done below without presenting proofs. For a deeper discussion of the theory and for the absent proofs we refer the reader to [1, 7, 8, 9].

Throughout this paper, let $h > 0$, $n \in \mathbb{N}$ and let $\|\cdot\|_{\mathbb{R}^n}$ denote a norm in $\mathbb{R}^n$. Further, we write C for the Banach space of all continuous functions from the interval $[-h, 0]$ into $\mathbb{R}^n$, provided with the usual norm $\|\varphi\|_C := \sup_{s\in[-h,0]} \|\varphi(s)\|_{\mathbb{R}^n}$ of uniform convergence. Similarly, let C^1 denote the Banach space of all continuously differentiable functions $\varphi\colon [-h, 0] \to \mathbb{R}^n$ with the norm $\|\varphi\|_{C^1} := \|\varphi\|_C + \|\varphi'\|_C$. Given some function $x\colon I \to \mathbb{R}^n$ defined on some interval $I \subset \mathbb{R}$, and some real $t \in \mathbb{R}$ with $[t-h, t] \subset I$, the *segment* x_t of x at t is defined by $x_t(s) := x(t+s)$, $-h \le s \le 0$.

From now on, we consider the functional differential equation

$$\dot{x}(t) = f(x_t) \tag{1.1}$$

given by some function $f\colon U \to \mathbb{R}^n$ defined on some open neighborhood $U \subset C^1$ of the origin $0 \in C^1$ and satisfying $f(0) = 0$. A *solution* of Eq. (1.1) is either a continuously differentiable function $x\colon [t_0 - h, t_e) \to \mathbb{R}^n$ with $t_0 < t_e \le \infty$ such that $x_t \in U$ for all $t_0 \le t < t_e$ and Eq. (1.1) holds for all $t_0 < t < t_e$, or a continuously differentiable function $x\colon \mathbb{R} \to \mathbb{R}^n$ satisfying $x_t \in U$ and Eq. (1.1) for all $t \in \mathbb{R}$, or a continuously differentiable function $x\colon (-\infty, t_r] \to \mathbb{R}^n$, $t_r \in \mathbb{R}$, such that $x_t \in U$ for all $t \le t_r$ and Eq. (1.1) holds as $t < t_r$.

As $f(0) = 0$ by assumption, it is clear that $x(t) = 0$, $t \in \mathbb{R}$, is a solution of Eq. (1.1) in the sense above. In particular, the subset

$$X_f := \{\varphi \in U \mid \varphi'(0) = f(\varphi)\}$$

of C^1 is not empty. We impose that the function f additionally satisfies the following conditions:

(S1) f is continuously differentiable, and

(S2) for each $\varphi \in U$ the derivative $Df(\varphi)\colon C^1 \to \mathbb{R}^n$ extends to a linear map $D_e f(\varphi)\colon C \to \mathbb{R}^n$ such that the map $U \times C \ni (\varphi, \chi) \mapsto D_e f(\varphi)\chi \in \mathbb{R}^n$ is continuous.

Then the results of the framework [7, 8, 9] show that the subset X_f of U is a C^1-smooth submanifold of codimension n. Moreover, for each $\varphi \in X_f$ there is a unique real $t_+(\varphi) > 0$ and a unique solution $x^\varphi\colon [-h, t_+(\varphi)) \to \mathbb{R}^n$ of Eq. (1.1) such that $x_0^\varphi = \varphi$ and x^φ is not continuable in the forward time direction. For all $\varphi \in X_f$ and all $0 \le t < t_+(\varphi)$ the segments x_t^φ belong to X_f, which is therefore called the *solution manifold* of Eq. (1.1). By assigning

$$F(t,\varphi) := x_t^\varphi$$

for all $(t,\varphi) \in \Omega$ where

$$\Omega := \{(s,\psi) \in [0,\infty) \times X_f \mid 0 \le s < t_+(\psi)\},$$

we obtain a continuous semiflow $F\colon \Omega \to X_f$ with continuously differentiable time-t-maps.

Since $x(t) = 0$, $t \in \mathbb{R}$, is a solution of Eq. (1.1), it is clear that $\varphi_0 := 0 \in U$ is a stationary point of the semiflow F such that $F(t,0) = 0$ for all $t \ge 0$. The linearization of F at $\varphi_0 = 0$ is the strongly continuous semigroup $T = \{T(t)\}_{t\ge 0}$ of bounded linear operators $T(t) := D_2F(t,0)$ on the Banach space

$$T_0X_f := \{\chi \in C^1 \mid \chi'(0) = Df(0)\chi\}$$

with the norm $\|\cdot\|_{C^1}$ of C^1. The action of an operator $T(t)$, $t \ge 0$, on $\chi \in T_0X_f$ is given by $T(t)\chi = v_t^\chi$, where $v^\chi\colon [-h,\infty) \to \mathbb{R}^n$ is the uniquely determined solution of the variational equation

$$\dot v(t) = Df(0)v_t$$

with initial value $v_0 = \chi$. The infinitesimal generator G of the strongly continuous semigroup T is given by the linear operator

$$G\colon \mathcal{D}(G) \ni \chi \mapsto \chi' \in T_0X_f$$

defined on the subset

$$\mathcal{D}(G) := \{\chi \in C^2 \mid \chi'(0) = Df(0)\chi,\ \chi''(0) = Df(0)\chi'\}$$

of the space C^2 of all twice continuously differentiable functions from $[-h,0]$ into $\mathbb{R}^n$.

The semigroup T is closely related to another strongly continuous semigroup. In order to clarify this point, recall that, due to assumption (S2) on f, the operator $Df(0)$ may be extended to a bounded linear operator $D_ef(0)\colon C \to \mathbb{R}^n$ on C. In particular, the operator $L_e := Df_e(0)$ defines the linear retarded functional differential equation

$$v'(t) = L_e v_t.$$

The solutions of the associated initial value problems

$$\begin{cases} v'(t) = L_e v_t \\ \quad v_0 = \chi \in C \end{cases} \tag{1.2}$$

induce a strongly continuous semigroup $T_e = \{T_e(t)\}_{t\ge 0}$ on C. The generator of T_e is defined by

$$G_e\colon \mathcal{D}(G_e) \ni \chi \mapsto \chi' \in C$$

on the domain

$$\mathcal{D}(G_e) := \{\chi \in C^1 \mid \chi'(0) = L_e\chi\}.$$

We have $\mathcal{D}(G_e) = T_0X_f$ and $T(t)\varphi = T_e(t)\varphi$ for all $t \geq 0$ and all $\varphi \in \mathcal{D}(G_e)$.

The relation between the semigroups T, T_e notably has an effect on the spectra $\sigma(G), \sigma(G_e)$ of the two generators G, G_e: they coincide as shown in [1]. The spectrum $\sigma(G_e) \subset \mathbb{C}$ of the generator G_e of T_e is given by the zeros of a familiar characteristic equation. In particular, it is discrete and contains only eigenvalues of finite rank, that is, all generalized eigenspaces are finite-dimensional. Moreover, for each $\beta \in \mathbb{R}$ the intersection $\sigma(G_e) \cap \{\lambda \in \mathbb{C} \mid \operatorname{Re}\lambda > \beta\}$ is finite. Therefore, the spectral parts

$$\sigma_c(G_e) := \{\lambda \in \sigma(G_e) \mid \operatorname{Re}\lambda = 0\}$$

and

$$\sigma_u(G_e) := \{\lambda \in \sigma(G_e) \mid \operatorname{Re}\lambda > 0\}$$

of $\sigma(G_e)$ are empty or finite. The associated realified generalized eigenspaces C_c and C_u are called the *center* and *unstable space* of G_e, respectively, and each of them is a finite dimensional subspace of C. In contrast, the *stable space* C_s of G_e, that is, the realified generalized eigenspace associated with the spectral part

$$\sigma_s(G_e) := \{\lambda \in \sigma(G_e) \mid \operatorname{Re}\lambda < 0\},$$

is an infinite-dimensional subspace of C. All the spaces C_u, C_c, and C_s are closed and invariant under $T_e(t)$, $t \geq 0$, and provide the decomposition

$$C = C_u \oplus C_c \oplus C_s \tag{1.3}$$

of the Banach space C. The semigroup T_e may be extended to a one-parameter group on each of the two finite dimensional spaces C_u, C_c, and the decomposition of C also leads to a decomposition of the smaller Banach space C^1:

$$C^1 = C_u \oplus C_c \oplus C^1_s \tag{1.4}$$

with the closed subspace $C^1_s := C_s \cap C^1$ of C^1. With respect to the semigroup T and its generator G, it turns out that both C_u and C_s belong to $\mathcal{D}(G_e) = T_0X_f$ and coincide with the unstable and center space of G, respectively. The stable space of G is given by the intersection $C_s \cap T_0X_f$ and we get the decomposition

$$T_0X_f = C_u \oplus C_c \oplus (C_s \cap T_0X_f)$$

of the Banach space T_0X_f. All the spaces C_u, C_c, and $C_s \cap T_0X_f$ are closed in T_0X_f and invariant under the action of the semigroup T. In addition, T is extendable to a one-parameter group on both C_u and C_c.

After the preparatory steps, we are now in the position to recall the main result from [5] about the existence of local center-unstable manifolds for the semiflow F at the stationary point $\varphi_0 = 0$. In doing so, we write C_{cu} for the so-called *center-unstable space* $C_c \oplus C_u$ of G.

Theorem 1.1 (Theorems 1 & 2 in [5]). *Given $f\colon U \to \mathbb{R}^n$, $U \subset C^1$ open, with $f(0) = 0$ and satisfying assumptions (S1) and (S2), suppose that $\{\lambda \in \sigma(G_e) \mid \operatorname{Re}\lambda \geq 0\} \neq \emptyset$ or, equivalently, $C_{cu} \neq \{0\}$.*

Then there exist open neighborhoods $C_{cu,0}$ of 0 in C_{cu} and $C^1_{s,0}$ of 0 in C^1_s with $N_{cu} := C_{cu,0} + C^1_{s,0}$ contained in U, and a continuously differentiable map $w_{cu}\colon C_{cu,0} \to C^1_{s,0}$ with $w_{cu}(0) = 0$ and $Dw_{cu}(0) = 0$ such that

$$W_{cu} := \{\varphi + w_{cu}(\varphi) \mid \varphi \in C_{cu,0}\}$$

has the following properties.

(i) W_{cu} *is a continuously differentiable submanifold of the solution manifold* X_f *of Eq.* (1.1) *and* $\dim W_{cu} = \dim C_{cu}$.

(ii) *If* $x\colon (-\infty, 0] \to \mathbb{R}^n$ *is a solution Eq.* (1.1) *and if* $x_t \in N_{cu}$ *for all* $t \leq 0$, *then* $x_t \in W_{cu}$ *as* $t \leq 0$.

(iii) W_{cu} *is positively invariant with respect to* F *relative to* N_{cu}*; that is, for all* $\varphi \in W_{cu}$ *and all* $t > 0$ *with* $\{F(s,\varphi) \mid 0 \leq s \leq t\} \subset N_{cu}$ *we have* $\{F(s,\varphi) \mid 0 \leq s \leq t\} \subset W_{cu}$.

The goal of this paper is to prove the following additional attraction property of local center-unstable manifolds.

Theorem 1.2. *Under the assumptions of Theorem 1.1, there exists an open neighborhood* U_A *of* 0 *in* U*, and reals* $K_A > 0$ *and* $\eta_A > 0$ *with the following property: If* $\varphi \in U_A$ *and if the solution* x^φ *of Eq.* (1.1) *does exist for all* $t \geq 0$ *and its segments* x_t^φ *belongs to* U_A *as long as* $t \geq 0$*, then there is some* $\psi \in X_f$ *with* $x_t^\psi \in W_{cu}$ *for all* $t \geq 0$ *such that*

$$\|x_t^\varphi - x_t^\psi\|_{C^1} = \|F(t,\varphi) - F(t,\psi)\|_{C^1} \leq K_A e^{-\eta_A t},$$

as $t \geq 0$.

In the next sections, we establish this statement. The main idea of the proof is to consider the global center-unstable manifolds of some smooth modifications of Eq. (1.1) and to show an attraction property for these manifolds – compare Theorem 4.1 – first. This is done constructively by adopting the ideas contained in Vanderbauwhede [6], where a similar result for ordinary differential equations is given. An essential ingredient of the method is to deduce certain integral equations and then to solve these equations by the contraction principle on suitable Banach spaces. Having the attraction property of the global center-unstable manifolds, the main result easily follows by a cut-off technique.

This paper is organized in detail as follows. The next section contains some preliminaries. There, we recall the variation-of-constants formula and some integral operators for inhomogeneous linear functional differential equations. Further, we introduce some smooth modifications of Eq. (1.1) and describe the construction of global center-unstable manifolds.

The third section is devoted to the study of some global semiflows of the modified equations. Apart from the modifications introduced in the second section, in this section we consider further auxiliary modifications of (1.1).

Section 4 begins with a statement about an attraction property of global center-unstable manifolds. Thereafter, we develop step by step a strategy for a proof of this statement. It turns out that the claimed attraction property may be characterized in an alternative way, which notably involves global solutions of certain parameter-dependent integral equations.

In Section 5 we prepare the last arguments for a proof of the attraction of global center-unstable manifolds: we construct parameter-dependent contractions on Banach spaces to solve the parameter-dependent integral equations obtained in Section 4. In addition, we show that the resulting fixed points depend continuously on the parameter. At the end of Section 5, we finally give a proof of the statement claimed at the beginning of Section 4.

The last section contains the proof of our main result.

2 Preliminaries

In this section we recapitulate some standard facts on delay differential equations and discuss some basics results needed for a proof of the main statement.

2.1 Sun-reflexifity

For each $t \geq 0$, let $T_e^*(t)$ denote the adjoint operator of the bounded linear operator $T_e(t)$ induced by the solutions of the initial value problems (1.2). The family $T_e^* = \{T_e^*(t)\}_{t\geq 0}$ forms a semigroup of bounded linear operators on the dual space C^* of C. But in general T_e^* does not constitute a strongly continuous semigroup on C^* with respect to the topology induced by the norm $\|\varphi^*\|_{C^*} := \sup_{\|\varphi\|_C \leq 1} |\varphi^*(\varphi)|$. However, let $C^\odot$ denote the set of all $\varphi^\odot \in C^*$ with the property that the curve $[0,\infty) \ni t \mapsto T_e^*(t)\varphi^\odot \in C^*$ is continuous. Then $C^\odot$ is a closed subspace of C^* and for all $t \geq 0$ we have $T_e^*(t)(C^\odot) \subset C^\odot$. As a consequence, the family $T_e^\odot = \{T_e^\odot(t)\}_{t\geq 0}$ of operators $T_e^\odot(t)\colon C^\odot \ni \varphi^\odot \mapsto T_e^*(t)\varphi^\odot \in C^\odot$ becomes a strongly continuous semigroup on the Banach space $C^\odot$.

Similarly, carrying out the process above with the semigroup $T_e^\odot$ on $C^\odot$ instead of T_e on C, we first obtain the family $T_e^{\odot *} = \{T_e^{\odot *}(t)\}_{t\geq 0}$ of adjoint operators of $T_e^\odot$ on the dual space $C^{\odot *}$ of $C^\odot$ and then the Banach space $C^{\odot\odot} \subset C^{\odot *}$, on which the restriction of $T^{\odot *}$ is strongly continuous. The original Banach space C and semigroup T_e are sun-reflexive: There is an isometric linear map $j\colon C \to C^{\odot *}$ such that $j(C) = C^{\odot\odot}$ and $T_e^{\odot *}(t)(j\varphi) = j(T_e(t)\varphi)$ for all $t \geq 0$ and all $\varphi \in C$. For simplicity, we identify C with $C^{\odot\odot}$ and omit the embedding operator j in the following.

For the spectrum $\sigma(G_e^{\odot *})$ of the infinitesimal generator $G_e^{\odot *}$ of the semigroup $T_e^{\odot *}$ we have $\sigma(G_e^{\odot *}) = \sigma(G_e)$. By analogy to the decomposition (1.3) of C, $C^{\odot *}$ can be decomposed as

$$C^{\odot *} = C_u \oplus C_c \oplus C_s^{\odot *} \tag{2.1}$$

and the subspaces C_u, C_u, and $C_s^{\odot *}$ are closed and invariant under $T_e^{\odot *}$. We have continuous projection operators $P_u^{\odot *}$, $P_c^{\odot *}$, and $P_s^{\odot *}$ of $C^{\odot *}$ onto C_u, C_c, and $C_s^{\odot *}$, respectively. Further, there exist real constants $K \geq 1$, $c_s < 0 < c_u$, and $0 < c_c < \min\{-c_s, c_u\}$ such that

$$\begin{aligned} &\|T_e(t)\varphi\|_C \leq Ke^{c_u t}\|\varphi\|_C, && t \leq 0,\ \varphi \in C_u, \\ &\|T_e(t)\varphi\|_C \leq Ke^{c_c |t|}\|\varphi\|_C, && t \in \mathbb{R},\ \varphi \in C_c, \\ &\|T_e^{\odot *}(t)\varphi\|_{C^{\odot *}} \leq Ke^{c_s t}\|\varphi^{\odot *}\|_{C^{\odot *}}, && t \geq 0,\ \varphi^{\odot *} \in C_s^{\odot *}. \end{aligned} \tag{2.2}$$

From the decomposition (1.4) of C^1, we also get continuous projection operators P_u, P_c, and P_s of C^1 onto subspaces C_u, C_c, and C_s^1, respectively. By the identification of C and $C^{\odot\odot}$ it easily follows that $C_s^1 = C^1 \cap C_s^{\odot *}$. Finally, in analogy to (2.2), for the action of T on subspaces of $T_0 X_f$ we also have

$$\begin{aligned} &\|T(t)\varphi\|_{C^1} \leq Ke^{c_u t}\|\varphi\|_{C^1}, && t \leq 0,\ \varphi \in C_u, \\ &\|T(t)\varphi\|_{C^1} \leq Ke^{c_c |t|}\|\varphi\|_{C^1}, && t \in \mathbb{R},\ \varphi \in C_c, \\ &\|T(t)\varphi\|_{C^1} \leq Ke^{c_s t}\|\varphi\|_{C^1}, && t \geq 0,\ \varphi \in C_s \cap \mathcal{D}(G_e). \end{aligned} \tag{2.3}$$

2.2 Variation-of-constants formula

Next, we are going to recall the variation-of-constants formula for solutions of the inhomogeneous linear differential equation

$$\dot{x}(t) = L_e x_t + q(t) \tag{2.4}$$

with a function $q\colon I \to \mathbb{R}^n$ defined on some interval $I \subset \mathbb{R}$. Here, a solution is a continuous function $x\colon I + [-h, 0] \to \mathbb{R}^n$ satisfying (2.4) for all $t \in I$. In order to state the variation-of-constants formula and its properties, we need some preparations and notation. To begin with, let $L^\infty([-h,0],\mathbb{R}^n)$ denote the Banach space of all measurable and essentially bounded functions from $[-h,0]$ into $\mathbb{R}^n$, equipped with the usual norm $\|\cdot\|_{L^\infty}$ of essential least upper bound. Then the product space $\mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$ equipped with the norm

$$\|(\alpha,\varphi)\|_{\mathbb{R}^n\times L^\infty} := \max\{\|\alpha\|_{\mathbb{R}^n}, \|\varphi\|_{L^\infty}\}$$

is isometrically isomorphic to the Banach space $C^{\odot *}$. After fixing a norm-preserving isomorphism $k\colon C^{\odot *} \to \mathbb{R}^n \times L^\infty([-h,0],\mathbb{R}^n)$, let for each $i = 1,\dots,n$ the element $r_i^{\odot *} \in C^{\odot *}$ be defined by $r_i^{\odot *} := k^{-1}(e_i, 0)$, where e_i is the i-th canonical basis vector of $\mathbb{R}^n$. The family $\{r_1^{\odot *},\dots,r_n^{\odot *}\}$ clearly forms a basis of the subspace $Y^{\odot *} := k^{-1}(\mathbb{R}^n \times \{0\})$ of $C^{\odot *}$ and by claiming $l(e_i) = r_i^{\odot *}$ we find a unique linear bijective mapping $l\colon \mathbb{R}^n \to Y^{\odot *}$ with $\|l\| = \|l^{-1}\| = 1$.

Given reals $a \le b \le c$ and a continuous function $w\colon [a,b] \to C^{\odot *}$, define the *weak-star-integral*

$$\int_a^b T_e^{\odot *}(c-\tau)w(\tau)\,d\tau \in C^{\odot *}$$

by

$$\left(\int_a^b T^{\odot *}(c-\tau)w(\tau)\,d\tau\right)(\varphi^\odot) := \int_a^b \left(T^{\odot *}(c-\tau)w(\tau)\right)(\varphi^\odot)\,d\tau$$

for all $\varphi^\odot \in C^\odot$. In addition, set

$$\int_b^a T^{\odot *}(c-\tau)w(\tau)\,d\tau := -\int_a^b T^{\odot *}(c-\tau)w(\tau)\,d\tau.$$

Then it follows that the weak-star-integral lies in C. We have

$$T_e^{\odot *}(t)\int_a^b T_e^{\odot *}(c-\tau)w(\tau)\,d\tau = \int_a^b T_e^{\odot *}(t+c-\tau)w(\tau)\,d\tau$$

as $t \ge 0$, and for any of the continuous projections $P_\lambda^{\odot *}$ with $\lambda \in \{s,c,u\}$ the identity

$$P_\lambda^{\odot *}\int_a^b T_e^{\odot *}(c-\tau)w(\tau)\,d\tau = \int_a^b T_e^{\odot *}(c-\tau)P_\lambda^{\odot *}w(\tau)\,d\tau$$

holds. In addition, as usual the norm of the weak-star-integral is bounded by the integral of the norm:

$$\left\|\int_a^b T_e^{\odot *}(c-\tau)w(\tau)\,d\tau\right\|_{C^{\odot *}} \le \int_a^b \left\|T_e^{\odot *}(c-\tau)w(\tau)\right\|_{C^{\odot *}}\,d\tau.$$

We return to Eq. (2.4). If $q\colon I \to \mathbb{R}^n$ is continuous and if $x\colon I + [-h,0] \to \mathbb{R}^n$ is a solution of Eq. (2.4) then the curve $u\colon I \ni \tau \mapsto x_\tau \in C$ satisfies the abstract integral equation

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau)Q(\tau)\,d\tau \tag{2.5}$$

with $Q\colon I \ni \tau \mapsto l(q(\tau)) \in Y^{\odot *}$ for all $s,t \in I$, $s \le t$. Conversely, if $Q\colon I \to Y^{\odot *}$ is continuous and if $u\colon I \to C$ is a solution of Eq. (2.5) then there exists a continuous $x\colon I + [-h,0] \to \mathbb{R}^n$ such that $x_t = u(t)$ for all $t \in I$ and that x is a solution of Eq. (2.4) on I for $q\colon I \ni \tau \mapsto l^{-1}(Q(\tau)) \in \mathbb{R}^n$. In this way we have a one-to-one correspondence between solutions of Eq. (2.4) and Eq. (2.5).

