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Stability of positive solutions of local partial differential equations with a nonlinear integral delay term[1]

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Abstract. Stability properties of positive stationary solutions to local partial differential equations with delay are studied. The results are applied to equations with not necessarily convex (concave) nonlinearities, for example, to the diffusive Nicholson's blowflies equation.

Key words : Partial functional differential equation, delay equation, positive stationary solution.

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1. Introduction

Stability properties of stationary solutions play a central role in the qualitative analysis of differential equations. There are many methods and approaches developed for different types of differential equations (ordinary, partial, delay and mixed). For more details on delay equations we refer to the classical monographs [9, 5, 24, 11].

This note is inspired by the work [7] where authors study the stability properties of positive stationary solutions to semi-linear delay partial differential equations (P.D.E.s) (see [7] for the history of the problem and references). In [7] authors start with a result for a general convex (concave) delay term and then present a more detailed analysis for the case of discrete delays. Taking into account that in some cases (see e.g. [14, 15, 16]) for partial differential equations the distributed delay has some essential advantages, we present in this note a slight generalization of the general result given in [7] and then study a wide class of partial differential eqautions (P.D.E.s), including equations with a distributed delay. In a sense, this note is a supplement of the results in [7]. Another motivation is to extend the technics to cover the case of non-convex (non-concave) nonlinearities to treat the diffusive Nicholson's blowflies equation (see e.g. [17, 19]).

Consider the following semi-linear partial differential equation with local (in space variable) delay term (we use the notations of [7])

$$\frac{\partial}{\partial t}u(t,x) - \Delta u(t,x) + d\cdot u(t,x) = f(u_t(x)), \qquad t>0, \quad x\in\Omega, \quad d\geq 0, \tag{1}$$

with the Dirichlet boundary condition

$$u(t,x)=0, \quad t\geq 0, \quad x\in\partial\Omega \tag{2}$$

[1]This paper is in final form and no version of it is submitted for publication elsewhere.

and the initial condition

$$u(\theta, x) = \varphi(\theta, x), \quad (\theta, x) \in [-r, 0] \times \Omega, \tag{3}$$

where Ω is a bounded domain in R^n with a smooth boundary (see e.g. [12]). In [7] the constant $d = 0$. As usually for delay equations [8], we denote by u_t the function of $\theta \in [-r, 0]$ by the formula $u_t \equiv u_t(\theta) \equiv u(t + \theta)$.

The property of the nonlinear function $f : C([-r, 0]; R) \to R$ will be given below.

As in [7] we denote by $\bar{\cdot}$ the embedding $\bar{\cdot} : R \to C([-r, 0]; R)$ as

$$\overline{a}(\theta) \equiv a, \quad \text{for all} \quad \theta \in [-r, 0], a \in R.$$

The C^2 function $U : \Omega \to R$ is called a positive stationary solution of (1)-(3) if

$$\Delta U(x) - d \cdot U(x) + f(\overline{U(x)}) = 0, \quad x \in \Omega, \tag{4}$$

$$U(x) = 0, \quad x \in \partial\Omega, \qquad U(x) > 0, \quad x \in \Omega. \tag{5}$$

The linearisation of (1)-(3) around U reads

$$\frac{\partial}{\partial t} w(t, x) - \Delta w(t, x) + d \cdot w(t, x) = Df(\overline{U(x)}) \cdot [w_t(x)], \quad t > 0, \quad x \in \Omega, \tag{6}$$

where $Df(\overline{U(x)})$ is the Fréchet derivative of f at $\overline{U(x)}$.

The characteristic equation is given as

$$\Delta v(x) - d \cdot v(x) + Df(\overline{U(x)}) \cdot [e^{\lambda \cdot} v(x)] = \lambda v(x), \quad x \in \Omega. \tag{7}$$

Denote the dominant characteristic root of (7) by Λ, i.e. there exists a function $V(x)$ satisfying the Dichichlet boundary condition such that

$$\Delta V(x) - d \cdot V(x) + Df(\overline{U(x)}) \cdot [e^{\Lambda \cdot} V(x)] = \Lambda V(x), \quad x \in \Omega \tag{8}$$

and for all solutions (λ, v) of (7) one has $Re\, \lambda \leq Re\, \Lambda$. To get the stability of $U(x)$ by the principle of the linearized stability one needs $Re\, \Lambda < 0$. In the case $Re\, \Lambda > 0$ the solution $U(x)$ is unstable.

As in [7], we assume

$$\Lambda \in R \quad \text{and} \quad V(x) > 0, \quad x \in \Omega. \tag{\textbf{H}}$$

This assumption is satisfied, for example, for positive semigroups (see [6] for more details and for delay equations [6, Section 6 of Chapter VI]).

