Electronic Journal of Qualitative Theory of Differential Equations
2009, No. **24**, p. 1-23; **http://www.math.u-szeged.hu/ejqtde/**

Structure of Solutions Sets and a Continuous Version of Filippov's Theorem for First Order Impulsive Differential Inclusions with Periodic Conditions

John R Graef[1] **and Abdelghani Ouahab**[2]

[1]Department of Mathematics, University of Tennessee at Chattanooga
Chattanooga, TN 37403-2504 USA
e-mail: John-Graef@utc.edu
[2] Laboratoire de Mathématiques, Université de Sidi Bel Abbès
BP 89, 22000 Sidi Bel Abbès, Algérie
e-mail: agh_ouahab@yahoo.fr

Abstract

In this paper, the authors consider the first-order nonresonance impulsive differential inclusion with periodic conditions

$$\begin{array}{rcll} y'(t)-\lambda y(t) & \in & F(t,y(t)), & \text{a.e. } t\in J\backslash\{t_1,\ldots,t_m\},\\ y(t_k^+)-y(t_k^-) & = & I_k(y(t_k^-)), & k=1,2,\ldots,m,\\ y(0) & = & y(b), & \end{array}$$

where $J=[0,b]$ and $F : J\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ is a set-valued map. The functions I_k characterize the jump of the solutions at impulse points t_k ($k=1,2,\ldots,m$). The topological structure of solution sets as well as some of their geometric properties (contractibility and R_δ-sets) are studied. A continuous version of Filippov's theorem is also proved.

Key words and phrases: Impulsive differential inclusions, periodic conditions, contractible, R_δ-set, acyclic, continuous selection, Filippov's theorem.

AMS (MOS) Subject Classifications: 34A60, 34K45, 34B37, 54C60.

1 Introduction

Many processes in engineering, physics, biology, population dynamics, medicine, and other areas are subject to abrupt changes such as shocks or perturbations (see for instance [1, 34] and the references therein). These changes may be viewed as impulses. For example, in the treatment of some diseases, periodic impulses correspond to the administration of a drug. In environmental sciences such impulses correspond to seasonal changes of the water level of artificial reservoirs. Such models can be described by impulsive differential equations. Contributions to the study of the mathematical aspects of such equations can be found, for example, in the works of Bainov and Simeonov [9], Lakshmikantham, Bainov, and Simeonov [35], Pandit and Deo [38], and Samoilenko and Perestyuk [41].

Impulsive ordinary differential inclusions and functional differential inclusions with different conditions have been intensely studied in the last several years, and we refer the reader to the monographs by Aubin [6] and Benchohra *et al.* [11], as well as the thesis of Ouahab [37], and the references therein.

We will consider the problem

$$y'(t) - \lambda y(t) \in F(t, y(t)), \quad \text{a.e. } t \in J\backslash\{t_1, \ldots, t_m\}, \tag{1}$$

$$y(t_k^+) - y(t_k) = I_k(y(t_k)), \quad k = 1, 2, \ldots, m, \tag{2}$$

$$y(0) = y(b), \tag{3}$$

where $\lambda \neq 0$ is a parameter, $J = [0, b]$, $F : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a multi-valued map, $I_k \in C(\mathbb{R}^n, \mathbb{R}^n)$, $k = 1, 2, \ldots, m$, $t_0 = 0 < t_1 < \ldots < t_m < t_{m+1} = b$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$, and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$.

In 1923, Kneser proved that the Peano existence theorem can be formulated in such a way that the set of all solutions is not only nonempty but is also compact and connected (see [39, 40]). Later, in 1942, Aronszajn [5] improved Kneser's theorem by showing that the set of all solutions is even an R_δ–set. It should also be clear that the characterization of the set of fixed points for some operators implies the corresponding result for the solution sets.

Lasry and Robert [36] studied the topological properties of the sets of solutions for a large class of differential inclusions including differential difference inclusions. The present paper is a continuation of their work but for a general class of impulsive differential inclusions with periodic conditions. Aronszajn's results for differential inclusions with difference conditions was improved by several authors, for example, see [2, 3, 4, 21, 23, 24]. Very recently, properties of the solutions of impulsive differential inclusions with initial conditions were study by Djebali *et al.* [18].

The main goal of this paper is to examine some properties of solutions sets for impulsive differential inclusions with periodic conditions and to present a continuous version of Filippov's theorem.

2 Preliminaries

Here, we introduce notations, definitions, and facts from multi-valued analysis that will be needed throughout this paper. We let $C(J, \mathbb{R})$ denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the Tchebyshev norm

$$\|x\|_\infty = \sup\{|x(t)| : t \in J\},$$

and we let $L^1(J, \mathbb{R})$ be the Banach space of measurable functions $x : J \longrightarrow \mathbb{R}$ that are Lebesgue integrable with the norm

$$|x|_1 = \int_0^b |x(s)| ds.$$

By $AC^i(J,\mathbb{R}^n)$, we mean the space of functions $y : J \to \mathbb{R}^n$ that are i-times differentiable and whose i^{th} derivative, $y^{(i)}$, is absolutely continuous.