2.3 Preparatory results on inhomogeneous linear equations

Let X denote a Banach space and $\|\cdot\|_X$ its norm. Then for each $\eta \geq 0$ the linear spaces

$$C_\eta((-\infty,0],X) := \left\{ g \in C((-\infty,0],X) \mid \sup_{s\in(-\infty,0]} e^{\eta s}\|g(s)\|_X < \infty \right\}$$

and

$$C_{\eta,\mathbb{R}}(\mathbb{R},X) := \left\{ g \in C(\mathbb{R},X) \mid \sup_{s\in\mathbb{R}} e^{\eta s}\|g(s)\|_X < \infty \right\}$$

provided with the weighted supremum norms

$$\|g\|_{C_\eta} = \sup_{s\leq 0} e^{\eta s}\|g(s)\|_X \quad \text{and} \quad \|g\|_{C_{\eta,\mathbb{R}}} = \sup_{s\in\mathbb{R}} e^{\eta s}\|g(s)\|_X,$$

respectively, become Banach spaces as well. Some of these spaces we will consider repeatedly in the sequel. In order to simplify notation, we shall use the following abbreviations throughout the paper:

$$\begin{aligned} &C^0_\eta := C_\eta((-\infty,0],C), && C^1_\eta := C_\eta((-\infty,0],C^1), && Y_\eta := C_\eta((-\infty,0],Y^{\odot *}),\\ &C^0_{\eta,\mathbb{R}} := C_{\eta,\mathbb{R}}(\mathbb{R},C), && C^1_{\eta,\mathbb{R}} := C_{\eta,\mathbb{R}}(\mathbb{R},C^1), \quad \text{and} && Y_{\eta,\mathbb{R}} := C_{\eta,\mathbb{R}}(\mathbb{R},Y^{\odot *}). \end{aligned}$$

Moreover, we write $P_{cu} := P_c + P_u$ for the projection of C^1 along C^1_s and $P^{\odot *}_{cu} := P^{\odot *}_u + P^{\odot *}_c$ for the projection of $C^{\odot *}$ along $C^{\odot *}_s$.

Definition 2.1. Given $Q\colon (-\infty,0] \to Y^{\odot *}$ and $Q_{\mathbb{R}}\colon \mathbb{R} \to Y^{\odot *}$, set

$$(\mathcal{K}^{cu}Q)(t) := \int_0^t T^{\odot *}_e(t-\tau)P^{\odot *}_{cu}Q(\tau)\,d\tau + \int_{-\infty}^t T^{\odot *}_e(t-\tau)P^{\odot *}_s Q(\tau)\,d\tau \tag{2.6}$$

for all $t \leq 0$, and

$$(\mathcal{K}^1 Q_{\mathbb{R}})(t) := \int_\infty^t T^{\odot *}_e(t-\tau)P^{\odot *}_s Q_{\mathbb{R}}(\tau)\,d\tau \tag{2.7}$$

and

$$(\mathcal{K}^2 Q_{\mathbb{R}})(t) := \int_\infty^t T^{\odot *}_e(t-\tau)P^{\odot *}_{cu} Q_{\mathbb{R}}(\tau)\,d\tau \tag{2.8}$$

for all $t \in \mathbb{R}$.

Proposition 2.2 (compare Proposition 3.2 in [5])**.** *Let $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$ be given.*

(i) Eq. (2.6) defines a bounded linear map $\hat{\mathcal{K}}\colon Y_\eta \ni Q \to \mathcal{K}^{cu}Q \in C^0_\eta$ with

$$\|\hat{\mathcal{K}}\| \leq K\left(\frac{\|P^{\odot *}_c\|}{\eta - c_c} + \frac{\|P^{\odot *}_u\|}{c_u+\eta} - \frac{\|P^{\odot *}_s\|}{c_s+\eta} \right).$$

Moreover, $u := (\hat{\mathcal{K}}Q)$, $Q \in Y_\eta$, is a solution of the integral equation

$$u(t) = T_e(t-s)u(s) + \int_s^t T^{\odot *}_e(t-\tau)Q(\tau)\,d\tau \tag{2.9}$$

as $-\infty < s \leq t < 0$, and it is the only one in C^0_η with the property $P^{\odot *}_{cu}u(0) = 0$.

(ii) *Eq. (2.7) defines a bounded linear map* $\hat{\mathcal{K}}^1\colon Y_{\eta,\mathbb{R}} \ni Q \mapsto (\mathcal{K}^1 Q) \in C^0_{\eta,\mathbb{R}}$ *with*

$$\|\hat{\mathcal{K}}^1\| \leq \frac{-K\|P_s^{\odot *}\|}{c_s + \eta}.$$

Moreover, $(\hat{\mathcal{K}}^1 Q)$, $Q \in Y_{\eta,\mathbb{R}}$, *is a solution of the integral equation*

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau)P_s^{\odot *}Q(\tau)\,d\tau \tag{2.10}$$

as $-\infty < s \leq t < \infty$, *and* $(\hat{\mathcal{K}}^1 Q)(t) \in C_s$ *for all* $t \in \mathbb{R}$.

(iii) *Eq. (2.8) defines a bounded linear map* $\hat{\mathcal{K}}^2\colon Y_{\eta,\mathbb{R}} \ni Q \mapsto (\mathcal{K}^2 Q) \in C^0_{\eta,\mathbb{R}}$ *with*

$$\|\hat{\mathcal{K}}^2\| \leq K\left(\frac{\|P_c^{\odot *}\|}{\eta - c_c} + \frac{\|P_u^{\odot *}\|}{c_u + \eta}\right).$$

Moreover, $(\hat{\mathcal{K}}^2 Q)$, $Q \in Y_{\eta,\mathbb{R}}$, *is a solution of the integral*

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}Q(\tau)\,d\tau \tag{2.11}$$

as $-\infty < s \leq t < \infty$, *and* $(\hat{\mathcal{K}}^2 Q)(t) \in C_{cu}$ *for all* $t \in \mathbb{R}$.

Proof. For the first part of the statement we refer the reader to Proposition 3.2 in [5] where the proof is carried out in full detail. Following these lines, one easily concludes parts (ii) and (iii) of the statement. □

Remark 2.3. Under the assumption on η from the last proposition, it is not difficult to see that $(\mathcal{K}^1 Q)$ is actually well-defined for all $Q \in C(\mathbb{R}, Y^{\odot *})$ satisfying $Q\,|_{(-\infty,0]} \in Y_\eta$. Furthermore, for such $Q \in C(\mathbb{R}, Y^{\odot *})$ the image $u := (\mathcal{K}^1 Q)$ is a continuous map from $\mathbb{R}$ into C with $u\,|_{(-\infty,0]} \in C^0_\eta$, solves Eq. (2.10) for all $-\infty < s \leq t < \infty$, and satisfies $u(t) \in C_s$ as $t \in \mathbb{R}$.

All the functions $\hat{\mathcal{K}}Q$, $\hat{\mathcal{K}}^1 Q$ and $\hat{\mathcal{K}}^2 Q$ are not only continuous but continuously differentiable, as established in our next result.

Proposition 2.4 (compare Corollary 3.4 in [5])**.** *Consider* $\eta \in \mathbb{R}$ *as in the last result. Then the following holds.*

(i) *Eq. (2.6) induces a bounded linear map*

$$\mathcal{K}_\eta\colon Y_\eta \ni Q \mapsto \mathcal{K}^{cu} Q \in C^1_\eta$$

with

$$\|\mathcal{K}_\eta\| \leq K(1 + e^{\eta h}\|L_e\|)\left(\frac{\|P_c^{\odot *}\|}{\eta - c_c} + \frac{\|P_u^{\odot *}\|}{c_u + \eta} - \frac{\|P_s^{\odot *}\|}{c_s + \eta}\right) + e^{\eta h}.$$

Moreover, given $Q \in Y_\eta$, $u := \mathcal{K}_\eta Q$ *is the only solution of Eq. (2.9) in* C^1_η *with the property* $P_{cu}^{\odot *}u(0) = 0$.

(ii) *Eq.* (2.7) *induces a bounded linear mapping*

$$\mathcal{K}_\eta^1\colon Y_{\eta,\mathbb{R}} \ni Q \mapsto \mathcal{K}^1 Q \in C^1_{\eta,\mathbb{R}}$$

with

$$\|\mathcal{K}_\eta^1\| \leq -K(1+e^{\eta h}\|L_e\|)\frac{\|P_s^{\odot *}\|}{c_s+\eta} + e^{\eta h}\|P_s^{\odot *}\|.$$

Moreover, given $Q \in Y_\eta$*, all segments of the solution* $\mathcal{K}_\eta^1 Q$ *of Eq.* (2.10) *belong to* C_s^1.

(iii) *Eq.* (2.8) *induces a bounded linear mapping*

$$\mathcal{K}_\eta^2\colon Y_{\eta,\mathbb{R}} \ni Q \mapsto \mathcal{K}^2 Q \in C^1_{\eta,\mathbb{R}}$$

with

$$\|\mathcal{K}_\eta^2\| \leq K(1+e^{\eta h}\|L_e\|)\left(\frac{\|P_c^{\odot *}\|}{\eta - c_c} + \frac{\|P_u^{\odot *}\|}{c_u+\eta}\right) + e^{\eta h}\|P_{cu}^{\odot *}\|.$$

Moreover, given $Q \in Y_\eta$*, all segments of the solution* $\mathcal{K}_\eta^2 Q$ *of Eq.* (2.11) *belong to* C_{cu}.

Proof. For the proof of the first assertion compare Corollary 3.4 and its proof in [5], whereas the proofs of assertions (ii) and (iii) immediately follows from Proposition 2.2 in combination with Proposition 4.2.1 in Hartung et al. [1]. □

Remark 2.5. An important ingredient of the proof of the last statement is a smoothing property of the integral equation (2.5). For example, if $Q \in C(\mathbb{R}, Y^{\odot *})$ and if $u \in C(\mathbb{R}, C)$ satisfies the integral equation (2.5) for all $-\infty < s \leq t < \infty$, then $u \in C(\mathbb{R}, C^1)$. For a proof, compare for instance the proof of Proposition 4.2.1 in Hartung et al. [1].

Corollary 2.6. *For given* $\eta \in \mathbb{R}$ *with* $c_c < \eta < \min\{-c_s, c_u\}$*, let*

$$\mathcal{K}_\eta^+\colon Y_{\eta,\mathbb{R}} \to C^1_{\eta,\mathbb{R}}$$

denote the map $Q \mapsto (\mathcal{K}_\eta^1 + \mathcal{K}_\eta^2)(Q)$*, where the operators* $\mathcal{K}_\eta^1$ *and* $\mathcal{K}_\eta^2$ *are defined as in the last proposition. Then* $\mathcal{K}_\eta^+$ *forms a bounded linear operator with*

$$\|\mathcal{K}_\eta^+\| \leq K(1+e^{\eta h}\|L_e\|)\left(\frac{\|P_u^{\odot *}\|}{c_u+\eta} + \frac{\|P_c^{\odot *}\|}{\eta - c_c} - \frac{\|P_s^{\odot *}\|}{c_s+\eta}\right) + e^{\eta h}(\|P_s^{\odot *}\| + \|P_{cu}^{\odot *}\|),$$

and for each $Q \in Y_{\eta,\mathbb{R}}$ *the function* $u = (\mathcal{K}_\eta^1 + \mathcal{K}_\eta^2)(Q)$ *satisfies*

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau)Q(\tau)\,d\tau$$

for all $-\infty < s \leq t < \infty$.

Proof. In view of Proposition 2.4, it is clear that the sum $\mathcal{K}_\eta^+$ of the two bounded linear operators $\mathcal{K}_\eta^1$ and $\mathcal{K}_\eta^2$ from $Y_{\eta,\mathbb{R}}$ into $C^1_{\eta,\mathbb{R}}$ is a bounded linear operator from $Y_{\eta,\mathbb{R}}$ into $C^1_{\eta,\mathbb{R}}$ as well. Furthermore, from the estimates for $\|\mathcal{K}_\eta^1\|$ and for $\|\mathcal{K}_\eta^2\|$ we get

$$\begin{aligned}\|\mathcal{K}_\eta^+\| &\leq \|\mathcal{K}_\eta^1\| + \|\mathcal{K}_\eta^2\| \\ &\leq K(1+e^{\eta h}\|L_e\|)\left(\frac{\|P_u^{\odot *}\|}{c_u+\eta} + \frac{\|P_c^{\odot *}\|}{\eta - c_c} - \frac{\|P_s^{\odot *}\|}{c_s+\eta}\right) + e^{\eta h}(\|P_s^{\odot *}\| + \|P_{cu}^{\odot *}\|).\end{aligned}$$

For the remaining part of the assertion, consider $u = \mathcal{K}_\eta^+ Q$ for some fixed $Q \in Y_{\eta,\mathbb{R}}$. Using Proposition 2.4 again, it follows that

$$\begin{aligned}
u(t) &= (\mathcal{K}_\eta^+ Q)(t)\\
&= (\mathcal{K}_\eta^1 Q)(t) + (\mathcal{K}_\eta^2 Q)(t)\\
&= (\mathcal{K}^1 Q)(t) + (\mathcal{K}^2 Q)(t)\\
&= T_e(t-s)(\mathcal{K}^1 Q)(s) + \int_s^t T_e^{\odot *}(t-\tau)P_s^{\odot *}Q(\tau)\,d\tau + T_e(t-s)(\mathcal{K}^2 Q)(s)\\
&\quad + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}Q(\tau)\,d\tau\\
&= T_e(t-s)\left[(\mathcal{K}^1 Q)(s) + (\mathcal{K}^2 Q)(s)\right] + \int_s^t T_e^{\odot *}(t-\tau)\left[P_s^{\odot *}Q(\tau) + P_{cu}^{\odot *}Q(\tau)\right]\,d\tau\\
&= T_e(t-s)(\mathcal{K}_\eta^+ Q)(s) + \int_s^t T_e^{\odot *}(t-\tau)Q(\tau)\,d\tau\\
&= T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau)Q(\tau)\,d\tau
\end{aligned}$$

for all $-\infty < s \leq t < \infty$, which proves the corollary. $\square$

2.4 Smooth modifications of the nonlinearity and the global center-unstable manifolds of the modified equations

Below we recapitulate some essential ingredients of the proof of Theorem 1.1. In particular, we describe the construction of global center-unstable manifolds for smooth modifications of Eq. (1.1). For the details we refer the reader to [5]. Compare also the construction of local center manifolds contained in Hartung et al. [1].

Introducing the maps

$$L := Df(0) \quad \text{and} \quad r\colon U \ni \varphi \mapsto f(\varphi) - L\varphi \in \mathbb{R}^n,$$

we may rewrite Eq. (1.1) as

$$\dot{x}(t) = Lx_t + r(x_t)$$

where the right-hand side is separated into a linear and a nonlinear term. It is easily seen that r inherits conditions (S1) and (S2) from f. In particular, we have $r(0) = 0$ and $Dr(0) = 0$.

In view of $\dim C_{cu} < \infty$, there exists a norm $\|\cdot\|_{cu}$ on C_{cu} such that its restriction to $C_{cu} \setminus \{0\}$ is C^∞-smooth. Using this norm, define

$$\|\varphi\|_1 := \max\{\|P_{cu}\varphi\|_{cu}, \|P_s\varphi\|_{C^1}\}$$

for all $\varphi \in C^1$. In this way we obtain another norm $\|\cdot\|_1$ on C^1, which is equivalent to $\|\cdot\|_{C^1}$. Choose next a C^∞-smooth map $\rho\colon [0,\infty) \to \mathbb{R}$ satisfying $\rho(t) = 1$ as $0 \leq t \leq 1$, $0 < \rho(t) < 1$ as $1 < t < 2$, and $\rho(t) = 0$ for all $t \geq 2$, and set

$$\hat{r}(\varphi) = \begin{cases} r(\varphi), & \text{for } \varphi \in U,\\ 0, & \text{for } \varphi \notin U. \end{cases}$$

For each $\delta > 0$ we introduce by

$$r_\delta\colon C^1 \ni \varphi \mapsto \rho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right)\rho\left(\frac{\|\varphi_s\|_{C^1}}{\delta}\right)\hat{r}(\varphi)$$

a modification of r that is defined on all of C^1. Here, φ_{cu} denotes the component $P_{cu}\varphi$ of $\varphi \in C^1$, and analogously φ_s the component $P_s\varphi$ of $\varphi \in C^1$.

For all sufficiently small $\delta > 0$ the restriction of r_δ to a small neighborhood of $0 \in C^1$ is continuously differentiable, bounded, and has a bounded derivative. More precisely, there exists some $\delta_0 > 0$ with $\{\psi \in C^1 \mid \|\psi_s\|_1 < \delta_0\} \subset U$ such that for each $0 < \delta < \delta_0$ the restriction $r_\delta\,|_{\{\psi\in C^1 | \|\psi_s\|_1<\delta\}}$ is a bounded and continuously differentiable function with a bounded derivative. Furthermore, for all $\varphi \in \{\psi \in C^1 \mid \|\psi_s\|_1 < \delta\}$ we have

$$r_\delta|_{\{\psi\in C^1 | \|\psi_s\|_1<\delta\}}(\varphi) = \hat{r}(\varphi)\,\rho\left(\frac{\|\varphi_{cu}\|_{cu}}{\delta}\right).$$

Next, there is some $0 < \delta_1 < \delta_0$ and a monotone increasing $\lambda\colon [0, \delta_1] \to [0, 1]$ satisfying $\lambda(0) = 0$ and $\lim_{\delta\searrow 0} \lambda(\delta) = 0$ such that

$$\|r_\delta(\varphi)\|_{\mathbb{R}^n} \leq \delta\,\lambda(\delta) \tag{2.12}$$

and

$$\|r_\delta(\varphi) - r_\delta(\psi)\|_{\mathbb{R}^n} \leq \lambda(\delta)\|\varphi - \psi\|_{C^1} \tag{2.13}$$

for all $0 < \delta \leq \delta_1$ and all $\varphi, \psi \in C^1$.

The proof for the existence part of Theorem 1.1 begins with the construction of global center-unstable manifolds for the modified equations

$$\dot{x}(t) = Lx_t + r_\delta(x_t) \tag{2.14}$$

where $0 < \delta \leq \delta_1$. Recall that these equations are closely related with the integral equations

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau)l(r_\delta(u(\tau)))\,d\tau. \tag{2.15}$$

For instance, if $x\colon (-\infty, 0] \to \mathbb{R}^n$ is a continuously differentiable solution of Eq. (2.14), then we obtain a solution of Eq. (2.15) by $u\colon (\infty, 0] \ni t \mapsto x_t \in C^1$. On the other hand, if $u\colon (-\infty, 0] \to C^1$ satisfies (2.15), then $x\colon (-\infty, 0] \to \mathbb{R}^n$ given by $x(t) := u(t)(0)$ as $-\infty < t \leq 0$ is a C^1-smooth solution of Eq. (2.14).

Consider now some fixed $\eta \in \mathbb{R}$ satisfying

$$c_c < \eta < \min\{-c_s, c_u\}.$$

There clearly exists some $0 < \delta < \delta_1$ ensuring

$$\lambda(\delta) \cdot \|\mathcal{K}_\eta\| < \frac{1}{2}.$$

With this choice of δ, let us temporarily denote by $R\colon C((-\infty, 0], C^1) \to C((-\infty, 0], Y^{\odot *})$ the substitution operator of the map $C^1 \ni \varphi \mapsto l(r_\delta(\varphi)) \in Y^{\odot *}$, that is, $R(u)(t) = l(r_\delta(u(t)))$ for all $u \in C((-\infty, 0], C^1)$ and all $t \leq 0$. Then R maps C^1_η into Y_η and thus induces a mapping

$$R_{\delta\eta}\colon C^1_\eta \ni u \mapsto R(u) \in Y_\eta,$$

which particularly satisfies

$$\|R_{\delta\eta}(u)\|_{Y_\eta} \leq \delta\lambda(\delta) \quad \text{and} \quad \|R_{\delta\eta}(u) - R_{\delta\eta}(v)\|_{Y_\eta} \leq \lambda(\delta)\|u - v\|_{C^1_\eta} \tag{2.16}$$

for all $u, v \in C^1_\eta$.

Given some $\varphi \in C_{cu}$, the curve $(-\infty, 0] \ni t \mapsto T_e(t)\varphi \in C^1$ belongs to C^1_η. Therefore, we may define a map $S_\eta\colon C^1 \supset C_{cu} \to C^1_\eta$ by $(S_\eta\varphi)(t) := T_e(t)\varphi$ as $\varphi \in C_{cu}$ and $t \le 0$. It easily follows that S_η forms a bounded linear operator with

$$\|S_\eta\| \le K(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|). \tag{2.17}$$

Using the mappings $\mathcal{K}_\eta$ from Proposition 2.4, $R_{\delta\eta}$ and S_η, we introduce another map $\mathcal{G}_\eta\colon C^1_\eta \times C_{cu} \to C^1_\eta$ given by

$$\mathcal{G}_\eta(u, \varphi) := S_\eta\varphi + \mathcal{K}_\eta \circ R_{\delta\eta}(u).$$

Under the given assumptions, for each $\varphi \in C_{cu}$ the induced map $\mathcal{G}_\eta(\cdot, \varphi)\colon C^1_\eta \to C^1_\eta$ has an uniquely determined fixed point $u(\varphi)$ since it forms a contraction of a sufficiently large ball about the origin into itself. The associated solution operator

$$\tilde{u}_\eta\colon C_{cu} \ni \varphi \mapsto u(\varphi) \in C^1_\eta$$

is globally Lipschitz-continuous, and for each $\varphi \in C_{cu}$ the function $\tilde{u}_\eta(\varphi)$ is a solution of Eq. (2.15) on $(-\infty, 0]$ with vanishing C_{cu} component at $t = 0$. Thus, in view of the one-to-one correspondence of solutions of Eq. (2.14) and Eq. (2.15), we see that for each $\varphi \in C_{cu}$ there exists a continuously differentiable function $x\colon (-\infty, 0] \to \mathbb{R}^n$ with $x_t = \tilde{u}(\varphi)(t)$ as $t \le 0$ such that x solves Eq. (2.14) for all $t \le 0$. The *global center-unstable manifold* of Eq. (2.14) at the stationary point $0 \in C^1$ is now the set

$$W^\eta := \{\tilde{u}_\eta(\varphi)(0) \mid \varphi \in C_{cu}\}.$$

We have

$$W^\eta = \{\varphi + w^\eta(\varphi) \mid \varphi \in C_{cu}\}$$

with the map

$$w^\eta\colon C_{cu} \ni \varphi \mapsto P_s(\tilde{u}_\eta(\varphi)(0)) \in C^1_s \tag{2.18}$$

and for each solution $v \in C^1_\eta$ of Eq. (2.15) we have $v(t) \in W^\eta$ as $t \le 0$.

3 Global semiflows of modified equations

The first step towards a proof of our main result Theorem 1.2 will be a similar statement for the modified equations (2.14) and the associated global center-unstable manifolds. As this statement will assert an asymptotic behaviour for $t \to \infty$ of certain solutions of Eq. (2.14), we need some preparations containing, among other things, a discussion about the existence of continuously differentiable solutions for $t \ge 0$. This is done below.