Now we present a slight generalization of theorem 2.1 [7] to cover the case of non-increasing and non-convex (non-concave) nonlinearities.

Theorem 1. *Assume (H) is satisfied. Consider a positive stationary solution* $U(x)$ *and denote* $s_{max} \equiv \max\{U(x), x \in \Omega\}$. *Consider the following conditions*

1) *the function* f *is increasing on* $[0, s_{max}]$ *i.e.* $Df(\overline{a}) \cdot [\varphi] \geq 0$ *for all* $a \in [0, s_{max}]$ *and* $\varphi \in C([-r, 0]; R)$, $\varphi \geq 0$;

2) *the function* $h(a) \equiv f(\overline{a})$ *is* C^2 *function* $h : R \to R$ *and satisfies*

3a) $h(0) \geq 0$ *and* $h'(a) < h'(0)$ *for all* $a \in (0, s_{max}]$

or

3b) *$h(0) \leq 0$ and $h'(a) > h'(0)$ for all $a \in (0, s_{max}]$.*

Then in the case 1),2),3a) the positive stationary solution $U(x)$ is stable while in the case 1),2),3b) the solution $U(x)$ is unstable.

Proof follows the line of arguments given in theorem 2.1 [7]. We indicate the moments where the proofs differ. Consider the function

$$g(a) \equiv Df(\overline{a}) \cdot [\overline{a}] - f(\overline{a}). \tag{9}$$

In the case 1),2),3a), we need the property $g(a) < 0$ for all $a \in (0, s_{max}]$. Using $g(a) = ah'(a) - h(a)$, we get $g'(a) = h''(a)$. Hence $g(a) = g(0) + \int_0^a h''(s)ds = g(0) + h'(a) - h'(0)$. The last property, $g(0) = -h(0)$ and 3a) imply $g(a) < 0$ for all $a \in (0, s_{max}]$. The same arguments give the property $g(a) > 0$ for all $a \in (0, s_{max}]$ in the case 1),2),3b). The rest of the proof follows the proof of theorem 2.1 [7] (see page 4 in [7]).

2. Partial differential equations with an integral delay term

Consider the semi-linear partial differential equation (1) with local (in space variable) and distributed (in time variable) nonlinear delay term of the form

$$f(\varphi) = \int_{-r}^{0} b(\varphi(\theta)) \cdot d\eta(\theta), \quad \varphi \in C([-r, 0]; R), \tag{10}$$

where $\eta : [-r, 0] \to R$ is of bounded variation.

So equation (1) takes the form (as before $d \geq 0$)

$$\frac{\partial}{\partial t} u(t, x) - \Delta u(t, x) + d \cdot u(t, x) = \int_{-r}^{0} b(u(t + \theta, x)) \cdot d\eta(\theta), \quad t > 0, \quad x \in \Omega. \tag{11}$$

We assume that

$$I_\xi \equiv \int_{-r}^{0} d\eta(\theta) \neq 0 \tag{12}$$

and

$$0 \leq \int_{-r}^{0} e^{\gamma\theta}\, d\eta(\theta) \leq I_\xi \quad \text{for all} \quad \gamma \geq 0. \tag{13}$$

Remark 1. *Assumption (13) is satisfied, for example, for any non-decreasing η, so $d\eta(\theta) \geq 0$.*

We also assume that function b in (10), (11) is a C^2 function $b : R \to R$, satisfying the property (c.f. 3a) in theorem 1)

$$b(a) \geq 0 \quad \text{for all} \quad a \geq 0 \quad \text{and} \quad b'(a) < b'(0) \quad \text{for all} \quad a > 0. \tag{14}$$

Now we prove the following

Theorem 2. *Assume (H) is satisfied. Let the function $\eta : [-r, 0] \to R$ be of bounded variation and satisfy (12), (13) and function b satisfy (14). Then any positive stationary solution $U(x)$ of (11),(2),(3) is stable.*

Remark 2. *Since in (10), (11) we use the Stiltjes integral, we may treat the cases of discrete, distributed and mixed discrete-distributed delays. In this connection it is interesting to compare our assumptions with the ones of part (ii) in [7, theorem 3.1] where a discrete delay is considered. It is easy to see that for the equation (11) our assumption* $\int_{-r}^{0} e^{\gamma\theta}\, d\eta(\theta) \le I_\xi$ *for all* $\gamma \ge 0$ *(see (13)) together with* $b(a) \ge 0$ *(see (14)) are less restrictive than the corresponding assumption* $\partial_\ell F(\overline{a}) \ge 0$ *for* $\ell = 1, ..., k, a \ge 0$ *in [7, theorem 3.1].*

Proof of theorem 2. One can check that the Fréchet derivative of f at $\overline{U(x)}$ reads

$$Df(\overline{U(x)}) \cdot [\psi] = \int_{-r}^{0} b'(U(x)) \cdot \psi(\theta) \cdot d\eta(\theta). \tag{15}$$

Hence equation (8) takes the form

$$\Delta V(x) - d \cdot V(x) + b'(U(x))V(x) \cdot \int_{-r}^{0} e^{\Lambda\theta} d\eta(\theta) = \Lambda V(x), \quad x \in \Omega. \tag{16}$$

Our goal is to show that $\Lambda < 0$.