To begin with, observe that Eq. (2.14) may be written as

$$\dot{x}(t) = f_\delta(x_t) \tag{3.1}$$

with the function $f_\delta\colon C^1 \ni \varphi \mapsto L\varphi + r_\delta(\varphi)$. By construction, the set

$$X_\delta := \{\varphi \in C^1 \mid \varphi'(0) = f_\delta(\varphi)\}$$

is clearly not empty since we have $f_\delta(0) = f(0) = 0$. Nevertheless, f_δ does not need to have the properties (S1) and (S2). For this reason, we can not use the results in Walther [7, 8, 9] in order to conclude the existence of solutions of the initial values problems

$$\dot{x}(t) = f_\delta(x_t)\,(= Lx_t + r_\delta(x_t)), \qquad x_0 = \varphi \in X_\delta, \tag{3.2}$$

for $t \geq 0$. However, this issue was already addressed by Qesmi and Walther in [3] where the authors prove that for all sufficiently small $\delta > 0$ the initial value problems have uniquely determined solutions. More precisely, the following holds:

Proposition 3.1. *Let $0 < \delta < \delta_1$ with $\lambda(\delta) < 1/5$ be given. Then for each $\varphi \in X_\delta$ there exists a unique continuously differentiable solution $x\colon [-h, \infty) \to \mathbb{R}^n$ of the initial value problem (3.2), and $x_t \in X_\delta$ for all $t \geq 0$.*

Moreover, the equations

$$F_\delta(t, \varphi) = x_t^\varphi, \qquad \varphi \in X_\delta,\ t \geq 0,$$

define a continuous semiflow $F_\delta\colon [0, \infty) \times X_\delta \to X_\delta$, and for each $s \geq 0$

$$\mathrm{Lip}_{0 \leq t \leq s} F_\delta(t, \cdot\,) < \infty.$$

Proof. Compare Corollary 6.2 and Proposition 6.3 in [3]. □

Now, recall once more the one-to-one correspondence between solutions of the differential equation defining the initial value problem (3.2) and solutions of the integral equation (2.15). Fixing some appropriate $\delta > 0$ and any $\varphi \in X_\delta$ and setting $u(t) := F_\delta(t, \varphi)$ for all $t \geq 0$, we first see

$$F_\delta(t, \varphi) = T_e(t - s)F_\delta(s, \varphi) + \int_s^t T_e^{\odot *}(t - \tau)l(r_\delta(F_\delta(\tau, \varphi)))\,d\tau \tag{3.3}$$

and then after application of P_{cu}

$$\begin{aligned}
P_{cu}F_\delta(t, \varphi) &= P_{cu}^{\odot *}F_\delta(t, \varphi) \\
&= P_{cu}^{\odot *}T_e(t - s)F_\delta(s, \varphi) + P_{cu}^{\odot *}\int_s^t T_e^{\odot *}(t - \tau)l(r_\delta(F_\delta(\tau, \varphi)))\,d\tau \\
&= T_e(t - s)P_{cu}^{\odot *}F_\delta(s, \varphi) + \int_s^t T_e^{\odot *}(t - \tau)P_{cu}^{\odot *}l(r_\delta(F_\delta(\tau, \varphi)))\,d\tau \\
&= T_e(t - s)P_{cu}F_\delta(s, \varphi) + \int_s^t T_e^{\odot *}(t - \tau)P_{cu}^{\odot *}l(r_\delta(F_\delta(\tau, \varphi)))\,d\tau,
\end{aligned}$$

that is,

$$P_{cu}F_\delta(t, \varphi) = T_e(t - s)P_{cu}F_\delta(s, \varphi) + \int_s^t T_e^{\odot *}(t - \tau)P_{cu}^{\odot *}l(r_\delta(F_\delta(\tau, \varphi)))\,d\tau, \tag{3.4}$$

for all $0 \leq s \leq t < \infty$.

Apart from the global semiflows F_δ generated by solutions of (3.2), we will need other auxiliary semiflows. In order to define these semiflows, we first have to study solutions of the modification

$$\dot{x}(t) = LP_{cu}x_t + (l^{-1} \circ P_{cu}^{\odot *} \circ l)(r_\delta(x_t + P_s\varphi)) \tag{3.5}$$

of Eq. (3.1) for $t \leq 0$ where $\varphi \in X_\delta$. We show that, for each $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$ and all sufficiently small $\delta > 0$, every $\varphi \in X_\delta$ uniquely determines a continuously differentiable solution $x\colon (-\infty, 0] \to \mathbb{R}^n$ of Eq. (3.5) with $x_0 = P_{cu}\varphi$ and $[(-\infty, 0] \ni t \mapsto x_t \in C^1] \in C^1_\eta$.

The proof of this statement is based on a fixed-point argument completely similar to the one used for the construction of the global center-unstable manifolds W^η. However, for the convenience of the reader, we carry out the details below.

For the remaining part of this section fix some $\eta \in \mathbb{R}$ with

$$c_c < \eta < \min\{-c_s, c_u\}$$

and choose $0 < \delta < \delta_1$ such that

$$\lambda(\delta)\|\mathcal{K}_\eta\| \leq \lambda(\delta)\|\mathcal{K}_\eta\|\,\|P_{cu}^{\odot *}\| < \frac{1}{2}. \tag{3.6}$$

We begin with a minor modification of Corollary 4.3 in [5].

Corollary 3.2. *Let $\overline{R}$ denote the operator which assigns to $u \in C((-\infty,0],C^1)$ the mapping*

$$(-\infty,0] \ni s \mapsto P_{cu}^{\odot *} l(r_\delta(u(s))) \in Y^{\odot *}$$

in $C((-\infty,0],Y^{\odot *})$.

Then $\overline{R}(C^1_\eta) \subset Y_\eta$, and the induced mapping $\overline{R}_{\delta\eta} \colon C^1_\eta \ni u \mapsto \overline{R}(u) \in Y_\eta$ satisfies

$$\|\overline{R}_{\delta\eta}(u)\|_{Y_\eta} \leq \|P_{cu}^{\odot *}\|\delta\lambda(\delta)$$

and

$$\|\overline{R}_{\delta\eta}(u) - \overline{R}_{\delta\eta}(v)\|_{Y_\eta} \leq \lambda(\delta)\|P_{cu}^{\odot *}\|\,\|u-v\|_{C^1_\eta}$$

for all $u, v \in C^1_\eta$.

Proof. Observe that for each $u \in C((-\infty,0],C^1)$ and all $s \leq 0$ we have

$$\overline{R}(u)(s) = P_{cu}^{\odot *} l(r_\delta(u(s))) = P_{cu}^{\odot *} R(u)(s)$$

where $R \colon C((-\infty,0],C^1) \to C((-\infty,0],Y^{\odot *})$ denotes the substitution operator of the map $C^1 \ni \varphi \mapsto l(r_\delta(\varphi)) \in Y^{\odot *}$. As $P_{cu}^{\odot *} \colon C^{\odot *} \to C_{cu}$ is continuous and $R(u) \in C((-\infty,0],Y^{\odot *})$, it becomes clear that $\overline{R}$ maps $C((-\infty,0],C^1)$ into $C((-\infty,0],Y^{\odot *})$.

Next, consider some $u \in C^1_\eta$. Using the first inequality of (2.16) we infer

$$\begin{aligned}
\sup_{t\leq 0} e^{\eta t}\|\overline{R}(u)(t)\|_{Y^{\odot *}} &= \sup_{t\leq 0} e^{\eta t}\|P_{cu}^{\odot *} R(u)(t)\|_{Y^{\odot *}} \\
&\leq \|P_{cu}^{\odot *}\| \cdot \sup_{t\leq 0} e^{\eta t}\|R(u)(t)\|_{Y^{\odot *}} \\
&\leq \|P_{cu}^{\odot *}\| \cdot \|R_{\delta\eta}(u)\|_{Y_\eta} \\
&\leq \|P_{cu}^{\odot *}\| \cdot \delta\lambda(\delta).
\end{aligned}$$

Hence, it follows that $\overline{R}(C^1_\eta) \subset Y_\eta$ and that $\overline{R}_{\delta\eta}$ is bounded by $\|P_{cu}^{\odot *}\|\,\delta\lambda(\delta)$.

Similarly, from the second inequality of (2.16) we get for all $u, v \in C^1_\eta$:

$$\begin{aligned}
\|\overline{R}_{\delta\eta}(u) - \overline{R}_{\delta\eta}(v)\|_{Y_\eta} &= \|\overline{R}(u) - \overline{R}(v)\|_{Y_\eta} \\
&= \sup_{t\leq 0} e^{\eta t}\|P_{cu}^{\odot *} R(u)(t) - P_{cu}^{\odot *} R(v)(t)\|_{Y^{\odot *}} \\
&\leq \|P_{cu}^{\odot *}\| \sup_{t\leq 0} e^{\eta t}\|R(u)(t) - R(v)(t)\|_{Y^{\odot *}} \\
&= \|P_{cu}^{\odot *}\|\,\|R_{\delta\eta}(u) - R_{\delta\eta}(v)\|_{Y_\eta} \\
&\leq \lambda(\delta)\|P_{cu}^{\odot *}\|\,\|u-v\|_{C^1_\eta}.
\end{aligned}$$

This proves the assertion. □

Next, we consider a slight modification of the bounded linear operator S_η used for the construction of W^η.

Corollary 3.3. *For each $\varphi \in C^1$ the curve $(-\infty, 0] \ni t \mapsto T_e(t)P_{cu}\varphi \in C^1$ belongs to C^1_η, and the mapping $\overline{S}_\eta \colon C^1 \to C^1_\eta$ defined by $(\overline{S}_\eta\varphi)(t) = T_e(t)P_{cu}\varphi$ for all $\varphi \in C^1$ and all $t \leq 0$ is a bounded linear operator with*

$$\|\overline{S}_\eta\| \leq K\,\|P_{cu}\|\,(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|).$$

Proof. First, by Corollary 4.5 in [5] it follows that for every $\varphi \in C^1$ the continuous curve $(-\infty, 0] \ni t \mapsto T_e(t)P_{cu}\varphi \in C^1$ is an element of the Banach space C^1_η. Moreover, we have $\overline{S}_\eta = S_\eta \circ P_{cu}$. Hence, as a composition of two bounded linear operators, $\overline{S}_\eta$ is a bounded linear operator as well, and the estimate (2.17) finally leads to

$$\|\overline{S}_\eta\| = \|S_\eta \circ P_{cu}\| \leq \|S_\eta\|\,\|P_{cu}\| \leq K\,\|P_{cu}\|(\|P_c^{\odot *}\| + \|P_u^{\odot *}\|).$$

□

For given $\varphi \in C^1$ consider the constant map

$$\mathcal{C}(\varphi) \colon (-\infty, 0] \to C^1$$

defined by

$$\mathcal{C}(\varphi)(t) := P_s\varphi. \tag{3.7}$$

Clearly, we have $\mathcal{C}(\varphi) \in C^1_\eta$. Using Corollary 3.4 in [5] and the last two corollaries, we obtain a well-defined map

$$\overline{\mathcal{G}}_\eta \colon C^1_\eta \times C^1 \to C^1_\eta$$

where

$$\overline{\mathcal{G}}_\eta(u, \varphi) := \overline{S}_\eta\varphi + \mathcal{K}_\eta \circ \overline{R}_{\delta\eta}(u + \mathcal{C}(\varphi))$$

with the bounded linear operator $\mathcal{K}_\eta \colon Y_\eta \to C^1_\eta$ from Proposition 2.4. Next, we show that for each fixed $\varphi \in C^1$ the induced map $\overline{\mathcal{G}}_\eta(\,\cdot\,, \varphi) \colon C^1_\eta \to C^1_\eta$ is a contraction such that the equation $u = \overline{\mathcal{G}}_\eta(u, \varphi)$ has exactly one solution in the Banach space C^1_η.

Proposition 3.4. *For any $\varphi \in C^1$ the mapping $\overline{\mathcal{G}}_\eta(\cdot, \varphi) \colon C^1_\eta \to C^1_\eta$ has exactly one fixed point $\overline{u} = \overline{u}(\varphi)$ and the associated solution operator*

$$\hat{u}_\eta \colon C^1 \ni \varphi \to \overline{u}(\varphi) \in C^1_\eta$$

of $\overline{u} = \overline{\mathcal{G}}_\eta(\overline{u}, \varphi)$ is Lipschitz continuous.

Proof. We mimic the proof of Proposition 4.6 in [5].

1. Let $\varphi \in C^1$ be given. Choose $\gamma = \gamma(\varphi) > 0$ such that $2\|\overline{S}_\eta\|\,\|\varphi\|_{C^1} < \gamma$. Then from Corollaries 3.2 and 3.3 we infer

$$\begin{aligned}
\|\overline{\mathcal{G}}_\eta(u, \varphi)\|_{C^1_\eta} &= \|\overline{S}_\eta\varphi + \mathcal{K}_\eta \circ \overline{R}_{\delta\eta}(u + \mathcal{C}(\varphi))\|_{C^1_\eta} \\
&\leq \|\overline{S}_\eta\varphi\|_{C^1_\eta} + \|(\mathcal{K}_\eta \circ \overline{R}_{\delta\eta})(u + P_s\varphi)\|_{C^1_\eta} \\
&\leq \|\overline{S}_\eta\|\|\varphi\|_{C^1} + \|\mathcal{K}_\eta\|\,\|\overline{R}_{\delta\eta}(u)\|_{Y_\eta} \\
&\leq \|\overline{S}_\eta\|\|\varphi\|_{C^1} + \lambda(\delta)\|\mathcal{K}_\eta\|\,\|P_{cu}^{\odot *}\|\,\|u\|_{C^1_\eta} \\
&\leq \frac{\gamma}{2} + \frac{\gamma}{2} \quad \text{(see (3.6))} \\
&= \gamma
\end{aligned}$$

for all $u \in C^1_\eta$ with $\|u\|_{C^1_\eta} \leq \gamma$. Consequently, the restriction of $\overline{\mathcal{G}}_\eta(\,\cdot\,,\varphi)$ to the closed ball $\{u \in C^1_\eta | \|u\|_{C^1_\eta} \leq \gamma\}$ in C^1_η is a self-map. Moreover, $\overline{\mathcal{G}}_\eta(\,\cdot\,,\varphi)$ is a contraction. Indeed, Corollary 3.2 in combination with condition (3.6) implies

$$\begin{aligned}
\|\overline{\mathcal{G}}_\eta(u,\varphi) - \overline{\mathcal{G}}_\eta(v,\varphi)\|_{C^1_\eta} &= \|\mathcal{K}_\eta \circ \overline{R}_{\delta\eta}(u + \mathcal{C}(\varphi)) - \mathcal{K}_\eta \circ \overline{R}_{\delta\eta}(v + \mathcal{C}(\varphi))\|_{C^1_\eta} \\
&\leq \|\mathcal{K}_\eta\| \, \|\overline{R}_{\delta\eta}(u + \mathcal{C}(\varphi)) - \overline{R}_{\delta\eta}(v + \mathcal{C}(\varphi))\|_{Y_\eta} \\
&\leq \lambda(\delta)\|\mathcal{K}_\eta\| \, \|P^{\odot *}_{cu}\| \, \|u - v\|_{C^1_\eta} \\
&\leq \frac{1}{2}\|u - v\|_{C^1_\eta}
\end{aligned}$$

for all $u, v \in C^1_\eta$. We conclude that there is a unique

$$\overline{u}(\varphi) \in \{u \in C^1_\eta \mid \|u\|_{C^1_\eta} \leq \gamma\} \subset C^\eta_1$$

satisfying $u = \overline{\mathcal{G}}_\eta(u,\varphi)$.

2. It remains to show the global Lipschitz continuity of the map $\hat{u}_\eta \colon C^1 \ni \varphi \mapsto \overline{u}(\varphi) \in C^1_\eta$. For this purpose, consider $\varphi, \psi \in C^1$. Using Corollaries 3.2 and 3.3 and the estimate (3.6) once more, we infer

$$\begin{aligned}
\|\hat{u}_\eta(\varphi) - \hat{u}_\eta(\psi)\|_{C^1_\eta} &= \|\overline{u}(\varphi) - \overline{u}(\psi)\|_{C^1_\eta} \\
&= \|\overline{\mathcal{G}}_\eta(\overline{u}(\varphi),\varphi) - \overline{\mathcal{G}}_\eta(\overline{u}(\psi),\psi)\|_{C^1_\eta} \\
&= \|\overline{S}_\eta(\varphi - \psi) + (\mathcal{K}_\eta \circ \overline{R}_{\delta\eta})(\overline{u}(\varphi)) - (\mathcal{K}_\eta \circ \overline{R}_{\delta\eta})(\overline{u}(\psi))\|_{C^1_\eta} \\
&\leq \|\overline{S}_\eta\| \, \|\varphi - \psi\|_{C^1} + \|\mathcal{K}_\eta\|\|\overline{R}_{\delta\eta}(\overline{u}(\varphi)) - \overline{R}_{\delta\eta}(\overline{u}(\psi))\|_{Y_\eta} \\
&\leq \|\overline{S}_\eta\| \, \|\varphi - \psi\|_{C^1} + \lambda(\delta)\|P^{\odot *}_{cu}\| \, \|\mathcal{K}_\eta\| \, \|\overline{u}(\varphi) - \overline{u}(\psi)\|_{C^1_\eta} \\
&\leq \|\overline{S}_\eta\| \, \|\varphi - \psi\|_{C^1} + \frac{1}{2}\|\overline{u}(\varphi) - \overline{u}(\psi)\|_{C^1_\eta} \\
&= \|\overline{S}_\eta\| \, \|\varphi - \psi\|_{C^1} + \frac{1}{2}\|\hat{u}_\eta(\varphi) - \hat{u}_\eta(\psi)\|_{C^1_\eta},
\end{aligned}$$

that is,

$$\|\hat{u}_\eta(\varphi) - \hat{u}_\eta(\psi)\|_{C^1_\eta} \leq 2\,\|\overline{S}_\eta\|\|\varphi - \psi\|_{C^1}.$$

Consequently, $\hat{u}$ has a global Lipschitz constant as claimed. □

For every $\varphi \in C^1$ the fixed point $\hat{u}_\eta(\varphi) \in C^1_\eta$ of $u = \overline{\mathcal{G}}_\eta(u,\varphi)$ from the last result is a special solution of the abstract integral equation associated with Eq. (3.5) by the variation-of-constants formula. More precisely, the following holds.

Corollary 3.5. *For all $\varphi \in C^1$ the mapping $\hat{u}_\eta(\varphi)$ from the last proposition is a solution of the abstract integral equation*

$$u(t) = T_e(t-s)P_{cu}u(s) + \int_s^t T_e^{\odot *}(t-\tau)P^{\odot *}_{cu} l(r_\delta(u(\tau) + P_s\,\varphi))d\tau \tag{3.8}$$

for $-\infty < s \leq t \leq 0$.

In particular, $\hat{u}_\eta(\varphi)(0) = P_{cu}\varphi$ and $\hat{u}_\eta(\varphi)(t) \in C_{cu}$ for all $t \leq 0$.

Proof. Following the proof of Corollary 4.7 in [5], consider for given $\varphi \in C^1$ the map

$$z := \hat{u}_\eta(\varphi) - \overline{S}_\eta \varphi = \overline{\mathcal{G}}_\eta(\hat{u}_\eta(\varphi), \varphi) \in C^1_\eta.$$

By Corollary 3.4 in [5], we conclude that

$$z(t) = T_e(t-s)z(s) + \int_s^t T_e^{\odot *}(t-\tau)\overline{R}_{\delta\eta}(\hat{u}_\eta(\varphi) + \mathcal{C}(\varphi))(\tau)d\tau$$

as $-\infty < s \leq t \leq 0$ and that $P_{cu}z(0) = P_{cu}^{\odot *}z(0) = 0$. Hence, it follows that

$$\begin{aligned}
\hat{u}_\eta(\varphi)(t) - T_e(t)P_{cu}\varphi &= \hat{u}_\eta(\varphi)(t) - (\overline{S}_\eta\varphi)(t) \\
&= z(t) \\
&= T_e(t-s)z(s) + \int_s^t T_e^{\odot *}(t-\tau)\overline{R}_{\delta\eta}(\hat{u}_\eta(\varphi) + \mathcal{C}(\varphi))(\tau)d\tau \\
&= T_e(t-s)\hat{u}_\eta(\varphi)(s) - T_e(t-s)(\overline{S}_\eta\varphi)(s) \\
&\quad + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\hat{u}_\eta(\varphi)(\tau) + \mathcal{C}(\varphi))\, d\tau \\
&= T_e(t-s)\hat{u}_\eta(\varphi)(s) - T_e(t-s)T_e(s)P_{cu}\varphi \\
&\quad + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\hat{u}_\eta(\varphi)(\tau) + \mathcal{C}(\varphi)))\, d\tau,
\end{aligned}$$

that is,

$$\hat{u}_\eta(\varphi)(t) = T_e(t-s)\hat{u}_\eta(\varphi)(s) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\hat{u}_\eta(\varphi)(\tau) + \mathcal{C}(\varphi))\, d\tau, \tag{3.9}$$

for all $-\infty < s \leq t \leq 0$. In addition, we get

$$P_{cu}(\hat{u}_\eta(\varphi)(0)) = P_{cu}z(0) + P_{cu}((\overline{S}_\eta\varphi)(0)) = 0 + P_{cu}T_e(0)\varphi = P_{cu}\varphi. \tag{3.10}$$

Next, recall that $P_{cu}^{\odot *}P_{cu}^{\odot *} = P_{cu}^{\odot *}$, $P_{cu}^{\odot *}P_s^{\odot *} = 0$, and that C_{cu} is invariant under the action of the semigroup $T_e^{\odot *}$. Combining this facts with the definition of $\mathcal{K}_\eta$ from Proposition 2.4, we get

$$\begin{aligned}
z(t) &= \overline{\mathcal{G}}_\eta(\hat{u}(\varphi), \varphi)(t) \\
&= [(\mathcal{K}_\eta \circ \overline{R}_{\delta\eta})(\hat{u}_\eta(\varphi) + \mathcal{C}(\varphi))](t) \\
&= \int_0^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}\overline{R}(\hat{u}_\eta(\varphi) + \mathcal{C}(\varphi))(\tau)\, d\tau + \int_{-\infty}^t T_e^{\odot *}(t-\tau)P_s^{\odot *}\overline{R}(\hat{u}_\eta(\varphi) + \mathcal{C}(\varphi))(\tau)\, d\tau \\
&= \int_0^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}P_{cu}^{\odot *}l(r_\delta(\hat{u}_\eta(\varphi)(\tau) + P_s\varphi))\, d\tau \\
&\quad + \int_{-\infty}^t T_e^{\odot *}(t-\tau)P_s^{\odot *}P_{cu}^{\odot *}l(r_\delta(\hat{u}_\eta(\varphi)(\tau) + P_s\varphi))\, d\tau \\
&= \int_0^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\hat{u}_\eta(\varphi)(\tau) + P_s\varphi))\, d\tau \\
&= P_{cu}^{\odot *}\int_0^t T_e^{\odot *}(t-\tau)l(r_\delta(\hat{u}_\eta(\varphi)(\tau) + \mathcal{C}(\varphi))\, d\tau \in C_{cu}
\end{aligned}$$

for all $t \leq 0$. Hence,

$$\hat{u}_\eta(\varphi)(t) = z(t) + (\overline{S}_\eta\varphi)(t) = z(t) + T_e(t)P_{cu}\varphi \in C_{cu}$$

as $t \leq 0$ and thus

$$\hat{u}_\eta(\varphi)(0) = P_{cu}\hat{u}_\eta(\varphi)(0) = P_{cu}\varphi$$

due to Eq. (3.10). Moreover, we see

$$T_e(t-s)\hat{u}_\eta(\varphi)(s) = T_e(t-s)P_{cu}\hat{u}_\eta(\varphi)(s)$$

for all $-\infty < s \leq t \leq 0$ such that Eq. (3.9) takes the form

$$\hat{u}_\eta(\varphi)(t) = T_e(t-s)P_{cu}\hat{u}_\eta(\varphi)(s) + \int_s^t T_e^{\odot *}(t-s)P_{cu}^{\odot *}l(r_\delta(\hat{u}_\eta(\varphi)(\tau) + \mathcal{C}(\varphi)))\,d\tau$$

as $-\infty < s \leq t \leq 0$. This shows the assertion. □

From the one-to-one correspondence between solutions of the abstract integral equation (3.8) and solutions of the differential equation (3.5) we conclude that for each $\varphi \in C^1$ there is a continuously differentiable function $x\colon (-\infty, 0] \to \mathbb{R}^n$ satisfying Eq. (3.5) for all $t \leq 0$ and having the properties $x_0 = P_{cu}\varphi$ and $[(-\infty, 0] \ni t \mapsto x_t \in C^1] \in C^1_\eta$. Moreover, all the segments of x are contained in the center-unstable space C_{cu}, and in view of the uniqueness result from Proposition 3.4 there is no other solution $y\colon (-\infty, 0] \to \mathbb{R}^n$ of Eq. (3.5) having the two properties $y_0 = P_{cu}\varphi$ and $[(-\infty, 0] \ni t \mapsto y_t \in C^1] \in C^1_\eta$ (compare also the details in the proof of the next proposition).

Using the solution operator $\hat{u}_\eta$, we define a map

$$F^{cu}_\eta\colon (-\infty, 0] \times C^1 \to C^1$$

by

$$F^{cu}_\eta(t, \varphi) := \hat{u}_\eta(\varphi)(t) + \mathcal{C}(\varphi)(t).$$

Below we prove that F^{cu}_η defines a continuous dynamical system on C^1.

Proposition 3.6. *The map $F^{cu}_\eta\colon (-\infty, 0] \times C^1 \to C^1$ forms a continuous semiflow.*
More precisely, F^{cu}_η is continuous and satisfies

$$F^{cu}_\eta(0, \varphi) = \varphi \quad \textit{and} \quad F^{cu}_\eta(s+t, \varphi) = F^{cu}_\eta(s, F^{cu}_\eta(t, \varphi))$$

for all $s, t \in (-\infty, 0]$ and all $\varphi \in C^1$.

Proof. 1. (*Proof of the continuity of F^{cu}_η.*) Let $t \in (-\infty, 0]$, $\varphi \in C^1$ and $\varepsilon > 0$ be given. As $\hat{u}_\eta(\varphi) \in C^1_\eta$ is continuous, there is some $\tilde{\delta}_1 > 0$ such that for all $s \in (-\infty, 0]$ with $|t-s| < \tilde{\delta}_1$

$$\|\hat{u}_\eta(\varphi)(t) - \hat{u}_\eta(\varphi)(s)\|_{C^1} < \frac{\varepsilon}{3}$$

holds. In addition, we find $\tilde{\delta}_2 > 0$ such that for each $\psi \in C^1$ with $\|\varphi - \psi\|_{C^1} < \tilde{\delta}_2$ we have

$$\|P_s\|\|\varphi - \psi\|_{C^1} < \frac{\varepsilon}{3} \quad \text{and} \quad e^{-\eta(t-\tilde{\delta}_1)}\operatorname{Lip}(\hat{u}_\eta)\|\varphi - \psi\|_{C^1} < \frac{\varepsilon}{3}$$

where $\operatorname{Lip}(\hat{u}_\eta)$ is a global Lipschitz constant of $\hat{u}_\eta$ due to Proposition 3.4. Set $\tilde{\delta} := \min\{\tilde{\delta}_1, \tilde{\delta}_2\}$ and consider arbitrary $(s, \psi) \in (-\infty, 0] \times C^1$ with $|t-s| < \tilde{\delta}$ and $\|\varphi - \psi\|_{C^1} < \tilde{\delta}$. Then we infer

$$\begin{aligned}
\|F_\eta^{cu}(t,\varphi)-F_\eta^{cu}(s,\psi)\|_{C^1} &= \|\hat{u}_\eta(\varphi)(t)+\mathcal{C}(\varphi)(t)-\hat{u}_\eta(\psi)(s)-\mathcal{C}(\psi)(s)\|_{C^1}\\
&\le \|\hat{u}_\eta(\varphi)(t)-\hat{u}_\eta(\psi)(s)\|_{C^1}+\|\mathcal{C}(\varphi)(t)-\mathcal{C}(\psi)(s)\|_{C^1}\\
&\le \|\hat{u}_\eta(\varphi)(t)-\hat{u}_\eta(\varphi)(s)\|_{C^1}+\|\hat{u}_\eta(\varphi)(s)-\hat{u}_\eta(\psi)(s)\|_{C^1}\\
&\quad+\|P_s\varphi-P_s\psi\|_{C^1}\\
&\le \frac{\varepsilon}{3}+e^{-\eta s}e^{\eta s}\|\hat{u}_\eta(\varphi)(s)-\hat{u}_\eta(\psi)(s)\|_{C^1}+\|P_s\|\|\varphi-\psi\|_{C^1}\\
&\le \frac{\varepsilon}{3}+e^{-\eta(t-\tilde{\delta})}\|\hat{u}_\eta(\varphi)-\hat{u}_\eta(\psi)\|_{C^1_\eta}+\frac{\varepsilon}{3}\\
&\le \frac{\varepsilon}{3}+e^{-\eta(t-\tilde{\delta})}\operatorname{Lip}(\hat{u}_\eta)\|\varphi-\psi\|_{C^1}+\frac{\varepsilon}{3}\\
&< \varepsilon.
\end{aligned}$$

This proves the continuity of F_η^{cu} at (t,φ).

2. (*Proof of the algebraic properties of a semiflow.*) To begin with, observe that from the definition of F_η^{cu} and the last result it immediately follows that

$$F_\eta^{cu}(0,\varphi)=\hat{u}_\eta(\varphi)(0)+\mathcal{C}(\varphi)(0)=P_{cu}\varphi+P_s\varphi=\varphi$$

for all $\varphi\in C^1$. Therefore, the only thing remaining to prove is the additive property of F_η^{cu}. For this purpose, let $\hat{t},\hat{s}\in(-\infty,0]$ and $\varphi\in C^1$ be given. We have

$$F_\eta^{cu}(\hat{s}+\hat{t},\varphi)=\hat{u}_\eta(\varphi)(\hat{s}+\hat{t})+\mathcal{C}(\varphi)(\hat{s}+\hat{t})=\hat{u}_\eta(\varphi)(\hat{s}+\hat{t})+P_s\varphi$$

and

$$\begin{aligned}
F_\eta^{cu}(\hat{t},F_\eta^{cu}(\hat{s},\varphi)) &= \hat{u}_\eta(F_\eta^{cu}(\hat{s},\varphi))(\hat{t})+\mathcal{C}(F_\eta^{cu}(\hat{s},\varphi))(\hat{t})\\
&= \hat{u}_\eta(F_\eta^{cu}(\hat{s},\varphi))(\hat{t})+P_sF_\eta^{cu}(\hat{s},\varphi)\\
&= \hat{u}_\eta(F_\eta^{cu}(\hat{s},\varphi))(\hat{t})+P_s[\hat{u}_\eta(\varphi)(\hat{s})+\mathcal{C}(\varphi)(\hat{s})]\\
&= \hat{u}_\eta(F_\eta^{cu}(\hat{s},\varphi))(\hat{t})+P_s[\mathcal{C}(\varphi)(\hat{s})] \quad \text{(due to Corollary 3.5)}\\
&= \hat{u}_\eta(F_\eta^{cu}(\hat{s},\varphi))(\hat{t})+P_s\varphi.
\end{aligned}$$

Thus, it suffices to prove

$$\hat{u}_\eta(\varphi)(\hat{s}+\hat{t})=\hat{u}_\eta(F_\eta^{cu}(\hat{s},\varphi))(\hat{t})$$

in order to see $F_\eta^{cu}(\hat{s}+\hat{t},\varphi)=F_\eta^{cu}(\hat{t},F_\eta^{cu}(\hat{s},\varphi))$. To this end, define

$$v(t):=\hat{u}_\eta(\varphi)(t+\hat{s}) \quad\text{and}\quad w(t):=\hat{u}_\eta(F_\eta^{cu}(\hat{s},\varphi))(t)$$

for all $t\le 0$. Accordingly to the last two results, $v,w\in C^1_\eta$ and $v(t),w(t)\in C_{cu}$ as $t\le 0$. Moreover, in view of

$$\hat{u}_\eta(\varphi)(\hat{s})=P_{cu}[\hat{u}_\eta(\varphi)(\hat{s})+P_s\varphi]=P_{cu}[\hat{u}_\eta(\varphi)(\hat{s})+\mathcal{C}(\varphi)(\hat{s})]=P_{cu}F_\eta^{cu}(\hat{s},\varphi)=\hat{u}(F_\eta^{cu}(\hat{s},\varphi))(0),$$

we have $v(0)=w(0)$ and both v and w satisfy

$$u(t)=T_e(t-s)P_{cu}u(s)+\int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(u(\tau)+P_s\varphi))\,d\tau$$

for all $-\infty<s\le t\le 0$ as $P_sF_\eta^{cu}(\hat{s},\varphi)=P_s\varphi$.

Consider now the mapping $z\colon (-\infty, 0] \to C^1$ given by

$$z(t) := v(t) - T_e(t)P_{cu}w(0) = v(t) - T_e(t)w(0).$$

Using estimates (2.2), we see

$$\begin{aligned}
\sup_{t\leq 0} e^{\eta t}\|T_e(t)w(0)\|_{C^1} &= \sup_{t\leq 0} e^{\eta t}\|T_e(t)P_{cu}w(0)\|_{C^1} \\
&\leq \sup_{t\leq 0} e^{\eta t}\|T_e(t)P_c w(0)\|_{C^1} + \sup_{t\leq 0} e^{\eta t}\|T_e(t)P_u w(0)\|_{C^1} \\
&\leq K\sup_{t\leq 0} e^{-(c_c-\eta)t}\|P_c w(0)\|_{C^1} + K\sup_{t\leq 0} e^{(c_u+\eta)t}\|P_u w(0)\|_{C^1} \\
&\leq K\|P_c\|\|w(0)\|_{C^1} + K\|P_u\|\|w(0)\|_{C^1} \\
&< \infty.
\end{aligned}$$

Thus, $z \in C^1_\eta$. In addition, for all $s \leq t \leq 0$

$$\begin{aligned}
z(t) &= v(t) - T_e(t)w(0) \\
&= T_e(t-s)v(s) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(v(\tau) + P_s\varphi))d\tau - T_e(t-s)T_e(s)w(0) \\
&= T_e(t-s)z(s) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(v(\tau) + \mathcal{C}(\varphi)(\tau)))\,d\tau.
\end{aligned}$$

Combining this fact together with $\overline{R}_{\delta\eta}(v + \mathcal{C}(\varphi)) \in Y_\eta$ due to Corollary 3.2 and

$$P_{cu}^{\odot *}z(0) = P_{cu}z(0) = P_{cu}v(0) - T_e(0)P_{cu}w(0) = v(0) - w(0) = 0,$$

we obtain $z = \mathcal{K}_\eta \circ \overline{R}_{\delta\eta}(v + \mathcal{C}(\varphi))$ from Corollary 3.4 in [5]. Hence, it follows that

$$\begin{aligned}
v(t) &= z(t) + T_e(t)w(0) \\
&= (\mathcal{K}_\eta \circ \overline{R}_{\delta\eta}(v + \mathcal{C}(\varphi)))(t) + T_e(t)P_{cu}w(0) \\
&= (\mathcal{K}_\eta \circ \overline{R}_{\delta\eta})(v + \mathcal{C}(F_\eta^{cu}(\hat{s},\varphi)))(t) + T_e(t)P_{cu}\hat{u}_\eta(F_\eta^{cu}(\hat{s},\varphi))(0) \\
&= (\mathcal{K}_\eta \circ \overline{R}_{\delta\eta})(v + \mathcal{C}(F_\eta^{cu}(\hat{s},\varphi)))(t) + T_e(t)P_{cu}F_\eta^{cu}(\hat{s},\varphi)
\end{aligned}$$

for all $t \leq 0$. In the Banach space C^1_η the last equation reads

$$v = \mathcal{K}_\eta \circ \overline{R}_{\delta\eta}(v + \mathcal{C}(F_\eta^{cu}(\hat{s},\varphi))) + \overline{S}_\eta(F_\eta^{cu}(\hat{s},\varphi)) = \overline{\mathcal{G}}_\eta(v, F_\eta^{cu}(\hat{s},\varphi)).$$

Therefore, Proposition 3.4 implies

$$v = \hat{u}_\eta(F_\eta^{cu}(\hat{s},\varphi)) = w.$$

In particular, it follows that

$$\hat{u}_\eta(\varphi)(\hat{t}+\hat{s}) = v(\hat{t}) = w(\hat{t}) = \hat{u}_\eta(F_\eta^{cu}(\hat{s},\varphi))(\hat{t}),$$

which completes the proof. □

4 An attraction property of the global center-unstable manifolds of the modified equations: the statement and the main idea of the proof

After the preparations in the last sections, we are now in the position to state an attraction property of the global center-unstable manifolds W^η of the modified equations (2.14).

Theorem 4.1 (Attraction property of the global center-unstable manifolds)**.** *Let $f\colon U \to \mathbb{R}^n$, $U \subset C^1$ open, with $f(0) = 0$, satisfying the properties (S1) and (S2), and with $C_{cu} \neq \{0\}$ be given. Further, for fixed $\eta \in \mathbb{R}$ with $c_c < \eta < \min\{-c_s, c_u\}$, let $0 < \delta < \delta_1$ satisfy*

$$\lambda(\delta) < \frac{1}{5}, \tag{4.1}$$

$$\lambda(\delta) \cdot \|\mathcal{K}_\eta^+\| < \frac{1}{2}, \tag{4.2}$$

and

$$\lambda(\delta) \cdot \|\mathcal{K}_\eta\| \leq \lambda(\delta) \cdot \|\mathcal{K}_\eta\| \cdot \|P_{cu}^{\odot *}\| < \frac{1}{2}. \tag{4.3}$$

Then there exist a continuous map $H_{cu}^\eta\colon X_\delta \to W^\eta$ such that for all $(\psi, \varphi) \in W^\eta \times X_\delta$ the following holds:

$$\sup_{t \geq 0} e^{\eta t} \|F_\delta(t, \varphi) - F_\delta(t, \psi)\|_{C^1} < \infty \tag{4.4}$$

if and only if $\psi = H_{cu}^\eta(\varphi)$.

From now on and until the end of the next section, we suppose that the assumptions of this theorem are satisfied. For a proof we adopt the ideas of Vanderbauwhede [6], where the assertion for the case of ordinary differential equations is discussed. The initial point of this strategy is an alternative characterization of property (4.4).

Lemma 4.2. *Suppose that $\overline{F}\colon \mathbb{R} \times X_\delta \to C^1$ is continuous and satisfies*

(a) $\overline{F}(t, \varphi) = F_\delta(t, \varphi)$ for all $t \geq 0$ and all $\varphi \in X_\delta$, and

(b) $\overline{F}(\,\cdot\,, \varphi)\,|_{(-\infty,0]} \in C_\eta^1$ for each $\varphi \in X_\delta$.

Let $\varphi, \psi \in X_\delta$ be given. Then the following statements are equivalent:

(i) $\psi \in W^\eta$ and

$$\sup_{t \geq 0} e^{\eta t} \|F_\delta(t, \psi) - F_\delta(t, \varphi)\|_{C^1} < \infty.$$

(ii) There exists some $z \in C_{\eta,\mathbb{R}}^1$ such that $\overline{F}(\,\cdot\,, \varphi) + z$ is a solution of

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau) l(r_\delta(u(\tau)))\, d\tau \tag{4.5}$$

as $-\infty < s \leq t < \infty$ and $\psi = \varphi + z(0)$.

Proof. We follow the proof of Lemma 5.6 in [6].

1. To begin with, assume that under given assumptions, property (i) holds. Then Corollary 4.7 in [5] in combination with the definition of W^{η} shows that $\psi = \tilde{u}_{\eta}(P_{cu}\psi)(0)$. Moreover, $\tilde{u}_{\eta}(P_{cu}\psi) \in C^1_{\eta}$ and $\tilde{u}_{\eta}(P_{cu}\psi)$ is a solution of Eq. (4.5) as $-\infty < s \leq t \leq 0$. Setting

$$v(t) := \begin{cases} \tilde{u}_{\eta}(P_{cu}\psi)(t), & \text{for } t \leq 0, \\ F_{\delta}(t,\psi), & \text{for } t \geq 0, \end{cases}$$

we obtain a continuous function $v\colon \mathbb{R} \to C^1$, which satisfies Eq. (4.5) for all $-\infty < s \leq t < \infty$. While the last point is clear for the cases $-\infty < s \leq t \leq 0$ and $0 \leq s \leq t < \infty$, in the situation $-\infty < s < 0 < t < \infty$ this results from the following straightforward calculation:

$$\begin{aligned} v(t) &= F_{\delta}(t,\psi) \\ &= T_e(t)F_{\delta}(0,\psi) + \int_0^t T_e^{\odot *}(t-\tau)l(r_{\delta}(F_{\delta}(\tau,\varphi)))\,d\tau \\ &= T_e(t)\tilde{u}_{\eta}(P_{cu}\psi)(0) + \int_0^t T_e^{\odot *}(t-\tau)l(r_{\delta}(v(\tau)))\,d\tau \\ &= T_e(t)T_e(-s)\tilde{u}_{\eta}(P_{cu}\psi)(s) + T_e(t)\int_s^0 T_e^{\odot *}(-\tau)l(r_{\delta}(\tilde{u}_{\eta}(P_{cu}\psi)(\tau)))\,d\tau \\ &\quad + \int_0^t T_e^{\odot *}(t-\tau)l(r_{\delta}(v(\tau)))\,d\tau \\ &= T_e(t-s)\tilde{u}_{\eta}(P_{cu}\psi)(s) + \int_s^0 T_e^{\odot *}(t-\tau)l(r_{\delta}(\tilde{u}_{\eta}(P_{cu}\psi)(\tau)))\,d\tau \\ &\quad + \int_0^t T_e^{\odot *}(t-\tau)l(r_{\delta}(v(\tau)))\,d\tau \\ &= T_e(t-s)v(s) + \int_s^0 T_e^{\odot *}(t-\tau)l(r_{\delta}(v(\tau)))\,d\tau + \int_0^t T_e^{\odot *}(t-\tau)l(r_{\delta}(v(\tau)))\,d\tau \\ &= T_e(t-s)v(s) + \int_s^t T_e^{\odot *}(t-\tau)l(r_{\delta}(v(\tau)))\,d\tau. \end{aligned}$$

Consider now $z\colon \mathbb{R} \to C^1$ given by $z(t) := v(t) - \overline{F}(t,\varphi)$. In view of property (b) and the above, it follows that

$$\begin{aligned} \sup_{t\leq 0} e^{\eta t}\|z(t)\|_{C^1} &= \sup_{t\leq 0} e^{\eta t}\|v(t) - \overline{F}(t,\varphi)\|_{C^1} \\ &= \sup_{t\leq 0} e^{\eta t}\|\tilde{u}_{\eta}(P_{cu}\psi)(t) - \overline{F}(t,\varphi)\|_{C^1} \\ &\leq \sup_{t\leq 0} e^{\eta t}\|\tilde{u}_{\eta}(P_{cu}\psi)(t)\|_{C^1} + \sup_{t\leq 0}\|\overline{F}(t,\varphi)\|_{C^1} \\ &< \infty, \end{aligned}$$

that is, $z \in C^1_{\eta}$. Moreover, combining (a) and (i) we see

$$\begin{aligned} \sup_{t\geq 0} e^{\eta t}\|z(t)\|_{C^1} &= \sup_{t\geq 0} e^{\eta t}\|v(t) - \overline{F}(t,\varphi)\|_{C^1} \\ &= \sup_{t\geq 0} e^{\eta t}\|F_{\delta}(t,\psi) - \overline{F}(t,\varphi)\|_{C^1} \\ &= \sup_{t\geq 0} e^{\eta t}\|F_{\delta}(t,\psi) - F_{\delta}(t,\varphi)\|_{C^1} \\ &< \infty. \end{aligned}$$

Therefore, $z \in C^1_{\eta,\mathbb{R}}$. As, in addition,

$$\psi = v(0) = \overline{F}(0,\varphi) + z(0) = \varphi + z$$

we conclude that property (i) indeed implies (ii).

2. Suppose now that (ii) holds. Then, in consideration of the one-to-one correspondence between solutions of Eq. (2.14) and of Eq. (2.15) and in consideration of the uniqueness result contained in Proposition 3.1, we have $F_\delta(t,\psi) = \overline{F}(t,\varphi) + z(t)$ as $t \geq 0$. Using property (a) and the fact $z \in C^1_{\eta,\mathbb{R}}$, we also infer

$$\sup_{t\geq 0} e^{\eta t}\|F_\delta(t,\psi) - F_\delta(t,\varphi)\|_{C^1} = \sup_{t\geq 0} e^{\eta t}\|F_\delta(t,\psi) - \overline{F}(t,\varphi)\|_{C^1} = \sup_{t\geq 0} e^{\eta t}\|z(t)\|_{C^1} < \infty.$$

Hence, it remains to prove $\psi \in W^\eta$. For this purpose, observe that $z\,|_{(-\infty,0]} \in C^1_\eta$ and $\overline{F}(\,\cdot\,,\varphi)|_{(-\infty,0]} \in C^1_\eta$. Thus, for $v\colon (-\infty,0] \to C^1$ given by $v(t) := \overline{F}(t,\varphi) + z(t)$ as $t \leq 0$, we also have $v \in C^1_\eta$. As, in addition, v is a solution of Eq. (4.5) for all $-\infty < s \leq t \leq 0$, Proposition 4.8 in [5] shows $v(0) = \psi \in W^\eta$ and the assertion follows. □

In view of the assumptions of the last result, it becomes clear that we need some continuous map $\overline{F}\colon \mathbb{R} \times X_\delta \to C^1$ with properties (a) and (b) in order to be able to use property (ii) for a proof of Theorem 4.1. Below we construct such a map. The key ingredients here are the global semiflows F_δ and F^{cu}_η discussed in the last section. Indeed, defining

$$\overline{F}\colon \mathbb{R} \times X_\delta \to C^1$$

by

$$\overline{F}(t,\varphi) := \begin{cases} F_\delta(t,\varphi), & \text{as } t \geq 0, \\ F^{cu}_\eta(t,\varphi), & \text{otherwise,} \end{cases} \tag{4.6}$$

the map $\overline{F}$ has the desired properties as shown next.

Proposition 4.3. *The mapping $\overline{F}\colon \mathbb{R} \times X_\delta \to C^1$ defined by Eq. (4.6) is continuous, possesses properties (a) and (b) from Lemma 4.2, and satisfies*

$$P_{cu}\overline{F}(t,\varphi) = T_e(t-s)P_{cu}\overline{F}(s,\varphi) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau \tag{4.7}$$

for all $\varphi \in X_\delta$ and all $-\infty < s \leq t < \infty$.