We multiply (16) by $U(x)$ and add (4), multiplied by $-V(x)$. Integration over Ω, using the symmetric Green formula, gives

$$\int_\Omega b'(U(x))V(x)U(x)\, dx \cdot \int_{-r}^{0} e^{\Lambda\theta} d\eta(\theta) - \int_\Omega b(U(x))V(x)\, dx \cdot I_\xi = \Lambda \int_\Omega U(x)V(x)\, dx.$$

Using $I_\xi \ne 0$ (see (12)), one can write

$$\int_\Omega b'(U(x))V(x)U(x)\, dx \cdot I_\xi I_\xi^{-1} \int_{-r}^{0} e^{\Lambda\theta} d\eta(\theta) - \int_\Omega b(U(x))V(x)\, dx \cdot I_\xi I_\xi^{-1} \int_{-r}^{0} e^{\Lambda\theta} d\eta(\theta)$$

$$+ \int_\Omega b(U(x))V(x)\, dx \cdot I_\xi \cdot \left[I_\xi^{-1} \int_{-r}^{0} e^{\Lambda\theta} d\eta(\theta) - 1 \right] = \Lambda \int_\Omega U(x)V(x)\, dx. \tag{17}$$

The function $g(a)$ (see (9)) reads

$$g(a) = \int_{-r}^{0} ab'(a) d\eta(\theta) - \int_{-r}^{0} b(a) d\eta(\theta) = [ab'(a) - b(a)] \cdot I_\xi, \tag{18}$$

so we can rewrite (17) as

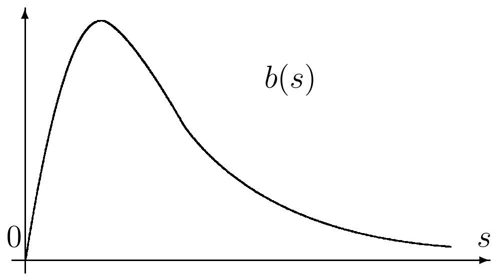
$$\int_\Omega g(U(x))V(x)\, dx \cdot I_\xi^{-1} \cdot \int_{-r}^{0} e^{\Lambda\theta} d\eta(\theta) + \int_\Omega b(U(x))V(x)\, dx \cdot \left[\int_{-r}^{0} e^{\Lambda\theta} d\eta(\theta) - I_\xi \right]$$

$$= \Lambda \int_\Omega U(x)V(x)\, dx. \tag{19}$$

In our case the function $h(a) = b(a) \cdot I_\xi$, so $h''(a) = b''(a) \cdot I_\xi$ and $g(a) = [-b(0) + b'(a) - b'(0)] \cdot I_\xi$. As in the proof of theorem 1, property (14) and (18) imply $g(a) < 0$ for all $a > 0$ (since $I_\xi > 0$).

Finally, if we assume that $\Lambda \ge 0$, then taking into account $I_\xi^{-1} \cdot \int_{-r}^{0} e^{\Lambda\theta} d\eta(\theta) \ge 0$ (by assumption (13)), $b(U(x)) \ge 0$ (by (14)), $\left[\int_{-r}^{0} e^{\Lambda\theta} d\eta(\theta) - I_\xi \right] \le 0$ (by (13)), we conclude that the left hand side of (19) is strictly negative. Hence so for the right hand side, which is a contradiction. Hence $\Lambda < 0$. Now we may apply the classical principle of linearized stability (for the delay case see e.g. [24, theorem 4.1, p.123]) to get the stability of U. The proof is complete. ∎

The diffusive Nicholson's blowflies equation (see e.g. [17, 19] and also [14, 15]) is the equation (11) with the nonlinear (birth) function $b(s) = \alpha s \cdot e^{-\beta s}$, where $\alpha, \beta > 0$.



One can easily check that such function b is neither convex nor concave, which implies the same for $h(a) \equiv f(\overline{a})$. Hence in this case one cannot apply the results of [7]. On the other hand, b does satisfy property (14) and as a result we have conditions on the function η (see (12), (13)) when one may apply theorem 2 to the diffusive Nicholson's blowflies equation with a distributed delay.