Proof. 1. Recall that, by Proposition 3.1, F_δ is continuous on $[0,\infty) \times X_\delta$, and that, by Proposition 3.6, F^{cu}_η is continuous on $(-\infty,0] \times X_\delta$. As for all $\varphi \in X_\delta$ we have

$$F_\delta(0,\varphi) = \varphi = F^{cu}_\eta(0,\varphi),$$

it is obvious that $\overline{F}$ is continuous on all of $\mathbb{R} \times X_\delta$. Moreover, in consideration of the definition, it is also clear that $\overline{F}$ has property (a) from Lemma 4.2. Next, observe that for each $\varphi \in X_\delta$ we have

$$\overline{F}(\,\cdot\,,\varphi)|_{(-\infty,0]} = F^{cu}_\eta(\,\cdot\,,\varphi) = \hat{u}(\varphi)(\,\cdot\,) + \mathcal{C}(\varphi)(\,\cdot\,)$$

and both $\hat{u}(\varphi)(\,\cdot\,)$ and $\mathcal{C}(\varphi)(\,\cdot\,)$ belong to C^1_η due to Proposition 3.4 and the introduction in front of it. For this reason, $\overline{F}(\,\cdot\,,\varphi) \in C^1_\eta$, which shows that $\overline{F}$ also has property (b) from Lemma 4.2.

2. It remains to prove that Eq. (4.7) holds. In order to show this, consider $\varphi \in X_\delta$ and $-\infty < s \leq t < \infty$. If $s \leq t \leq 0$ then Eq. (4.7) follows from Corollary 3.5:

$$\begin{aligned}
P_{cu}\overline{F}(t,\varphi) &= P_{cu}F^{cu}_\eta(t,\varphi)\\
&= P_{cu}\left[\hat{u}_\eta(\varphi)(t) + \mathcal{C}(\varphi)(t)\right]\\
&= P_{cu}\hat{u}_\eta(\varphi)(t) + P_{cu}P_s\varphi\\
&= P_{cu}\hat{u}_\eta(\varphi)(t)\\
&= T_e(t-s)P_{cu}\hat{u}_\eta(\varphi)(s) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\hat{u}_\eta(\varphi)(\tau) + P_s\varphi))\,d\tau\\
&= T_e(t-s)P_{cu}\left[\hat{u}_\eta(\varphi)(s) + P_s\varphi\right] + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(F^{cu}_\eta(\tau,\varphi))\,d\tau\\
&= T_e(t-s)P_{cu}F^{cu}_\eta(s,\varphi) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau\\
&= T_e(t-s)P_{cu}\overline{F}(s,\varphi) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau.
\end{aligned}$$

In case $0 \leq s \leq t < \infty$, Eq. (3.4) implies

$$\begin{aligned}
P_{cu}\overline{F}(t,\varphi) &= P_{cu}F_\delta(t,\varphi)\\
&= T_e(t-s)P_{cu}F_\delta(s,\varphi) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(F_\delta(\tau,\varphi)))\,d\tau\\
&= T_e(t-s)P_{cu}\overline{F}(s,\varphi) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau,
\end{aligned}$$

so formula (4.7) holds again. Finally, for $s < 0 < t$ we get

$$\begin{aligned}
P_{cu}\overline{F}(t,\varphi) &= T_e(t)P_{cu}\overline{F}(0,\varphi) + \int_0^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau\\
&= T_e(t)T_e(-s)P_{cu}\overline{F}(s,\varphi) + T_e(t)\int_s^0 T_e^{\odot *}(-\tau)P_{cu}^{\odot *}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau\\
&\quad + \int_0^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau\\
&= T_e(t-s)P_{cu}\overline{F}(s,\varphi) + \int_s^t T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau
\end{aligned}$$

by combining the two preceding cases. This establishes the formula. □

Given $\varphi \in X_\delta$, we would like to find some $z \in C^1_{\eta,\mathbb{R}}$ such that $\overline{F}(\,\cdot\,,\varphi) + z$ is a solution of Eq. (4.5). For this purpose, we deduce a necessary and sufficient condition for $z \in C^1_{\eta,\mathbb{R}}$ to turn $\overline{F}(\,\cdot\,,\varphi) + z$ into a solution of Eq. (4.5).

Lemma 4.4. *Let $\varphi \in X_\delta$ and $z \in C^1_{\eta,\mathbb{R}}$ be given. Then $\overline{F}(\,\cdot\,,\varphi) + z$ satisfies*

$$u(t) = T_e(t-s)u(s) + \int_s^t T_e^{\odot *}(t-\tau)l(r_\delta(u(\tau)))\,d\tau \tag{4.8}$$

for all $-\infty < s \leq t < \infty$ if and only if

$$\begin{aligned}
z(t) = -P_s\overline{F}(t,\varphi) &+ \int_{-\infty}^t T_e^{\odot *}(t-\tau)P_s^{\odot *}l(r_\delta(\overline{F}(\tau,\varphi) + z(\tau)))\,d\tau\\
&- \int_t^\infty T_e^{\odot *}(t-\tau)P_{cu}^{\odot *}\left[l(r_\delta(\overline{F}(\tau,\varphi) + z(\tau))) - l(r_\delta(\overline{F}(\tau,\varphi)))\right]\,d\tau
\end{aligned} \tag{4.9}$$

for all $t \in \mathbb{R}$.

This result is motivated by Vanderbauwhede [6, Lemma 5.7]. For its proof we need two corollaries, which both are easy consequences of the exponential trichotomy (2.2) of the strongly continuous semigroup T_e.

Corollary 4.5. *Let $z \in C^1_{\eta,\mathbb{R}}$ and $t \in \mathbb{R}$ be given. Then*

$$\lim_{s\to\infty} T_e(t-s)P_{cu}^{\odot *}z(s) = 0$$

in C.

Proof. Recall that T_e defines a group on the center-unstable space $C_{cu} \subset C$. In particular, $\mathbb{R} \ni s \mapsto T_e(t-s)P_{cu}^{\odot *}z(s) \in C_{cu}$ is a continuous mapping from $\mathbb{R}$ into C. Furthermore, combining the estimates $\|z(s)\|_C \leq e^{-\eta s}\|z\|_{C^0_{\eta,\mathbb{R}}}$ and $\|z\|_{C^0_{\eta,\mathbb{R}}} \leq \|z\|_{C^1_{\eta,\mathbb{R}}}$ together with (2.2), we conclude

$$\begin{aligned}
\|T_e(t-s)P_{cu}^{\odot *}z(s)\|_C &\leq \|T_e(t-s)P_c^{\odot *}z(s)\|_C + \|T_e(t-s)P_u^{\odot *}z(s)\|_C \\
&\leq Ke^{c_c|t-s|}\|P_c^{\odot *}z(s)\|_C + Ke^{c_u(t-s)}\|P_u^{\odot *}z(s)\|_C \\
&\leq Ke^{c_c(s-t)}\|P_c^{\odot *}\|\|z(s)\|_C + Ke^{c_u(t-s)}\|P_u^{\odot *}\|\|z(s)\|_C \\
&\leq Ke^{c_c(s-t)}e^{-\eta s}\|P_c^{\odot *}\|\|z\|_{C^0_{\eta,\mathbb{R}}} + Ke^{c_u(t-s)}e^{-\eta s}\|P_u^{\odot *}\|\|z\|_{C^0_{\eta,\mathbb{R}}} \\
&\leq e^{(c_c-\eta)s}Ke^{-c_c t}\|P_c^{\odot *}\|\|z\|_{C^0_{\eta,\mathbb{R}}} + e^{-(c_u+\eta)s}Ke^{c_u t}\|P_u^{\odot *}\|\|z\|_{C^0_{\eta,\mathbb{R}}} \\
&\leq e^{(c_c-\eta)s}\underbrace{Ke^{-c_c t}\|P_c^{\odot *}\|\|z\|_{C^1_{\eta,\mathbb{R}}}}_{<\infty} + e^{-(c_u+\eta)s}\underbrace{Ke^{c_u t}\|P_u^{\odot *}\|\|z\|_{C^1_{\eta,\mathbb{R}}}}_{<\infty}
\end{aligned}$$

for all $s, t \in \mathbb{R}$ with $t - s \leq 0$. As $0 < c_c < \eta < c_u$, taking the limit for $s \to \infty$ indeed shows $\lim_{s\to\infty} T_e(t-s)P_{cu}^{\odot *}z(s) = 0$ as claimed. □

Corollary 4.6. *Let $\varphi \in X_f$, $z \in C^1_{\eta,\mathbb{R}}$, and $t \in \mathbb{R}$ be given. Then*

$$\lim_{s\to-\infty} T_e(t-s)\left[P_s^{\odot *}\overline{F}(s,\varphi) - P_s^{\odot *}z(s)\right] = 0$$

in C.

Proof. Recall that we have $\overline{F}(\,\cdot\,,\varphi)|_{(-\infty,0]} \in C^1_\eta$. Consequently, using the estimates (2.2) we infer

$$\begin{aligned}
\|T_e(t-s)P_s^{\odot *}(\overline{F}(s,\varphi)-z(s))\|_C &\leq Ke^{c_s(t-s)}\|P_s^{\odot *}\overline{F}(s,\varphi) + P_s^{\odot *}z(s)\|_C \\
&\leq Ke^{c_s(t-s)}\|P_s^{\odot *}\|\left(\|\overline{F}(s,\varphi)\|_C + \|z(s)\|_C\right) \\
&\leq Ke^{c_s(t-s)}\|P_s^{\odot *}\|e^{-\eta s}\left(e^{\eta s}\|\overline{F}(s,\varphi)\|_C + e^{\eta s}\|z(s)\|_C\right) \\
&\leq e^{-(c_s+\eta)s}Ke^{c_s t}\|P_s^{\odot *}\|\left(\|\overline{F}(\cdot,\varphi)|_{(-\infty,0]}\|_{C^0_\eta} + \|z|_{(-\infty,0]}\|_{C^0_\eta}\right) \\
&\leq e^{-(c_s+\eta)s}Ke^{c_s t}\|P_s^{\odot *}\|\left(\|\overline{F}(\cdot,\varphi)|_{(-\infty,0]}\|_{C^0_\eta} + \|z\|_{C^1_{\eta,\mathbb{R}}}\right)
\end{aligned}$$

for all $s \leq \min\{0,t\}$. Since $c_s < 0 < \eta < -c_s$ it becomes clear that

$$\|T_e(t-s)(P_s^{\odot *}\overline{F}(s,\varphi) - P_s^{\odot *}z(s))\|_C \to 0$$

as $s \to -\infty$. □

Having established the auxiliary results, we are now in position to prove Lemma 4.4.

Proof of Lemma 4.4. We adopt the proof of Lemma 5.7 in Vanderbauwhede [6].

1. Assume that, given $\varphi \in X_\delta$ and $z \in C^1_{\eta,\mathbb{R}}$, $\overline{F}(\,\cdot\,,\varphi) + z$ is a globally defined solution of Eq. (4.8) for all $-\infty < s \leq t < \infty$. Then, in view of Proposition 4.3 and Eq. (4.7), we get

$$\begin{aligned}
z(t) &= T_e(t-s)[\overline{F}(s,\varphi)+z(s)] + \int_s^t T_e^{\odot *}(t-\tau) l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau - \overline{F}(t,\varphi)\\
&= -P_s\overline{F}(t,\varphi) - P_{cu}\overline{F}(t,\varphi) + T_e(t-s)[\overline{F}(s,\varphi)+z(s)]\\
&\quad + \int_s^t T_e^{\odot *}(t-\tau) l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))d\tau\\
&= -P_s\overline{F}(t,\varphi) + T_e(t-s)[P_s\overline{F}(s,\varphi)+z(s)] - P_{cu}\overline{F}(t,\varphi) + T_e(t-s)P_{cu}\overline{F}(s,\varphi)\\
&\quad + \int_s^t T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))d\tau\\
&\quad + \int_s^t T_e^{\odot *}(t-\tau) P_{s}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau\\
&= -P_s\overline{F}(t,\varphi) + T_e(t-s)[P_s\overline{F}(s,\varphi)+z(s)] - \int_s^t T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau\\
&\quad + \int_s^t T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau\\
&\quad + \int_s^t T_e^{\odot *}(t-\tau) P_{s}^{\odot *} l(r_\delta(\overline{F}(s,\varphi)+z(\tau)))\,d\tau,
\end{aligned}$$

that is,

$$\begin{aligned}
z(t) &= -P_s\overline{F}(t,\varphi) + T_e(t-s)[P_s\overline{F}(s,\varphi)+z(s)]\\
&\quad + \int_s^t T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} \left[l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau))) - l(r_\delta(\overline{F}(\tau,\varphi)))\right]\,d\tau\\
&\quad + \int_s^t T_e^{\odot *}(t-\tau) P_{s}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau
\end{aligned}$$

as $-\infty < s \leq t < \infty$. Hence, the application of the projections $P_{cu}^{\odot *}$ and $P_s^{\odot *}$ shows that

$$\begin{aligned}
P_{cu}^{\odot *} z(t) &= T_e(t-s) P_{cu}^{\odot *} z(s)\\
&\quad + \int_s^t T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} [l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau))) - l(r_\delta(\overline{F}(\tau,\varphi)))]\,d\tau
\end{aligned} \tag{4.10}$$

and that

$$\begin{aligned}
P_s^{\odot *} z(t) &= -P_s^{\odot *}\overline{F}(t,\varphi) + T_e(t-s)[P_s^{\odot *}\overline{F}(s,\varphi) - P_s^{\odot *} z(s)]\\
&\quad + \int_s^t T_e^{\odot *}(t-\tau) P_{s}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau
\end{aligned} \tag{4.11}$$

as $-\infty < s \leq t < \infty$. Moreover, we claim that Eq. (4.10) holds for all $s, t \in \mathbb{R}$. Indeed, as T_e defines a group on the center-unstable space C_{cu} we may apply the operator $T_e(s-t)$ to both sides of Eq. (4.10) in order to see that

$$\begin{aligned}
T_e(s-t) P_{cu}^{\odot *} z(t) &= T_e(s-t) T_e(t-s) P_{cu}^{\odot *} z(s)\\
&\quad + T_e(s-t) \int_s^t T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} [l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau))) - l(r_\delta(\overline{F}(\tau,\varphi)))]\,d\tau\\
&= P_{cu}^{\odot *} z(s) + \int_s^t T_e^{\odot *}(s-\tau) P_{cu}^{\odot *} [l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau))) - l(r_\delta(\overline{F}(\tau,\varphi)))]\,d\tau,
\end{aligned}$$

that is,

$$P_{cu}^{\odot*}z(s) = T_e(s-t)P_{cu}^{\odot*}z(t) - \int_s^t T_e^{\odot*}(s-\tau)P_{cu}^{\odot*}[l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau))) - l(r_\delta(\overline{F}(\tau,\varphi)))]\,d\tau,$$

for all $-\infty < s \leq t < \infty$. Thus, formula (4.10) holds for all $s, t \in \mathbb{R}$ as claimed. In particular, this proves that for fixed $t \in \mathbb{R}$, we may take the limit for $s \to \infty$ in Eq. (4.10). Then, in consideration of Corollary 4.5, we get

$$P_{cu}^{\odot*}z(t) = -\int_t^\infty T_e^{\odot*}(t-\tau)P_{cu}^{\odot*}[l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau))) - l(r_\delta(\overline{F}(\tau,\varphi)))]\,d\tau.$$

Similarly, carrying out the limit process $s \to -\infty$ in Eq. (4.11) in combination with Corollary 4.6 leads to

$$P_s^{\odot*}z(t) = -P_s^{\odot*}\overline{F}(t,\varphi) + \int_{-\infty}^t T_e^{\odot*}(t-\tau)P_s^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau.$$

Hence, it follows that

$$\begin{aligned}
z(t) &= P_s^{\odot*}z(t) + P_{cu}^{\odot*}z(t)\\
&= -P_s^{\odot*}\overline{F}(t,\varphi) + \int_{-\infty}^t T_e^{\odot*}(t-\tau)P_s^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau\\
&\quad - \int_t^\infty T_e^{\odot*}(t-\tau)P_{cu}^{\odot*}\left[l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau))) - l(r_\delta(\overline{F}(\tau,\varphi)))\right]\\
&= -P_s\overline{F}(t,\varphi) + \int_{-\infty}^t T_e^{\odot*}(t-\tau)P_s^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau\\
&\quad - \int_t^\infty T_e^{\odot*}(t-\tau)P_{cu}^{\odot*}\left[l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau))) - l(r_\delta(\overline{F}(\tau,\varphi)))\right]
\end{aligned}$$

for each $t \in \mathbb{R}$. This proves one direction of the assertion.

2. Suppose, conversely, that for given $\varphi \in X_\delta$ and $z \in C^1_{\eta,\mathbb{R}}$ Eq. (4.9) holds, and let $s \leq t$ be given. Obviously,

$$\begin{aligned}
P_s\overline{F}(t,\varphi) + z(t) &= \int_{-\infty}^t T_e^{\odot*}(t-\tau)P_s^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau\\
&\quad - \int_t^\infty T_e^{\odot*}(t-\tau)P_{cu}^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau\\
&\quad + \int_t^\infty T_e^{\odot*}(t-\tau)P_{cu}^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau
\end{aligned}\tag{4.12}$$

and

$$\begin{aligned}
z(s) &= -P_s\overline{F}(s,\varphi) + \int_{-\infty}^s T_e^{\odot*}(s-\tau)P_s^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau\\
&\quad - \int_s^\infty T_e^{\odot*}(s-\tau)P_{cu}^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau\\
&\quad + \int_s^\infty T_e^{\odot*}(s-\tau)P_{cu}^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau.
\end{aligned}$$

Applying $T_e(t-s)$ on both sides of the last equation gives

$$\begin{aligned}
T_e(t-s)z(s) &= -T_e(t-s)P_s\overline{F}(s,\varphi) + \int_{-\infty}^s T_e^{\odot*}(t-\tau)P_s^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau\\
&\quad - \int_s^\infty T_e^{\odot*}(t-\tau)P_{cu}^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\,d\tau\\
&\quad + \int_s^\infty T_e^{\odot*}(t-\tau)P_{cu}^{\odot*}l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau,
\end{aligned}$$

that is,

$$\begin{aligned}
0 = &- \int_s^\infty T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau + T_e(t-s)z(s) + T_e(t-s)P_s\overline{F}(s,\varphi) \\
&- \int_{-\infty}^s T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\, d\tau \\
&+ \int_s^\infty T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\, d\tau.
\end{aligned}$$

Next, after adding zero in the way represented above to the right-hand side of Eq. (4.12), a simple calculation leads to

$$\begin{aligned}
P_s\overline{F}(t,\varphi) + z(t) = T_e(t-s)z(s) + T_e(t-s)P_s\overline{F}(s,\varphi) + \int_s^t T_e^{\odot *}(t-\tau) l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\, d\tau \\
- \int_s^t T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau.
\end{aligned}$$

Hence, by combining the last equation with Eq. (4.7), we finally obtain

$$\begin{aligned}
\overline{F}(t,\varphi) + z(t) &= P_{cu}\overline{F}(t,\varphi) + P_s\overline{F}(t,\varphi) + z(t) \\
&= T_e(t-s)P_{cu}\overline{F}(s,\varphi) + \int_s^t T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau \\
&\quad + T_e(t-s)z(s) + T_e(t-s)P_s\overline{F}(s,\varphi) + \int_s^t T_e^{\odot *}(t-\tau) l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\, d\tau \\
&\quad - \int_s^t T_e^{\odot *}(t-\tau) P_{cu}^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau \\
&= T_e(t-s)\overline{F}(s,\varphi) + T_e(t-s)z(s) + \int_s^t T_e^{\odot *}(t-\tau) l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\, d\tau \\
&= T_e(t-s)(\overline{F}(s,\varphi)+z(s)) + \int_s^t T_e^{\odot *}(t-\tau) l(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)))\, d\tau.
\end{aligned}$$

As $s \le t$ were arbitrary given, we conclude that $\overline{F}(\,\cdot\,,\varphi) + z$ is a solution of Eq. (4.8). This completes the proof. □

Now, consider some fixed $\varphi \in X_\delta$. If we find some $z \in C^1_{\eta,\mathbb{R}}$ satisfying Eq. (4.9) then Lemma 4.4 implies that $\overline{F}(\,\cdot\,,\varphi) + z$ is a global solution of the abstract integral equation (4.8). Hence, in turn, by application of Lemma 4.2 it follows that $\psi := \overline{F}(0,\varphi) + z(0) = \varphi + z(0)$ belongs to W^η and that φ and ψ satisfy (4.4). Therefore, in the next step towards a proof of Theorem 4.1 we would like to solve Eq. (4.9) in $C^1_{\eta,\mathbb{R}}$ for each given $\varphi \in X_\delta$. Moreover, under the assumption that these solutions are uniquely determined, in this way we also would obtain a possible choice for the map H^η_{cu} from Theorem 4.1, namely, $X_\delta \ni \varphi \mapsto \varphi + z(0) = \psi \in W^\eta$.

5 The remaining part of the proof for the attraction property of the global center-unstable manifolds

Our next goal is to show that for each fixed $\varphi \in X_\delta$ Eq. (4.9) has a uniquely determined solution in $C^1_{\eta,\mathbb{R}}$. This will be done by construction of a parameter-dependent contraction on the Banach space $C^1_{\eta,\mathbb{R}}$ below.

To begin with, observe that Eq. (4.9) may be formally written as

$$\begin{aligned} z(t) = -P_s\overline{F}(t,\varphi) &+ \int_{-\infty}^{t} T_e^{\odot*}(t-\tau)P_s^{\odot*} l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau \\ &+ \int_{-\infty}^{t} T_e^{\odot*}(t-\tau)P_s^{\odot*} l\left(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)) - r_\delta(\overline{F}(\tau,\varphi))\right)\,d\tau \\ &- \int_{t}^{\infty} T_e^{\odot*}(t-\tau)P_{cu}^{\odot*} l\left(r_\delta(\overline{F}(\tau,\varphi)+z(\tau)) - r_\delta(\overline{F}(\tau,\varphi))\right)\,d\tau. \end{aligned}$$

Thus, after introducing the mapping $\overline{r}_\delta \colon \mathbb{R}\times X_\delta \times C^1 \to \mathbb{R}^n$ given by

$$\overline{r}_\delta(t,\varphi,z) := r_\delta(\overline{F}(t,\varphi)+z) - r_\delta(\overline{F}(t,\varphi)),$$

we get the representation

$$\begin{aligned} z(t) = -P_s\overline{F}(t,\varphi) &+ \int_{-\infty}^{t} T_e^{\odot*}(t-\tau)P_s^{\odot*} l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau \\ &+ \int_{-\infty}^{t} T_e^{\odot*}(t-\tau)P_s^{\odot*} l(\overline{r}_\delta(\tau,\varphi,z(\tau)))\,d\tau \\ &- \int_{t}^{\infty} T_e^{\odot*}(t-\tau)P_{cu}^{\odot*} l(\overline{r}_\delta(\tau,\varphi,z(\tau)))\,d\tau \end{aligned} \tag{5.1}$$

of Eq. (4.9). Note that the involved map $\overline{r}_\delta$ is continuous. Moreover, using (2.12) and (2.13), it easily follows that

$$\|\overline{r}_\delta(t,\varphi,z)\|_{\mathbb{R}^n} \leq \lambda(\delta)\|z\|_{C^1} \tag{5.2}$$

and

$$\|\overline{r}_\delta(t,\varphi,z_1) - \overline{r}_\delta(t,\varphi,z_2)\|_{\mathbb{R}^n} \leq \lambda(\delta)\|z_1-z_2\|_{C^1} \tag{5.3}$$

for all $(t,\varphi)\in\mathbb{R}\times X_\delta$ and all $z,z_1,z_2\in C^1$.

In the first instance, representation (5.1) of Eq. (4.9) is purely formal. But next we are going to prove that all the improper integrals on the right-hand side of (5.1) indeed exist. We begin with consideration of the first integral.

Corollary 5.1. *Consider $\tilde{\eta}\in\mathbb{R}$ with $c_c<\eta\leq\tilde{\eta}<\min\{-c_s,c_u\}$. Then for each $\varphi\in X_\delta$*

$$t \mapsto \int_{-\infty}^{t} T_e^{\odot*}(t-\tau)P_s^{\odot*} l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau$$

defines a continuous map $v(\varphi)$ from $\mathbb{R}$ into C^1, and its restriction to $(-\infty,0]$ belongs to $C^1_{\tilde{\eta}}$.

Proof. Set $Q(t) := l(r_\delta(\overline{F}(t,\varphi)))\in Y^{\odot*}$ as $t\in\mathbb{R}$. Then, in view of the continuity of the maps l, r_δ and $\overline{F}(\,\cdot\,,\varphi)$, Q defines a continuous map from $\mathbb{R}$ into $Y^{\odot*}$ as well. Furthermore, we claim that $Q|_{(-\infty,0]}\in Y_{\tilde{\eta}}$. Indeed, by Proposition 4.3 we have

$$\overline{F}(\,\cdot\,,\varphi)|_{(-\infty,0]}\in C^1_\eta \subseteq C^1_{\tilde{\eta}}$$

and thus, by (2.13),

$$\sup_{t\leq 0} e^{\tilde{\eta}t}\|Q(t)\|_{C^{\odot*}} = \sup_{t\leq 0} e^{\tilde{\eta}t}\|l(r_\delta(\overline{F}(t,\varphi)))\|_{C^{\odot*}} \leq \sup_{t\leq 0} e^{\tilde{\eta}t}\lambda(\delta)\|\overline{F}(t,\varphi)\|_{C^1} < \infty.$$

For this reason, from Remark 2.3 it follows that $u := (\mathcal{K}^1 Q)$, that is,

$$t \mapsto \int_{-\infty}^{t} T_e^{\odot*}(t-\tau)P_s^{\odot*} l(r_\delta(\overline{F}(\tau,\varphi)))\,d\tau,$$

defines a continuous map from $\mathbb{R}$ into C, that its restriction to the interval $(-\infty, 0]$ belongs to $C^0_{\tilde{\eta}}$, and that it additionally satisfies Eq. (2.10) for all $-\infty < s \leq t < \infty$. Hence, using Remark 2.5 from the second section and Proposition 3.3 in [5], we see that $u = (\mathcal{K}^1 Q) \in C(\mathbb{R}, C^1)$ and $(\mathcal{K}^1 Q)|_{(-\infty,0]} \in C^1_{\tilde{\eta}}$. This proves the assertion. $\square$

Proposition 5.2. *Let $\tilde{\eta} > 0$ be as in Corollary 5.1, and let $\mathcal{Z}_{\tilde{\eta}}$ denote the map, which assigns to $\varphi \in X_\delta$ the mapping*

$$\mathbb{R} \ni t \mapsto -P_s \overline{F}(t, \varphi) + \int_{-\infty}^{t} T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau, \varphi)))\, d\tau \in C^1.$$

Then $\mathcal{Z}_{\tilde{\eta}}(X_\delta) \subset C^1_{\tilde{\eta}, \mathbb{R}}$.

Proof. 1. At first, note that by Proposition 4.3 and Corollary 5.1, for each $\varphi \in X_\delta$, $\mathcal{Z}_{\tilde{\eta}}(\varphi)$ forms a well-defined continuous map from $\mathbb{R}$ into C^1. Consequently, it remains to prove that for given $\varphi \in X_\delta$ we have

$$\sup_{t \in \mathbb{R}} e^{\tilde{\eta} t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t)\|_{C^1} < \infty.$$

For this purpose, let $\varphi \in X_\delta$ be given. Then

$$\sup_{t \in \mathbb{R}} e^{\tilde{\eta} t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t)\|_{C^1} \leq \sup_{t \leq 0} e^{\tilde{\eta} t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t)\|_{C^1} + \sup_{t \geq 0} e^{\tilde{\eta} t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t)\|_{C^1}.$$

and next we estimate the two terms on the right-hand side separately.

2. (*Estimate of* $\sup_{t \leq 0} e^{\tilde{\eta} t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t)\|_{C^1}$.) Using the triangle inequality together with the definition of $\overline{F}$ and Corollary 5.1, one obtains

$$\begin{aligned}
\sup_{t \leq 0} e^{\tilde{\eta} t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t)\|_{C^1} &= \sup_{t \leq 0} e^{\tilde{\eta} t} \| - P_s \overline{F}(t, \varphi) + v(\varphi)(t)\|_{C^1} \\
&\leq \sup_{t \leq 0} e^{\tilde{\eta} t} \|P_s \overline{F}(t, \varphi)\|_{C^1} + \sup_{t \leq 0} e^{\tilde{\eta} t} \|v(\varphi)(t)\|_{C^1} \\
&\leq \sup_{t \leq 0} e^{\tilde{\eta} t} \|P_s F^{cu}_{\eta}(t, \varphi)\|_{C^1} + \left\| v(\varphi)(\,\cdot\,)|_{(-\infty,0]} \right\|_{C^1_{\tilde{\eta}}} \\
&= \sup_{t \leq 0} e^{\tilde{\eta} t} \|P_s \hat{u}_\eta(\varphi)(t) + P_s \,\mathcal{C}(\varphi)(t)\|_{C^1} + \left\| v(\varphi)(\,\cdot\,)|_{(-\infty,0]} \right\|_{C^1_{\tilde{\eta}}} \\
&= \sup_{t \leq 0} e^{\tilde{\eta} t} \|P_s \varphi\|_{C^1} + \left\| v(\varphi)(\,\cdot\,)|_{(-\infty,0]} \right\|_{C^1_{\tilde{\eta}}} \\
&= \|P_s \varphi\|_{C^1} + \left\| v(\varphi)(\,\cdot\,)|_{(-\infty,0]} \right\|_{C^1_{\tilde{\eta}}} \\
&< \infty.
\end{aligned}$$

3. (*Estimate of* $\sup_{t \geq 0} e^{\tilde{\eta} t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t)\|_{C^1}$.) We begin with the observation that, in view of the definition of $\overline{F}$ and Eq. (3.3), we have

$$\begin{aligned}
\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) &= -P_s \overline{F}(t, \varphi) + \int_{-\infty}^{t} T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau, \varphi)))\, d\tau \\
&= -P_s \overline{F}(t, \varphi) + \int_{0}^{t} T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau, \varphi)))\, d\tau \\
&\quad + \int_{-\infty}^{0} T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau, \varphi)))\, d\tau
\end{aligned}$$

$$
\begin{aligned}
&= -P_s F_\delta(t,\varphi) + \int_0^t T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(F_\delta(\tau,\varphi))\, d\tau \\
&\quad + \int_{-\infty}^0 T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau \\
&= -T_e(t) P_s F_\delta(0,\varphi) + \int_{-\infty}^0 T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau \\
&= -T_e(t) P_s \varphi + \int_{-\infty}^0 T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau
\end{aligned}
$$

as $t \geq 0$. Set $u(t) := \mathcal{Z}_{\tilde{\eta}}(\varphi)(t)$, $t \geq 0$. We claim that

$$u(t) = T_e(t-s)u(s)$$

for all $0 \leq s \leq t < \infty$. Indeed, from the representation of $\mathcal{Z}_{\tilde{\eta}}(\varphi)$ derived above it follows that

$$
\begin{aligned}
u(t) - T_e(t-s)u(s) &= \mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - T_e(t-s)\mathcal{Z}_{\tilde{\eta}}(\varphi)(s) \\
&= -T_e(t) P_s \varphi + \int_{-\infty}^0 T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau \\
&\quad + T_e(t-s) T_e(s) P_s \varphi - T_e(t-s) \int_{-\infty}^0 T_e^{\odot *}(s-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau \\
&= -T_e(t) P_s \varphi + \int_{-\infty}^0 T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(t,\varphi))\, d\tau \\
&\quad + T_e(t) P_s \varphi - \int_{-\infty}^0 T_e^{\odot *}(t-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau \\
&= 0
\end{aligned}
$$

as $0 \leq s \leq t < \infty$. In particular, $u(t) = T_e(t)u(0)$ for all $t \geq 0$.

Next, we claim that, for each $t \geq 0$, $u(t)$ lies in the domain $\mathcal{D}(G_e)$ of the generator of the semigroup T_e. In order to see this, recall once more the one-to-one correspondence between solutions of Eq. (2.4) and Eq. (2.5). The map $x\colon [-h,\infty) \to \mathbb{R}^n$ given by

$$
x(t) := \begin{cases} u(0)(t), & \text{as } -h \leq t \leq 0, \\ u(t)(0), & \text{as } t \geq 0 \end{cases}
$$

is continuously differentiable, its segments x_t coincide with $u(t)$ for all $t \geq 0$, the mapping $[0,\infty) \ni t \mapsto x_t \in C^1$ is continuous, and additionally x satisfies the differential equation $x'(t) = L_e x_t$ as $t \geq 0$. In particular, the last point implies $u(t) = x_t \in \mathcal{D}(G_e) = T_0 X_f$ as claimed.

In addition, note that we also have

$$u(0) = \mathcal{Z}_{\tilde{\eta}}(\varphi)(0) = -P_s \varphi + \int_{-\infty}^0 T_e^{\odot *}(-\tau) P_s^{\odot *} l(r_\delta(\overline{F}(\tau,\varphi)))\, d\tau \in C_s^{\odot *}.$$

As a consequence,

$$u(t) = T_e(t)u(0) = T(t)u(0) \in (C_s^{\odot *} \cap T_0 X_f) = C_s \cap T_0 X_f$$

for all $t \geq 0$, such that the last estimate of (2.3) in combination with $\tilde{\eta} + c_s \leq 0$ finally implies

$$\begin{aligned}
\sup_{t\geq 0} e^{\tilde{\eta}t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t)\|_{C^1} &= \sup_{t\geq 0} e^{\tilde{\eta}t} \|u(t)\|_{C^1} \\
&= \sup_{t\geq 0} e^{\tilde{\eta}t} \|T(t)u(0)\|_{C^1} \\
&\leq \sup_{t\geq 0} e^{\tilde{\eta}t} e^{c_s t} \|u(0)\|_{C^1} \\
&= \|u(0)\|_{C^1} \\
&< \infty.
\end{aligned}$$

This is the desired conclusion. □

After analyzing the first improper integral on the right-hand side of Eq. (5.1), we now address the existence of the last two.

Corollary 5.3. *Suppose that $\tilde{\eta} > 0$ satisfies $c_c < \eta \leq \tilde{\eta} < \min\{-c_s, c_u\}$ and let G denote the map which assigns to each point $(\varphi, z) \in X_\delta \times C^1_{\tilde{\eta},\mathbb{R}}$ the curve $\mathbb{R} \ni t \mapsto l(\bar{r}_\delta(t, \varphi, z(t))) \in C^{\odot *}$. Then G maps $X_\delta \times C^1_{\tilde{\eta},\mathbb{R}}$ into $Y_{\tilde{\eta},\mathbb{R}}$.*

Moreover, the induced map

$$G_{\delta\tilde{\eta}} \colon X_\delta \times C^1_{\tilde{\eta},\mathbb{R}} \ni (\varphi, z) \mapsto G(\varphi, z) \in Y_{\tilde{\eta},\mathbb{R}}$$

satisfies

$$\|G_{\delta\tilde{\eta}}(\varphi, z) - G_{\delta\tilde{\eta}}(\varphi, \hat{z})\|_{Y_{\tilde{\eta},\mathbb{R}}} \leq \lambda(\delta)\|z - \hat{z}\|_{C^1_{\tilde{\eta},\mathbb{R}}}$$

for all $\varphi \in X_\delta$ and all $z, \hat{z} \in C^1_{\tilde{\eta},\mathbb{R}}$.

Proof. 1. As the maps l and $\bar{r}_\delta$ are both continuous it is clear that G indeed defines a map from $X_\delta \times C^1_{\tilde{\eta},\mathbb{R}}$ into $C(\mathbb{R}, Y^{\odot *})$. Furthermore, in view of (5.2), we have

$$\sup_{t\in\mathbb{R}} e^{\tilde{\eta}t} \|G(\varphi, z)(t)\|_{C^{\odot *}} = \sup_{t\in\mathbb{R}} e^{\tilde{\eta}t} \|l(\bar{r}_\delta(t, \varphi, z(t)))\|_{C^{\odot *}} \leq \lambda(\delta) \sup_{t\in\mathbb{R}} e^{\tilde{\eta}t} \|z(t)\|_{C^1} = \lambda(\delta)\|z\|_{C^1_{\tilde{\eta},\mathbb{R}}}$$

for all $(\varphi, z) \in X_\delta \times C^1_{\tilde{\eta},\mathbb{R}}$. Therefore, it follows that $G(X_\delta \times C^1_{\tilde{\eta},\mathbb{R}}) \subset Y_{\tilde{\eta},\mathbb{R}}$.

2. Given $\varphi \in X_\delta$ and $z, \hat{z} \in C^1_{\tilde{\eta},\mathbb{R}}$, estimate (5.3) implies

$$\begin{aligned}
\|G_{\delta\tilde{\eta}}(\varphi, z) - G_{\delta\tilde{\eta}}(\varphi, \hat{z})\|_{Y_{\tilde{\eta},\mathbb{R}}} &= \sup_{t\in\mathbb{R}} e^{\tilde{\eta}t} \|l(\bar{r}_\delta(t, \varphi, z(t))) - l(\bar{r}_\delta(t, \varphi, \hat{z}(t)))\|_{C^{\odot *}} \\
&\leq \lambda(\delta) \sup_{t\in\mathbb{R}} e^{\tilde{\eta}t} \|z(t) - \hat{z}(t)\|_{C^1} \\
&= \lambda(\delta)\|z - \hat{z}\|_{C^1_{\tilde{\eta},\mathbb{R}}}.
\end{aligned}$$

Hence, $G_{\delta\tilde{\eta}}$ is Lipschitz continuous with Lipschitz constant $\lambda(\delta)$, and the claim follows. □

Given $(\varphi, z) \in X_\delta \times C^1_{\eta,\mathbb{R}}$, the existence of the last two integrals on the right-hand side of Eq. (5.1) now follows from Proposition 2.4 and the corollary after it. Indeed, for those (φ, z) the sum of the two integrals coincides with the value $(\mathcal{K}^+_\eta \circ G_{\delta,\eta})(\varphi, z)(t)$ as $t \in \mathbb{R}$. Moreover, introducing the map

$$\mathcal{R}_\eta \colon C^1_{\eta,\mathbb{R}} \times X_\delta \to C^1_{\eta,\mathbb{R}}$$

where

$$\mathcal{R}_\eta(z, \varphi) := \mathcal{Z}_\eta(\varphi) + (\mathcal{K}^+_\eta \circ G_{\delta,\eta})(\varphi, z),$$

we obtain a representation of the right-hand side of Eq. (5.1) in the Banach space $C^1_{\eta,\mathbb{R}}$. Consequently, given $\varphi \in X_\delta$, a solution z of Eq. (5.1) in $C^1_{\eta,\mathbb{R}}$ is a fixed point of the map $\mathcal{R}_\eta(\,\cdot\,,\varphi)$. Below we prove that each $\varphi \in X_\delta$ leads to a uniquely determined solution z of $z = \mathcal{R}_\eta(z,\varphi)$ in $C^1_{\eta,\mathbb{R}}$.

Proposition 5.4. *For each $\varphi \in X_\delta$, the induced map $\mathcal{R}_\eta(\,\cdot\,,\varphi)\colon C^1_{\eta,\mathbb{R}} \to C^1_{\eta,\mathbb{R}}$ has a uniquely determined fixed point $z = z(\varphi)$.*

Proof. Let $\varphi \in X_\delta$ be given. As $\|\mathcal{Z}_\eta(\varphi)\|_{C^1_{\eta,\mathbb{R}}} < \infty$ due to Proposition 5.2 there clearly is some real $\gamma > 0$ with $2\gamma > \|\mathcal{Z}_\eta(\varphi)\|_{C^1_{\eta,\mathbb{R}}}$. Combining this with Corollary 5.3 and assumption (4.2), we conclude that

$$\begin{aligned}\|\mathcal{R}_\eta(z,\varphi)\|_{C^1_{\eta,\mathbb{R}}} &\le \|\mathcal{Z}_\eta(\varphi)\|_{C^1_{\eta,\mathbb{R}}} + \|(\mathcal{K}^+_\eta \circ G_{\delta,\eta})(\varphi,z)\|_{C^1_{\eta,\mathbb{R}}}\\ &\le \frac{\gamma}{2} + \|\mathcal{K}^+_\eta\|\|G_{\delta,\eta}(\varphi,z)\|_{Y_{\eta,\mathbb{R}}}\\ &\le \frac{\gamma}{2} + \lambda(\delta)\|\mathcal{K}^+_\eta\|\|z\|_{C^1_{\eta,\mathbb{R}}}\\ &\le \frac{\gamma}{2} + \frac{\gamma}{2}\\ &= \gamma\end{aligned}$$

as long as $z \in C^1_{\eta,\mathbb{R}}$ satisfies $\|z\|_{C^1_{\eta,\mathbb{R}}} \le \gamma$. Hence, the map $\mathcal{R}_\eta(\,\cdot\,,\varphi)$ maps the closed ball $\{z \in C^1_{\eta,\mathbb{R}} \mid \|z\|_{C^1_{\eta,\mathbb{R}}} \le \gamma\}$ of radius γ about $0 \in C^1_{\eta,\mathbb{R}}$ into itself. Similarly, we see

$$\begin{aligned}\|\mathcal{R}_\eta(z,\varphi) - \mathcal{R}_\eta(\tilde{z},\varphi)\|_{C^1_{\eta,\mathbb{R}}} &= \|(\mathcal{K}^+_\eta \circ G_{\delta,\eta})(\varphi,z) - (\mathcal{K}^+_\eta \circ G_{\delta,\eta})(\varphi,\tilde{z})\|_{C^1_{\eta,\mathbb{R}}}\\ &\le \|\mathcal{K}^+_\eta\|\|G_{\delta,\eta}(\varphi,z) - G_{\delta,\eta}(\varphi,\tilde{z})\|_{Y_{\eta,\mathbb{R}}}\\ &\le \lambda(\delta)\|\mathcal{K}^+_\eta\|\|z-\tilde{z}\|_{C^1_{\eta,\mathbb{R}}}\\ &\le \frac{1}{2}\|z-\tilde{z}\|_{C^1_{\eta,\mathbb{R}}}\end{aligned}$$

for all $z, \tilde{z} \in C^1_{\eta,\mathbb{R}}$. Consequently, $\mathcal{R}_\eta(\,\cdot\,,\varphi)$ is a contractive self-mapping of the closed subset $\{z \in C^1_{\eta,\mathbb{R}} \mid \|z\|_{C^1_{\eta,\mathbb{R}}} \le \gamma\}$ of the Banach space $C^1_{\eta,\mathbb{R}}$. Thus, the Banach contraction principle shows the existence of a unique $z = z(\varphi) \in C^1_{\eta,\mathbb{R}}$ with $z = \mathcal{R}_\eta(z,\varphi)$. □

Remark 5.5. Observe that the choice of the reals $c_c < \eta < \min\{-c_s, c_u\}$ and $0 < \delta < \delta_1$ satisfying condition (4.2), that is, $\lambda(\delta)\|\mathcal{K}^+_\eta\| < 1/2$, is the essential hypothesis for the proof of the last proposition. Now recall that by Corollary 2.6 we have

$$\|\mathcal{K}^+_\eta\| \le \tilde{c}(\eta)$$

where $\tilde{c}\colon (c_c, \min\{-c_s, c_u\}) \to [0,\infty)$ is given by

$$\tilde{c}(\tilde{\eta}) := K(1 + e^{\eta h}\|L_e\|)\left(\frac{\|P^{\odot *}_u\|}{c_u + \eta} + \frac{P^{\odot *}_c}{\eta - c_c} - \frac{P^{\odot *}_s}{c_s + \eta}\right) + e^{\eta h}(\|P^{\odot *}_s\| + \|P^{\odot *}_{cu}\|).$$

The function $\tilde{c}$ is clearly continuous. For this reason, under condition (4.2) we have

$$\lambda(\delta)\|\mathcal{K}^+_{\tilde{\eta}}\| < \frac{1}{2}$$

for all $\eta \leq \tilde{\eta} < \min\{-c_s, c_u\}$ with $\tilde{\eta} - \eta \geq 0$ small enough. Hence, after fixing some real $\eta \leq \tilde{\eta} < \min\{-c_s, c_u\}$ with $\tilde{\eta} - \eta \geq 0$ sufficiently small, one can draw exactly the same conclusion as in Proposition 5.4; that is, for each $\varphi \in X_\delta$ the map $\mathcal{R}_{\tilde{\eta}}(\,\cdot\,, \varphi) \colon C^1_{\tilde{\eta},\mathbb{R}} \to C^1_{\tilde{\eta},\mathbb{R}}$ given by

$$\mathcal{R}_{\tilde{\eta}}(z, \varphi) := \mathcal{Z}_{\tilde{\eta}}(\varphi) + (\mathcal{K}^+_{\tilde{\eta}} \circ G_{\delta,\tilde{\eta}})(\varphi, z),$$

as $z \in C^1_{\tilde{\eta},\mathbb{R}}$ has a uniquely determined fixed point in $C^1_{\tilde{\eta},\mathbb{R}}$.

Our next goal is to show that the fixed point $z(\varphi) \in C^1_{\eta,\mathbb{R}}$ from the last statement depends continuously on $\varphi \in X_\delta$. For this purpose, we need some auxiliary results. We begin with the proof that the map $\mathcal{Z}_{\tilde{\eta}} \colon X_\delta \to C^1_{\tilde{\eta},\mathbb{R}}$ is continuous.

Proposition 5.6. *The map $\mathcal{Z}_{\tilde{\eta}} \colon X_\delta \to C^1_{\tilde{\eta},\mathbb{R}}$ from Proposition 5.2 is continuous.*

Proof. 1. Given $\varphi, \psi \in X_\delta$, we trivially have

$$\|\mathcal{Z}_{\tilde{\eta}}(\varphi) - \mathcal{Z}_{\tilde{\eta}}(\psi)\|_{C^1_{\tilde{\eta},\mathbb{R}}} \leq \sup_{t\leq 0} e^{\tilde{\eta}t}\|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - \mathcal{Z}_{\tilde{\eta}}(\psi)(t)\|_{C^1} + \sup_{t\geq 0} e^{\tilde{\eta}t}\|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - \mathcal{Z}_{\tilde{\eta}}(\psi)(t)\|_{C^1}.$$

Hence, it suffices to show

$$\sup_{t\leq 0} e^{\tilde{\eta}t}\|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - \mathcal{Z}_{\tilde{\eta}}(\psi)(t)\|_{C^1} \to 0$$

and

$$\sup_{t\geq 0} e^{\tilde{\eta}t}\|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - \mathcal{Z}_{\tilde{\eta}}(\psi)(t)\|_{C^1} \to 0$$

as $\varphi \to \psi$. We consider the two expressions separately below.