Remark 3. *For the diffusive Nicholson's blowflies equation it is natural to take a differentiable* η *such that* $d\eta(\theta) = \xi(\theta)\, d\theta$ *with* $\xi(\theta) \geq 0, \xi \not\equiv 0$ *(c.f. [14, 15])) so the properties (12), (13) are satisfied.*

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References

[1] N.V. Azbelev, V.P. Maksimov and L.F. Rakhmatullina, Introduction to the theory of functional differential equations, Moscow, Nauka, 1991.

[2] A. V. Babin, and M. I. Vishik, Attractors of Evolutionary Equations, Amsterdam, North-Holland, 1992.

[3] I. D. Chueshov, Introduction to the Theory of Infinite-Dimensional Dissipative Systems, Acta, Kharkov (1999), (in Russian). English transl. Acta, Kharkov (2002) (see http://www.emis.de/monographs/Chueshov).

[4] I. D. Chueshov, On a certain system of equations with delay, occuring in aeroelasticity, J. Soviet Math. **58**, 1992, p.385-390.

[5] O. Diekmann, S. van Gils, S. Verduyn Lunel, H-O. Walther, Delay Equations: Functional, Complex, and Nonlinear Analysis, Springer-Verlag, New York, 1995.

[6] K.-J. Engel, R. Nagel, One-parameter semigroups for linear evolution equations, Springer, New York, 2000.

[7] G. Farkas, P.L. Simon, Stability properties of positive solutions to partial differential equations with delay, *Electron. J. Diff. Eqn.* **2001** No. 64. (2001) 1-8.

[8] J.K. Hale, Theory of Functional Differential Equations, Springer, Berlin-Heidelberg- New York, 1977.

[9] J.K. Hale and S. M. Verduyn Lunel, Theory of Functional Differential Equations, Springer-Verlag, New York, 1993.

[10] D. Henry, Geometric theory of semilinear parabolic equations. Lecture Notes in Mathematics, 840. Springer-Verlag, Berlin-New York, 1981.

[11] T. Krisztin, H.-O. Walther and J. Wu, Shape, Smoothness and Invariant Stratification of an Attracting Set for Delayed Monotone Positive Feedback, Fields Institute Monographs, 11, AMS, Providence, RI, 1999.

[12] J. L. Lions, Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires, Dunod, Paris, 1969.

[13] A.D. Mishkis, Linear differential equations with retarded argument. 2nd edition, Nauka, Moscow, 1972.

[14] A.V. Rezounenko, J. Wu, A non-local PDE model for population dynamics with state-selective delay: local theory and global attractors, Journal of Computational and Applied Mathematics, 190, Issues 1-2 (2006), P.99-113.

[15] A.V. Rezounenko, Partial differential equations with discrete and distributed state-dependent delays, Journal of Mathematical Analysis and Applications, 326, Issue **2**, (2007), 1031-1045. (see preprint version: "A.V. Rezounenko, Two models of partial differential equations with discrete and distributed state-dependent delays". *preprint.* March 22, 2005, http://arxiv.org/abs/math.DS/0503470).

[16] A.V. Rezounenko, On a class of P.D.E.s with nonlinear distributed in space and time state-dependent delay terms // Mathematical Methods in the Applied Sciences. -2008 (in press), published Online: Jan 21 2008.

[17] J. W. -H. So, J. Wu and Y. Yang, Numerical steady state and Hopf bifurcation analysis on the diffusive Nicholson's blowflies equation. Appl. Math. Comput. 111 (2000), no. 1, 33–51.

[18] J. W. -H. So, J. Wu and X.Zou, A reaction diffusion model for a single species with age structure. I. Travelling wavefronts on unbounded domains, Proc. Royal .Soc. Lond. A (2001) 457, 1841-1853.

[19] J. W.- H. So and Y. Yang, Dirichlet problem for the diffusive Nicholson's blowflies equation, J. Differential Equations 150 (1998), no. 2, 317–348.

[20] R. Temam, Infinite Dimensional Dynamical Systems in Mechanics and Physics, Springer, Berlin-Heidelberg-New York, 1988.

[21] C. C. Travis and G. F. Webb, Existence and stability for partial functional differential equations, Transactions of AMS, **200**, (1974), 395-418.

[22] H.-O. Walther, Stable periodic motion of a system with state dependent delay, Differential and Integral Equations 15 (2002), 923-944.

[23] H.-O. Walther, The solution manifold and C^1-smoothness for differential equations with state-dependent delay, J. Differential Equations 195 (2003), no. 1, 46–65.

[24] J. Wu, Theory and Applications of Partial Functional Differential Equations, Springer-Verlag, New York, 1996.

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