2. (*Estimate of* $\sup_{t\leq 0} e^{\tilde{\eta}t}\|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - \mathcal{Z}_{\tilde{\eta}}(\psi)(t)\|_{C^1}$.) Let $Q_1, Q_2 \colon \mathbb{R} \to Y^{\odot *}$ be given by

$$Q_1(t) := \begin{cases} l(r_\delta(\overline{F}(t, \varphi))), & t \leq 0, \\ e^{-\tilde{\eta}t} l(r_\delta(\overline{F}(0, \varphi))), & t \geq 0, \end{cases}$$

and

$$Q_2(t) := \begin{cases} l(r_\delta(\overline{F}(t, \psi))), & t \leq 0, \\ e^{-\tilde{\eta}t} l(r_\delta(\overline{F}(0, \psi))), & t \geq 0, \end{cases}$$

respectively. Clearly, both Q_1 and Q_2 are continuous. Moreover, we claim that $Q_1, Q_2 \in Y_{\tilde{\eta},\mathbb{R}}$. For a proof, consider Q_1 first. As shown in the proof of Corollary 5.1, we have

$$\sup_{t\leq 0} e^{\tilde{\eta}t}\|Q_1(t)\|_{C^{\odot *}} < \infty.$$

Next,

$$\sup_{t\geq 0} e^{\tilde{\eta}t}\|Q_1(t)\|_{C^{\odot *}} = \sup_{t\geq 0} e^{\tilde{\eta}t}\|e^{-\tilde{\eta}t} l(r_\delta(\overline{F}(0, \varphi)))\|_{C^{\odot *}} = \|l(r_\delta(\varphi))\|_{C^{\odot *}} < \infty.$$

Thus

$$\sup_{t\in\mathbb{R}} e^{\tilde{\eta}t}\|Q_1(t)\|_{C^{\odot *}} \leq \sup_{t\leq 0} e^{\tilde{\eta}t}\|Q_1(t)\|_{C^{\odot *}} + \sup_{t\geq 0} e^{\tilde{\eta}t}\|Q_2(t)\|_{C^{\odot *}} < \infty,$$

which implies $Q_1 \in Y_{\tilde{\eta},\mathbb{R}}$. Similarly, we see $Q_2 \in Y_{\tilde{\eta},\mathbb{R}}$. In particular, $Q_1 - Q_2 \in Y_{\tilde{\eta},\mathbb{R}}$ and, by combining estimate (2.13) with the fact $\tilde{\eta} \geq \eta > 0$ and Proposition 3.4, we infer

$$
\begin{aligned}
\|Q_1 - Q_2\|_{Y_{\tilde{\eta},\mathbb{R}}} &\le \sup_{t\le 0} e^{\tilde{\eta}t}\|Q_1(t) - Q_2(t)\|_{C^{\odot *}} + \sup_{t\ge 0} e^{\tilde{\eta}t}\|Q_1(t) - Q_2(t)\|_{C^{\odot *}} \\
&= \sup_{t\le 0} e^{\tilde{\eta}t}\|l(r_\delta(\overline{F}(t,\varphi))) - l(r_\delta(\overline{F}(t,\psi)))\|_{C^{\odot *}} \\
&\quad + \sup_{t\ge 0} e^{\tilde{\eta}t}e^{-\tilde{\eta}t}\|l(r_\delta(\overline{F}(0,\varphi))) - l(r_\delta(\overline{F}(0,\psi)))\|_{C^{\odot *}} \\
&\le \sup_{t\le 0} e^{\tilde{\eta}t}\|r_\delta(\overline{F}(t,\varphi)) - r_\delta(\overline{F}(t,\psi))\|_{\mathbb{R}^n} + \|r_\delta(\varphi) - r_\delta(\psi)\|_{\mathbb{R}^n} \\
&\le \sup_{t\le 0} \lambda(\delta)e^{\tilde{\eta}t}\|F^{cu}_\eta(t,\varphi) - F^{cu}_\eta(t,\psi)\|_{C^1} + \lambda(\delta)\|\varphi - \psi\|_{C^1} \\
&= \sup_{t\le 0} \lambda(\delta)e^{\tilde{\eta}t}\|\hat{u}_\eta(\varphi)(t) + \mathcal{C}(\varphi)(t) - \hat{u}_\eta(\psi)(t) + \mathcal{C}(\psi)(t)\|_{C^1} + \lambda(\delta)\|\varphi - \psi\|_{C^1} \\
&\le \lambda(\delta)\sup_{t\le 0} e^{\eta t}\|\hat{u}_\eta(\varphi)(t) - \hat{u}_\eta(\psi)(t)\|_{C^1} \\
&\quad + \sup_{t\le 0} \lambda(\delta)e^{\tilde{\eta}t}\|\mathcal{C}(\varphi)(t) - \mathcal{C}(\psi)(t)\|_{C^1} + \lambda(\delta)\|\varphi - \psi\|_{C^1} \\
&= \lambda(\delta)\left(\|\hat{u}_\eta(\varphi) - \hat{u}_\eta(\psi)\|_{C^1_\eta} + \sup_{t\le 0} e^{\tilde{\eta}t}\|P_s\varphi - P_s\psi\|_{C^1} + \|\varphi - \psi\|_{C^1}\right) \\
&\le \lambda(\delta)\Big(\operatorname{Lip}(\hat{u}_\eta)\|\varphi - \psi\|_{C^1} + \|P_s\|\|\varphi - \psi\|_{C^1} + \|\varphi - \psi\|_{C^1}\Big),
\end{aligned}
$$

that is,

$$
\|Q_1 - Q_2\|_{Y_{\tilde{\eta},\mathbb{R}}} \le \lambda(\delta)\left(\operatorname{Lip}(\hat{u}_\eta) + \|P_s\| + 1\right)\|\varphi - \psi\|_{C^1}.
$$

Now, observe that from Proposition 2.4 it follows that both $\mathcal{K}^1_{\tilde{\eta}}Q_1$ and $\mathcal{K}^1_{\tilde{\eta}}Q_2$ are well-defined and belong to $C^1_{\tilde{\eta},\mathbb{R}}$. Further, in view of the last estimate

$$
\|\mathcal{K}^1_{\tilde{\eta}}Q_1 - \mathcal{K}^1_{\tilde{\eta}}Q_2\|_{C^1_{\tilde{\eta},\mathbb{R}}} \le \|\mathcal{K}^1_{\tilde{\eta}}\|\|Q_1 - Q_2\|_{Y_{\tilde{\eta},\mathbb{R}}} \le \lambda(\delta)\|\mathcal{K}^1_{\tilde{\eta}}\|(\operatorname{Lip}(\hat{u}_\eta) + \|P_s\| + 1)\|\varphi - \psi\|_{C^1}.
$$

Therefore,

$$
\begin{aligned}
&\sup_{t\le 0} e^{\tilde{\eta}t}\|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - \mathcal{Z}_{\tilde{\eta}}(\psi)(t)\|_{C^1} \\
&\quad\le \sup_{t\le 0} e^{\tilde{\eta}t}\|P_s\overline{F}(t,\varphi) - P_s\overline{F}(t,\psi)\|_{C^1} \\
&\qquad + \sup_{t\le 0} e^{\tilde{\eta}t}\left\|\int_{-\infty}^t T_e^{\odot *}(t-\tau)P_s^{\odot *}l(r_\delta(\overline{F}(\tau,\varphi))\,d\tau - \int_{-\infty}^t T_e^{\odot *}(t-\tau)P_s^{\odot *}r_\delta(\overline{F}(\tau,\psi))\,d\tau\right\|_{C^1} \\
&\quad= \sup_{t\le 0} e^{\tilde{\eta}t}\|P_sF^{cu}_\eta(t,\varphi) - P_sF^{cu}_\eta(t,\psi)\|_{C^1} + \sup_{t\le 0} e^{\tilde{\eta}t}\left\|(\mathcal{K}^1Q_1)(t) - (\mathcal{K}^1Q_2)(t)\right\|_{C^1} \\
&\quad\le \sup_{t\le 0} e^{\tilde{\eta}t}\|P_s\,\mathcal{C}(\varphi)(t) - P_s\,\mathcal{C}(\psi)(t)\|_{C^1} + \sup_{t\in\mathbb{R}} e^{\tilde{\eta}t}\|(\mathcal{K}^1Q_1)(t) - (\mathcal{K}^1Q_2)(t)\|_{C^1} \\
&\quad= \sup_{t\le 0} e^{\tilde{\eta}t}\|P_s\varphi - P_s\psi\|_{C^1} + \|\mathcal{K}^1_{\tilde{\eta}}Q_1 - \mathcal{K}^1_{\tilde{\eta}}Q_2\|_{C^1_{\tilde{\eta},\mathbb{R}}} \\
&\quad\le \|P_s\|\|\varphi - \psi\|_{C^1} + \lambda(\delta)\|\mathcal{K}^1_{\tilde{\eta}}\|\left(\operatorname{Lip}(\hat{u}_\eta) + \|P_s\| + 1\right)\|\varphi - \psi\|_{C^1} \\
&\quad= \Big(\|P_s\| + \lambda(\delta)\|\mathcal{K}^1_{\tilde{\eta}}\|(\operatorname{Lip}(\hat{u}_\eta) + \|P_s\| + 1)\Big)\|\varphi - \psi\|_{C^1}.
\end{aligned}
$$

3. (*Estimate of* $\sup_{t\ge 0} e^{\tilde{\eta}t}\|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - \mathcal{Z}_{\tilde{\eta}}(\psi)(t)\|_{C^1}$.) Define the maps $u_1, u_2\colon [0,\infty) \to C^1$ by $u_1(t) := \mathcal{Z}_{\tilde{\eta}}(\varphi)(t)$ and $u_2(t) := \mathcal{Z}_{\tilde{\eta}}(\psi)(t)$, respectively. As shown in part 3 of the proof of

Proposition 5.2 both u_1 and u_2 are solutions of $u(t) = T_e(t-s)u(s)$ for all $0 \leq s \leq t < \infty$. Moreover, $u(t) := u_1(t) - u_2(t)$ satisfies $u(t) = T_e(t-s)u(s)$ for all $0 \leq s \leq t < \infty$ as well. Following the proof of Proposition 5.2 further, we first see

$$\sup_{t\geq 0} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - \mathcal{Z}_{\tilde{\eta}}(\psi)(t)\|_{C^1} \leq \|u(0)\|_{C^1} = \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(0) - \mathcal{Z}_{\tilde{\eta}}(\psi)(0)\|_{C^1}$$

and then, in view of the estimate derived in the last step,

$$\begin{aligned}\sup_{t\geq 0} e^{\tilde{\eta}t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - \mathcal{Z}_{\tilde{\eta}}(\psi)(t)\|_{C^1} &\leq \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(0) - \mathcal{Z}_{\tilde{\eta}}(\psi)(0)\|_{C^1} \\ &\leq \sup_{t\leq 0} e^{\tilde{\eta}t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi)(t) - \mathcal{Z}_{\tilde{\eta}}(\psi)(t)\|_{C^1} \\ &\leq c\|\varphi - \psi\|_{C^1}\end{aligned}$$

with some $c > 0$.

4. By part 2 and part 3 it follows that

$$\sup_{t\leq 0} e^{\tilde{\eta}t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi) - \mathcal{Z}_{\tilde{\eta}}(\psi)\|_{C^1} \to 0 \quad \text{and} \quad \sup_{t\geq 0} e^{\tilde{\eta}t} \|\mathcal{Z}_{\tilde{\eta}}(\varphi) - \mathcal{Z}_{\tilde{\eta}}(\psi)\|_{C^1} \to 0$$

as $\varphi \to \psi$. Hence, the first part of the proof implies $\mathcal{Z}_{\tilde{\eta}}(\varphi) \to \mathcal{Z}_{\tilde{\eta}}(\psi)$ in $C^1_{\tilde{\eta},\mathbb{R}}$ for $\varphi \to \psi$. This shows the continuity of $\mathcal{Z}_{\tilde{\eta}}$. □

Remark 5.7. Observe that in view of the proof the map $X_\delta \ni \varphi \mapsto \mathcal{Z}_{\tilde{\eta}}(\varphi) \in C^1_{\tilde{\eta},\mathbb{R}}$ is not only continuous but Lipschitz continuous with a global Lipschitz constant.

Next, we show that, given $\varphi \in X_\delta$, the uniquely determined fixed point of $\mathcal{R}_\eta(\,\cdot\,,\varphi)$ is also a fixed point of $\mathcal{R}_{\tilde{\eta}}(\,\cdot\,,\varphi)$ for $\tilde{\eta} > 0$ with $\tilde{\eta} - \eta > 0$ sufficiently small.

Proposition 5.8. *Let $0 < \eta \leq \tilde{\eta} < \min\{-c_s, c_u\}$ with $\lambda(\delta)\|\mathcal{K}^+_{\tilde{\eta}}\| < 1/2$ and $\varphi \in X_\delta$ be given. Further, suppose that $z_1 \in C^1_{\eta,\mathbb{R}}$ is the uniquely determined fixed point of $\mathcal{R}_\eta(\,\cdot\,,\varphi)\colon C^1_{\eta,\mathbb{R}} \to C^1_{\eta,\mathbb{R}}$, and that $z_2 \in C^1_{\tilde{\eta},\mathbb{R}}$ is the uniquely determined fixed point of $\mathcal{R}_{\tilde{\eta}}\colon (\,\cdot\,,\varphi)\colon C^1_{\tilde{\eta},\mathbb{R}} \to C^1_{\tilde{\eta},\mathbb{R}}$, which both exist due to Proposition 5.4 and the remark after it. Then $z_1(t) = z_2(t)$ for all $t \in \mathbb{R}$.*

Proof. Under the given assumptions, a straightforward calculation results in

$$\begin{aligned}z_1(t) - z_2(t) &= \int_{-\infty}^{t} T_e^{\odot*}(t-\tau)P_s^{\odot*}l(\bar{r}_\delta(\tau,\varphi,z_1(\tau)) - \bar{r}_\delta(\tau,\varphi,z_2(\tau)))\,d\tau \\ &\quad + \int_{\infty}^{t} T_e^{\odot*}(t-\tau)P_{cu}^{\odot*}l(\bar{r}_\delta(\tau,\varphi,z_1(\tau)) - \bar{r}_\delta(\tau,\varphi,z_2(\tau)))\,d\tau\end{aligned}$$

for all $t \in \mathbb{R}$. Thus, after defining $\tilde{Q}\colon \mathbb{R} \to C^{\odot*}$ by

$$\tilde{Q}(t) := l(\bar{r}_\delta(t,\varphi,z_1(t)) - \bar{r}_\delta(t,\varphi,z_2(t))),$$

we formally obtain

$$z_1(t) - z_2(t) = (\mathcal{K}^1\tilde{Q})(t) + (\mathcal{K}^2\tilde{Q})(t) \tag{5.4}$$

as $t \in \mathbb{R}$.

Next, define $u\colon \mathbb{R} \to C^1$ and $Q\colon \mathbb{R} \to C^{\odot*}$ by

$$u(t) := \begin{cases} z_1(t) - z_2(t), & \text{for } t \leq 0, \\ e^{-\tilde{\eta}t}\,[z_1(t) - z_2(t)]\,, & \text{for } t \geq 0, \end{cases}$$

and

$$Q(t) := \begin{cases} \tilde{Q}(t), & \text{for } t \le 0, \\ e^{-\tilde{\eta} t}\tilde{Q}(t), & \text{for } t \ge 0, \end{cases}$$

respectively. We claim that $u \in C^1_{\tilde{\eta},\mathbb{R}}$ and $Q \in Y_{\tilde{\eta},\mathbb{R}}$. Indeed, as $\tilde{\eta} \ge \eta$ we clearly have $z_1|_{(-\infty,0]}, z_2|_{(\infty,0]} \in C^1_{\tilde{\eta}}$ and so

$$\begin{aligned} \sup_{t\in\mathbb{R}} e^{\tilde{\eta} t}\|u(t)\|_{C^1} &\le \sup_{t\le 0} e^{\tilde{\eta} t}\|z_1(t) - z_2(t)\|_{C^1} + \sup_{t\ge 0} e^{\tilde{\eta} t}\|e^{-\tilde{\eta} t}(z_1(t) - z_2(t))\|_{C^1} \\ &\le \|(z_1 - z_2)|_{(-\infty,0]}(t)\|_{C^1_{\tilde{\eta}}} + \sup_{t\ge 0} e^{\eta t}\|z_1(t)\|_{C^1} + \sup_{t\ge 0} e^{\tilde{\eta} t}\|z_2(t)\|_{C^1} \\ &< \infty, \end{aligned}$$

that is, $u \in C^1_{\tilde{\eta},\mathbb{R}}$. Combining this with estimate (5.3) leads to

$$\begin{aligned} \sup_{t\le 0} e^{\tilde{\eta} t}\|Q(t)\|_{C^{\odot *}} &= \sup_{t\le 0} e^{\tilde{\eta} t}\|l\big(\bar{r}_\delta(t,\varphi,z_1(t)) - \bar{r}_\delta(t,\varphi,z_2(t))\big)\|_{C^{\odot *}} \\ &\le \sup_{t\le 0} e^{\tilde{\eta} t}\|\bar{r}_\delta(t,\varphi,z_1(t)) - \bar{r}_\delta(t,\varphi,z_2(t))\|_{\mathbb{R}^n} \\ &\le \sup_{t\le 0} e^{\tilde{\eta} t}\lambda(\delta)\|z_1(t) - z_2(t)\|_{C^1} \\ &\le \lambda(\delta)\sup_{t\le 0} e^{\tilde{\eta} t}\|u(t)\|_{C^1} \\ &\le \lambda(\delta)\sup_{t\in\mathbb{R}} e^{\tilde{\eta} t}\|u(t)\|_{C^1} \\ &= \lambda(\delta)\|u\|_{C^1_{\tilde{\eta},\mathbb{R}}} \end{aligned}$$

and similarly to

$$\sup_{t\ge 0} e^{\tilde{\eta} t}\|Q(t)\|_{C^{\odot *}} \le \lambda(\delta)\|u\|_{C^1_{\tilde{\eta},\mathbb{R}}}.$$

Hence,

$$\sup_{t\in\mathbb{R}} e^{\tilde{\eta} t}\|Q(t)\|_{C^{\odot *}} \le \max\{\sup_{t\le 0} e^{\tilde{\eta} t}\|Q(t)\|_{C^{\odot *}}, \sup_{t\ge 0} e^{\tilde{\eta} t}\|Q(t)\|_{C^{\odot *}}\} \le \lambda(\delta)\|u\|_{C^1_{\tilde{\eta},\mathbb{R}}}$$

and therefore $Q \in Y_{\tilde{\eta},\mathbb{R}}$ as claimed.

Using the arguments above, especially Eq. (5.4), together with the linearity of the integral operators $\mathcal{K}^i$ one easily finds $u = \mathcal{K}^+_{\tilde{\eta}} Q$ in $C^1_{\tilde{\eta},\mathbb{R}}$. It follows that

$$\|u\|_{C^1_{\tilde{\eta},\mathbb{R}}} = \|\mathcal{K}^+_{\tilde{\eta}} Q\|_{C^1_{\tilde{\eta},\mathbb{R}}} \le \|\mathcal{K}^+_{\tilde{\eta}}\|\|Q\|_{Y_{\tilde{\eta},\mathbb{R}}} \le \lambda(\delta)\|\mathcal{K}^+_{\tilde{\eta}}\|\|u\|_{C^1_{\tilde{\eta},\mathbb{R}}} \le \frac{1}{2}\|u\|_{C^1_{\tilde{\eta},\mathbb{R}}},$$

and so $u = 0 \in C^1_{\tilde{\eta},\mathbb{R}}$. For this reason, we conclude that $z_1(t) = z_2(t)$ for all $t \in \mathbb{R}$, and this finishes the proof. □

The following corollary is the last auxiliary result for the proof that the uniquely determined fixed point of $\mathcal{R}_\eta(\,\cdot\,,\varphi)$ depends continuously on $\varphi \in X_\delta$.

Corollary 5.9. *Suppose that $\tilde{\eta} > \eta$ and $z \in C^1_{\eta,\mathbb{R}} \cap C^1_{\tilde{\eta},\mathbb{R}}$. Then the map $X_\delta \ni \varphi \mapsto G_{\delta\eta}(\varphi, z) \in Y_{\eta,\mathbb{R}}$ with $G_{\delta\eta}$ defined in Corollary 5.3 is continuous.*

Proof. 1. Let $\varphi \in X_\delta$ and $\varepsilon > 0$ be given. Then, in view of $\eta - \tilde{\eta} < 0$ and $\|z\|_{C^1_{\tilde{\eta},\mathbb{R}}} < \infty$ by assumption, we clearly find some $R > 0$ with

$$c_1 := 2\lambda(\delta)e^{(\eta-\tilde{\eta})R}\|z\|_{C^1_{\tilde{\eta},\mathbb{R}}} < \varepsilon.$$

Next, recall from Proposition 3.1 that we have

$$\sup_{0\le t\le R} \mathrm{Lip}(F_\delta(t,\cdot)) < \infty.$$

Therefore, there is some sufficiently small $\delta(R,\varepsilon) > 0$ with the property that both

$$c_2 := 2\lambda(\delta)\cdot\delta(R,\varepsilon)\cdot(\|P_s\| + \mathrm{Lip}(\hat{u}_\eta)) < \varepsilon$$

and

$$c_3 := 2\lambda(\delta)\cdot\delta(R,\varepsilon)\cdot\sup_{0\le t\le R}\mathrm{Lip}(F_\delta(t,\cdot)) < \varepsilon$$

are satisfied. Now, we claim that

$$\|G_{\delta\eta}(\varphi,z) - G_{\delta,\eta}(\psi,z)\|_{Y_{\eta,\mathbb{R}}} < \varepsilon$$

for all $\psi \in X_\delta$ with $\|\varphi - \psi\|_{C^1} < \delta(R,\varepsilon)$. In order to see this claim and so the assertion of the corollary, we show that under given assumptions

$$t \mapsto e^{\eta t}\|G_{\delta\eta}(\varphi,z)(t) - G_{\delta\eta}(\psi,z)(t)\|_{C^{\odot *}}$$

is bounded by c_1 on (R,∞), is bounded by c_2 on $(-\infty,0]$, and bounded by c_3 on $[0,R]$.

2. (*Estimate of* $\sup_{t>R} e^{\eta t}\|G_{\delta\eta}(\varphi,z)(t) - G_{\delta\eta}(\psi,z)\|_{C^{\odot *}}$.) From the assumptions and estimate (5.2) it follows that

$$\begin{aligned}
\sup_{t>R} e^{\eta t}\|G_{\delta\eta}(\varphi,z)(t) - G_{\delta\eta}(\psi,z)(t)\|_{C^{\odot *}} &\le \sup_{t>R} e^{\eta t}\|\overline{r}_\delta(t,\varphi,z(t)) - \overline{r}_\delta(t,\psi,z(t))\|_{\mathbb{R}^n} \\
&\le 2\lambda(\delta)\sup_{t>R} e^{\eta t}\|z(t)\|_{C^1} \\
&= 2\lambda(\delta)\sup_{t>R} e^{(\eta-\tilde{\eta})t}e^{\tilde{\eta}t}\|z(t)\|_{C^1} \\
&\le 2\lambda(\delta)\sup_{t>R} e^{(\eta-\tilde{\eta})t}\|z\|_{C^1_{\tilde{\eta},\mathbb{R}}} \\
&= 2\lambda(\delta)e^{(\eta-\tilde{\eta})R}\|z\|_{C^1_{\tilde{\eta},\mathbb{R}}} \\
&= c_1
\end{aligned}$$

for all $\psi \in X_\delta$.

3. (*Estimate of* $\sup_{t\le 0} e^{\eta t}\|G_{\delta\eta}(\varphi,z) - G_{\delta\eta}(\psi,z)\|_{C^{\odot *}}$.) Let $\psi \in X_\delta$ with $\|\varphi - \psi\|_{C^1} < \delta(R,\varepsilon)$ be given. Using the Lipschitz continuity of r_δ given by (2.13), we first deduce that

$$\begin{aligned}
&\sup_{t\le 0} e^{\eta t}\|G_{\delta\eta}(\varphi,z)(t) - G_{\delta\eta}(\psi,z)(t)\|_{C^{\odot *}} \\
&\quad\le \sup_{t\le 0} e^{\eta t}\|\overline{r}_\delta(t,\varphi,z(t)) - \overline{r}_\delta(t,\psi,z(t))\|_{\mathbb{R}^n} \\
&\quad= \sup_{t\le 0} e^{\eta t}\|r_\delta(\overline{F}(t,\varphi) + z(t)) - r_\delta(\overline{F}(t,\varphi)) - r_\delta(\overline{F}(t,\psi) + z(t)) + r_\delta(\overline{F}(t,\psi))\|_{\mathbb{R}^n} \\
&\quad\le \sup_{t\le 0} e^{\eta t}\|r_\delta(\overline{F}(t,\varphi) + z(t)) - r_\delta(\overline{F}(t,\psi) + z(t))\|_{\mathbb{R}^n} + \sup_{t\le 0} e^{\eta t}\|r_\delta(\overline{F}(t,\varphi)) - r_\delta(\overline{F}(t,\psi))\|_{\mathbb{R}^n} \\
&\quad\le 2\lambda(\delta)\sup_{t\le 0} e^{\eta t}\|\overline{F}(t,\varphi) - \overline{F}(t,\psi)\|_{\mathbb{R}^n} \\
&\quad= 2\lambda(\delta)\sup_{t\le 0} e^{\eta t}\|F^{cu}_\eta(t,\varphi) - F^{cu}_\eta(t,\psi)\|_{\mathbb{R}^n}.
\end{aligned}$$

Now we may proceed similarly as in part 4 of the proof of Proposition 5.6 to conclude that

$$\sup_{t\leq 0} e^{\eta t}\|F_\eta^{cu}(t,\varphi) - F_\eta^{cu}(t,\psi)\|_{\mathbb{R}^n} \leq (\mathrm{Lip}(\hat{u}_\eta) + \|P_s\|)\|\varphi-\psi\|_{C^1}.$$

Combining these finally yields

$$\begin{aligned}\sup_{t\leq 0} e^{\eta t}\|G_{\delta\eta}(\varphi,z)(t) - G_{\delta\eta}(\psi,z)(t)\|_{C^{\odot *}} &\leq 2\lambda(\delta)(\|P_s\| + \mathrm{Lip}(\hat{u}_\eta))\|\varphi-\psi\|_{C^1}\\ &< 2\lambda(\delta)(\|P_s\| + \mathrm{Lip}(\hat{u}_\eta))\delta(R,\varepsilon)\\ &= c_2.\end{aligned}$$

4. (*Estimate of* $\sup_{0\leq t\leq R} e^{\eta t}\|G_{\delta\eta}(\varphi,z)(t) - G_{\delta\eta}(\psi,t)\|_{C^{\odot *}}$.) Using once more the Lipschitz continuity of the map r_δ, we get

$$\begin{aligned}&\sup_{0\leq t\leq R} e^{\eta t}\|G_{\delta\eta}(\varphi,z)(t) - G_{\delta\eta}(\psi,z)(t)\|_{C^{\odot *}}\\ &\quad\leq \sup_{0\leq t\leq R} e^{\eta t}\|\overline{r}_\delta(t,\varphi,z(t)) - \overline{r}_\delta(t,\psi,z(t))\|_{\mathbb{R}^n}\\ &\quad\leq e^{\eta R}\sup_{0\leq t\leq R}\|r_\delta(\overline{F}(t,\varphi)+z(t)) - r_\delta(\overline{F}(t,\varphi)) - r_\delta(\overline{F}(t,\psi)+z(t)) + r_\delta(\overline{F}(t,\psi))\|_{\mathbb{R}^n}\\ &\quad\leq e^{\eta R}\sup_{0\leq t\leq R}\left(\|r_\delta(\overline{F}(t,\varphi)+z(t)) - r_\delta(\overline{F}(t,\psi)+z(t))\|_{\mathbb{R}^n} + \|r_\delta(\overline{F}(t,\varphi)) - r_\delta(\overline{F}(t,\psi))\|_{\mathbb{R}^n}\right)\\ &\quad\leq 2\lambda(\delta)\sup_{0\leq t\leq R}\|\overline{F}(t,\varphi) - \overline{F}(t,\psi)\|_{C^1}\\ &\quad\leq 2\lambda(\delta)\sup_{0\leq t\leq R}\|F_\delta(t,\varphi) - F_\delta(t,\psi)\|_{C^1}\end{aligned}$$

for all $\psi\in X_\delta$. Hence, if $\|\varphi-\psi\|_{C^1} < \delta(\mathbb{R},\varepsilon)$ then

$$\begin{aligned}\sup_{0\leq t\leq R} e^{\eta t}\|G_{\delta\eta}(\varphi,z)(t) - G_{\delta\eta}(\psi,z)(t)\|_{C^{\odot *}} &\leq 2\lambda(\delta)\sup_{0\leq t\leq R}\|F_\delta(t,\varphi) - F_\delta(t,\psi)\|_{C^1}\\ &\leq 2\lambda(\delta)\sup_{0\leq t\leq R}\mathrm{Lip}(F_\delta(t,\cdot))\|\varphi-\psi\|_{C^1}\\ &\leq 2\lambda(\delta)\cdot\delta(R,\varepsilon)\cdot\sup_{0\leq t\leq R}\mathrm{Lip}(F_\delta(t,\cdot))\\ &= c_3\end{aligned}$$

as claimed. □

Now we are in the position to state and prove the continuous dependence of the fixed point of the map $\mathcal{R}_\eta(\,\cdot\,,\varphi)$ on the parameter $\varphi\in X_\delta$.

Proposition 5.10. *Let $z_\eta\colon X_\delta\to C^1_{\eta,\mathbb{R}}$ denote the solution operator of the parameter dependent contraction from Proposition 5.4; that is, $z_\eta(\varphi) = \mathcal{R}_\eta(z_\eta(\varphi),\varphi)$ for all $\varphi\in X_\delta$. Then z_η is continuous.*

Proof. Let $\varphi\in X_\delta$ be given. Then the definition of $\mathcal{R}_\eta$ together with Corollary 5.3 imply that for all $\psi \in X_\delta$

$$\begin{aligned}
&\|z_\eta(\varphi) - z_\eta(\psi)\|_{C^1_{\eta,\mathbb{R}}} \\
&\quad = \|\mathcal{Z}_\eta(\varphi) + (\mathcal{K}_\eta^+ \circ G_{\delta,\eta})(\varphi, z_\eta(\varphi)) - \mathcal{Z}_\eta(\psi) - (\mathcal{K}_\eta^+ \circ G_{\delta,\eta})(\psi, z_\eta(\psi))\|_{C^1_{\eta,\mathbb{R}}} \\
&\quad \leq \|(\mathcal{K}_\eta^+ \circ G_{\delta,\eta})(\varphi, z_\eta(\varphi)) - (\mathcal{K}_\eta^+ \circ G_{\delta,\eta})(\psi, z_\eta(\varphi))\|_{C^1_{\eta,\mathbb{R}}} \\
&\qquad + \|\mathcal{Z}_\eta(\varphi) + (\mathcal{K}_\eta^+ \circ G_{\delta,\eta})(\psi, z_\eta(\varphi)) - \mathcal{Z}_\eta(\psi) - (\mathcal{K}_\eta^+ \circ G_{\delta,\eta})(\psi, z_\eta(\psi))\|_{C^1_{\eta,\mathbb{R}}} \\
&\quad \leq \|\mathcal{K}_\eta^+\| \|G_{\delta,\eta}(\varphi, z_\eta(\varphi)) - G_{\delta,\eta}(\psi, z_\eta(\varphi))\|_{Y_{\eta,\mathbb{R}}} + \|\mathcal{Z}_\eta(\varphi) - \mathcal{Z}_\eta(\psi)\|_{C^1_{\eta,\mathbb{R}}} \\
&\qquad + \|\mathcal{K}_\eta^+\| \|G_{\delta,\eta}(\psi, z_\eta(\varphi)) - G_{\delta,\eta}(\psi, z_\eta(\psi))\|_{C^1_{\eta,\mathbb{R}}} \\
&\quad \leq \|\mathcal{K}_\eta^+\| \|G_{\delta,\eta}(\varphi, z_\eta(\varphi)) - G_{\delta,\eta}(\psi, z_\eta(\varphi))\|_{Y_{\eta,\mathbb{R}}} + \|\mathcal{Z}_\eta(\varphi) - \mathcal{Z}_\eta(\psi)\|_{C^1_{\eta,\mathbb{R}}} \\
&\qquad + \lambda(\delta) \|\mathcal{K}_\eta^+\| \|z_\eta(\varphi) - z_\eta(\psi)\|_{C^1_{\eta,\mathbb{R}}},
\end{aligned}$$

that is,

$$\begin{aligned}
\|z_\eta(\varphi) - z_\eta(\psi)\|_{C^1_{\eta,\mathbb{R}}} \leq{}& \frac{\|\mathcal{K}_\eta^+\|}{1 - \lambda(\delta)\|\mathcal{K}_\eta^+\|} \|G_{\delta,\eta}(\varphi, z_\eta(\varphi)) - G_{\delta,\eta}(\psi, z_\eta(\varphi))\|_{Y_{\eta,\mathbb{R}}} \\
&+ \frac{1}{1 - \lambda(\delta)\|\mathcal{K}_\eta^+\|} \|\mathcal{Z}_\eta(\varphi) - \mathcal{Z}_\eta(\psi)\|_{C^1_{\eta,\mathbb{R}}}.
\end{aligned}$$

Fix some $\eta < \tilde{\eta} < \min\{-c_s, c_u\}$ with $\|\mathcal{K}_{\tilde{\eta}}^+\|\lambda(\delta) < 1/2$, which is possible due to Remark 5.5. By Proposition 5.8, we see $z_\eta(\varphi) = z_{\tilde{\eta}}(\varphi) \in C^1_{\eta,\mathbb{R}} \cap C^1_{\tilde{\eta},\mathbb{R}}$. Hence, Corollary 5.9 yields that the map

$$X_\delta \ni \psi \mapsto G_{\delta,\eta}(\psi, z_\eta(\varphi)) \in Y_{\eta,\mathbb{R}}$$

is continuous. In addition, due to Proposition 5.6 the map $X_\delta \ni \psi \mapsto \mathcal{Z}_\eta(\psi) \in C^1_{\eta,\mathbb{R}}$ is continuous as well. Therefore, from the estimate above for $\|z_\eta(\varphi) - z_\eta(\psi)\|_{C^1_{\eta,\mathbb{R}}}$ it follows that $z_\eta(\psi) \to z_\eta(\varphi)$ as $\psi \to \varphi$. This proves that z_η is continuous at φ. □

After having established all the necessary preparations, we are now able to prove Theorem 4.1 about an attraction property of the global center-unstable manifolds.

Proof of Theorem 4.1. 1. For each $\varphi \in X_\delta$ let

$$H^\eta_{cu}(\varphi) := \varphi + z_\eta(\varphi)(0)$$

with $z_\eta(\varphi) \in C^1_{\eta,\mathbb{R}}$ introduced in the last proposition. Observe that we have $H^\eta_{cu}(\varphi) \in W^\eta$ for all $\varphi \in X_\delta$. Indeed, by Proposition 5.4, $z_\eta(\varphi) \in C^1_\eta$ is a global solution of Eq. (4.9). Therefore, Lemma 4.4 yields that $\overline{F}(\,\cdot\,, \varphi) + z_\eta(\varphi)$ with $\overline{F}$ from Proposition 4.3 satisfies Eq. (4.8), and so in view of Lemma 4.2 it follows that

$$H^\eta_{cu}(\varphi) = \varphi + z_\eta(\varphi)(0) = F_\delta(0, \varphi) + z_\eta(\varphi)(0) = \overline{F}(0, \varphi) + z_\eta(\varphi)(0) \in W^\eta.$$

Consequently, H^η_{cu} forms a map from X_δ into the global center-unstable manifold W^η. Moreover, as a sum of the two continuous maps $\varphi \mapsto \varphi$ and $\varphi \mapsto z_\eta(\varphi)$ it is clearly continuous as well.

2. Now consider some fixed $\varphi \in X_\delta$ and assume that $\psi \in W^\eta$ is such that estimate (4.4) is satisfied. By combining Proposition 4.3 with Lemma 4.2, we find some $z \in C^1_{\eta,\mathbb{R}}$ with the

property that $\psi = \varphi + z(0)$ and that $\overline{F}(\,\cdot\,, \varphi) + z$ is a solution of Eq. (4.5). Hence, by Lemma 4.4, z satisfies Eq. (4.9). But then Proposition 5.4 yields $z = z_\eta(\varphi)$ and so

$$\psi = \varphi + z(0) = \varphi + z_\eta(\varphi)(0) = H^\eta_{cu}(\varphi).$$

This shows one direction of the statement.

3. On the other hand, suppose that $\psi = H^\eta_{cu}(\varphi)$ for $(\varphi, \psi) \in X_\delta \times W^\eta$. Then it follows that $\psi = \varphi + z_\eta(\varphi)(0)$, where $z_\eta(\varphi) \in C^1_{\eta,\mathbb{R}}$ is such that $\overline{F}(\,\cdot\,, \varphi) + z_\eta(\varphi)$ is a solution of Eq. (4.8) due to Lemma 4.4. In particular, we have $(\overline{F}(\,\cdot\,, \varphi) + z_\eta(\varphi))|_{(-\infty,0]} \in C^1_\eta$ and, in consideration of the uniqueness of solutions, $\overline{F}(t, \varphi) + z_\eta(\varphi)(t) = \overline{F}(t, \psi)$ for all $t \geq 0$. Consequently,

$$\begin{aligned}
\sup_{t\geq 0} e^{\eta t}\|F_\delta(t,\psi) - F_\delta(t,\varphi)\|_{C^1} &= \sup_{t\geq 0} e^{\eta t}\|\overline{F}(t,\psi) - \overline{F}(t,\varphi)\|_{C^1} \\
&= \sup_{t\geq 0} e^{\eta t}\|\overline{F}(t,\varphi) + z_\eta(\varphi)(t) - \overline{F}(t,\varphi)\|_{C^1} \\
&= \sup_{t\geq 0} e^{\eta t}\|z_\eta(\varphi)(t)\|_{C^1} \\
&\leq \|z_\eta(\varphi)\|_{C^1_{\eta,\mathbb{R}}} \\
&< \infty,
\end{aligned}$$

which proves the other direction of the assertion. □

We close this section with a consequence of Theorem 4.1.

Corollary 5.11. *Under the assumptions of Theorem 4.1 and with the map $H^\eta_{cu}\colon X_\delta \to W^\eta$ defined in its proof, the following holds: For each $\psi \in W^\eta$ and for each $\varepsilon > 0$ there exists some $\tilde{\delta} > 0$ such that*

$$\|F_\delta(t,\varphi) - F_\delta(t, H^\eta_{cu}(\varphi))\|_{C^1} \leq \varepsilon e^{-\eta t}$$

for all $t \geq 0$ and all $\varphi \in X_\delta$ with $\|\varphi - \psi\|_{C^1} < \tilde{\delta}$.

Proof. To begin with, observe that in consideration of the definition of H^η_{cu}, of Lemma 4.4, and of the uniqueness of solutions we have

$$F_\delta(t, H^\eta_{cu}(\varphi)) - F_\delta(t,\varphi) = F_\delta(t,\varphi) + z_\eta(\varphi)(t) - F_\delta(t,\varphi) = z_\eta(\varphi)(t) \tag{5.5}$$

for all $t \geq 0$ and all $\varphi \in X_\delta$. Next, it is easily seen that $H^\eta_{cu}(\varphi) = \varphi$ for all $\varphi \in W^\eta$. Hence,

$$z_\eta(\varphi)(t) = 0 \tag{5.6}$$

as $(t, \varphi) \in [0, \infty) \times W^\eta$.

Now let $\psi \in W^\eta$ and $\varepsilon > 0$ be given. By the continuity of the map z_η due to Proposition 5.10, we clearly find some $\tilde{\delta} > 0$ such that, for all $\varphi \in X_\delta$ with $\|\varphi - \psi\|_{C^1} < \tilde{\delta}$,

$$\|z_\eta(\varphi) - z_\eta(\psi)\|_{C^1_{\eta,\mathbb{R}}} < \varepsilon$$

holds. Hence, in view of Eq. (5.6), it follows that

$$\sup_{t\geq 0} e^{\eta t}\|z_\eta(\varphi)(t)\|_{C^1} = \sup_{t\geq 0} e^{\eta t}\|z_\eta(\varphi)(t) - z_\eta(\psi)(t)\|_{C^1} \leq \|z_\eta(\varphi) - z_\eta(\psi)\|_{C^1_{\eta,\mathbb{R}}} \leq \varepsilon$$

for all $\varphi \in X_\delta$ satisfying $\|\varphi - \psi\|_{C^1} < \tilde{\delta}$. Combining this with Eq. (5.5) finally shows that for $\varphi \in X_\delta$ with $\|\varphi - \psi\|_{C^1} < \tilde{\delta}$,

$$\begin{aligned}\|F_\delta(t, H^\eta_{cu}(\varphi)) - F_\delta(t,\varphi)\|_{C^1} &\leq e^{-\eta t} \sup_{t\geq 0} e^{\eta t} \|F_\delta(t, H^\eta_{cu}(\varphi)) - F_\delta(t,\varphi)\|_{C^1} \\ &= e^{-\eta t} \sup_{t\geq 0} e^{\eta t} \|z_\eta(\varphi)(t)\|_{C^1} \\ &\leq \varepsilon e^{-\eta t}.\end{aligned}$$

□

6 Proof of Theorem 1.2

In the following we use the attraction property of the global center-unstable manifolds obtained in the last sections to give a proof for Theorem 1.2 asserting an attraction property of local center-unstable manifolds.

Given the assumptions of Theorem 1.1, and thus of Theorem 1.2 as well, we clearly find constants $\eta > 0$ with $c_c < \eta < \min\{-c_s, c_u\}$ and $0 < \delta < \delta_1$ such that the conditions of Theorem 4.1 are satisfied. Now, set

$$\begin{aligned} C_{cu,0} &:= \{\varphi \in C_{cu} \mid \|\varphi\|_1 < \delta\}, \\ C^1_{s,0} &:= \left\{\varphi \in C^1_s \mid \|\varphi\|_1 < \delta\right\}, \\ N_{cu} &:= C_{cu,0} + C^1_{s,0}, \\ w_{cu} &:= w^\eta|_{C_{cu,0}}, \end{aligned}$$

and

$$W_{cu} := \{\varphi + w_{cu}(\varphi) \mid \varphi \in C_{cu,0}\},$$

where the map w^η is defined by Eq. (2.18). With these definitions Theorem 1.1 follows as shown in [5] in detail. In particular, we have $\varphi_0 = 0 \in W_{cu} \subset W^\eta$ and $r_\delta(\varphi) = r(\varphi)$ for all $\varphi \in N_{cu}$. The proof of our main result is now straightforward.

Proof of Theorem 1.2. 1. As $N_{cu} \subset U$ is open and both norms $\|\cdot\|_{C^1}$ and $\|\cdot\|_1$ are equivalent, there clearly exists some $\tilde{\varepsilon} > 0$ with $\{\varphi \in C^1 \mid \|\varphi\|_{C^1} < 2\tilde{\varepsilon}\} \subset N_{cu}$. Next, using Corollary 5.11 with $\psi = \varphi_0 = 0 \in W^\eta$ we find some $0 < \tilde{\delta} < \tilde{\varepsilon}$ such that for all $\varphi \in X_\delta$ with $\|\varphi\|_{C^1} < \tilde{\delta}$ we have

$$\|F_\delta(t,\varphi) - F_\delta(t, H^\eta_{cu}(\varphi))\|_{C^1} \leq \tilde{\varepsilon} e^{-\eta t}, \tag{6.1}$$

and so

$$\|F_\delta(t, H^\eta_{cu}(\varphi))\|_{C^1} \leq \tilde{\varepsilon} e^{-\eta t} + \|F_\delta(t,\varphi)\|_{C^1}$$

as $t \geq 0$.

2. Suppose now that $x\colon [-h,\infty) \to \mathbb{R}^n$ is a solution of Eq. (1.1) with $\|x_t\|_{C^1} \leq \tilde{\delta}$ for all $t \geq 0$. Set $\tilde{\varphi} := x_0$ and note that $r_\delta(x_t) = r(x_t)$ for each $t \geq 0$, since the segments of x stay in N_{cu} for all $t \geq 0$. Hence, we have $x_t \in X_\delta \cap X_f$ as $t \geq 0$ and x is a solution of the smoothed equation (3.1) as well. In particular, $F_\delta(t,\tilde{\varphi}) = F(t,\tilde{\varphi})$ for all $t \geq 0$.

Next, observe that the last inequality of the first part shows that for $t \geq 0$

$$\begin{aligned}\|F_\delta(t, H^\eta_{cu}(\tilde{\varphi}))\|_{C^1} &\leq \tilde{\varepsilon} e^{-\eta t} + \|F_\delta(t, \tilde{\varphi})\|_{C^1}\\ &= \tilde{\varepsilon} e^{-\eta t} + \|F(t, \tilde{\varphi})\|_{C^1}\\ &= \tilde{\varepsilon} e^{-\eta t} + \|x_t\|_{C^1}\\ &\leq \tilde{\varepsilon} + \tilde{\delta}\\ &< 2\tilde{\varepsilon}.\end{aligned}$$

Consequently, all the segments $y_t = F_\delta(t, H^\eta_{cu}(\tilde{\varphi}))$ of the unique solution $y\colon [-h, \infty) \to \mathbb{R}^n$ of Eq. (3.1) with initial value $y_0 = H^\eta_{cu}(\tilde{\varphi}) \in W^\eta$ are contained in the neighborhood N_{cu} of $\varphi_0 = 0 \in C^1$. Therefore, for each $t \geq 0$ we have $r_\delta(y_t) = r(y_t)$ and thus y is also a solution of Eq. (1.1) with segments $y_t \in X_\delta \cap X_f$. In particular, $y_0 = H^\eta_{cu}(\tilde{\varphi}) \in W_{cu}$ and $F(t, H^\eta_{cu}(\tilde{\varphi})) = F_\delta(t, \tilde{\varphi})$ as $t \geq 0$. Now the positive invariance of W_{cu} with respect to F relative to N_{cu}, that is, property (iii) of Theorem 1.1, shows $y_t = F(t, H^\eta_{cu}(\varphi)) \in W_{cu}$ as $t \geq 0$. Furthermore, estimate (6.1) implies

$$\|x_t - y_t\|_{C^1} = \|F(t, \tilde{\varphi}) - F(t, H^\eta_{cu}(\tilde{\varphi}))\|_{C^1} = \|F_\delta(t, \tilde{\varphi}) - F_\delta(t, H^\eta_{cu}(\tilde{\varphi}))\|_{C^1} \leq \tilde{\varepsilon} e^{-\eta t}$$

for all $t \geq 0$.

3. Setting $K_A := \tilde{\varepsilon}$, $\eta_A := \eta$, and

$$U_A := \left\{\psi \in C^1 \mid \|\psi\|_{C^1} < \tilde{\delta}\right\}$$

completes the proof. □

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