For a metric space (X,d), the following notations will be used throughout this paper:

- $\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$.
- $\mathcal{P}_p(X) = \{Y \in \mathcal{P}(X) : Y$ has the property "p"$\}$ where p could be: cl = closed, b = bounded, cp = compact, cv = convex, etc. Thus,
- $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y$ closed$\}$.
- $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y$ bounded$\}$.
- $\mathcal{P}_{cv}(X) = \{Y \in \mathcal{P}(X) : Y$ convex,$\}$ where X is a Banach spaces.
- $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y$ compact$\}$.
- $\mathcal{P}_{cv,cp}(X) = \mathcal{P}_{cv}(X) \cap \mathcal{P}_{cp}(X)$.

Let $(X, \|.\|)$ be a separable Banach space and $F : J \to \mathcal{P}_{cl}(X)$ be a multi-valued map. We say that F is *measurable* provided for every open $U \subset X$, the set $F^{+1}(U) = \{t \in J : F(t) \subset U\}$ is Lebesgue measurable in J. We will need the following lemma.

Lemma 2.1 ([15, 20]) *The mapping F is measurable if and only if for each $x \in X$, the function $\zeta : J \to [0,+\infty)$ defined by*

$$\zeta(t) = \text{dist}(x, F(t)) = \inf\{\|x-y\| : y \in F(t)\}, \quad t \in J,$$

is Lebesgue measurable.

Let $(X, \|\cdot\|)$ be a Banach space and $F : X \to \mathcal{P}(X)$ be a multi-valued map. We say that F has a *fixed point* if there exists $x \in X$ such that $x \in F(x)$. The set of fixed points of F will be denoted by $Fix\,F$. We say that F has *convex (closed) values* if $F(x)$ is convex (closed) for all $x \in X$, and that F is *totally bounded* if $F(A) = \bigcup_{x\in A}\{F(x)\}$ is bounded in X for each bounded set A of X, i.e.,

$$\sup_{x\in A}\{\sup\{\|y\| : y \in F(x)\}\} < \infty.$$

Let (X,d) and (Y,ρ) be two metric spaces and let $F : X \to \mathcal{P}_{cl}(Y)$ be a multi-valued mapping. We say that F is *upper semi-continuous (u.s.c. for short)* on X if for each $x_0 \in X$ the set $F(x_0)$ is a nonempty, closed subset of X, and if for each open set N of Y containing $F(x_0)$, there exists an open neighborhood M of x_0 such that $G(M) \subseteq Y$. That is, if the set $F^{-1}(V) = \{x \in X,\ F(x) \cap V \neq \emptyset\}$ is closed for any closed set V in Y. Equivalently, F is *u.s.c.* if the set $F^{+1}(V) = \{x \in X,\ F(x) \subset V\}$ is open for any open set V in Y. The mapping F is said to be *lower semi-continuous* (*l.s.c.*) if the inverse image of V by F

$$F^{-1}(V) = \{x \in X : F(x) \cap V \neq \emptyset\}$$

is open for any open set V in Y. Equivalently, F is *l.s.c.* if the core of V by F

$$F^{+1}(V) = \{x \in X \,:\, F(x) \subset V\}$$

is closed for any closed set V in Y. Finally, for a multi-valued function $G : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$, we take

$$\|G(t,z)\|_{\mathcal{P}} := \sup\{|v| \,:\, v \in G(t,z)\}.$$

Definition 2.2 *A mapping G is a multi-valued Carathéodory function if:*

(a) *The function $t \mapsto G(t,z)$ is measurable for each $z \in \mathbb{R}^n$;*

(b) *For a.e. $t \in J$, the map $z \mapsto G(t,z)$ is upper semi-continuous.*

Furthermore, it is L^1-Carathéodory if it is locally integrably bounded, i.e., for each positive real number r, there exists $h_r \in L^1(J,\mathbb{R}^+)$ such that

$$\|G(t,z)\|_{\mathcal{P}} \leq h_r(t) \;\; \text{for a.e. } \; t \in J \;\; \text{and all } \; \|z\| \leq r.$$

Consider the Hausdorf pseudo-metric distance $H_d\colon\ \mathcal{P}(\mathbb{R}^n)\times\mathcal{P}(\mathbb{R}^n) \longrightarrow \mathbb{R}^+\cup\{\infty\}$ defined by

$$H_d(A,B) = \max\left\{\sup_{a\in A} d(a,B),\, \sup_{b\in B} d(A,b)\right\}$$

where $d(A,b) = \inf_{a\in A} d(a,b)$, $d(a,B) = \inf_{b\in B} d(a,b)$. Then $(\mathcal{P}_{b,cl}(\mathbb{R}^n), H_d)$ is a metric space and $(\mathcal{P}_{cl}(X), H_d)$ is a generalized metric space (see [33]). Moreover, H_d satisfies the triangle inequality. Note that if $x_0 \in \mathbb{R}^n$, then

$$d(x_0,A) = \inf_{x\in A} d(x_0,x) \;\text{ while }\; H_d(\{x_0\},A) = \sup_{x\in A} d(x_0,x).$$

Definition 2.3 *A multivalued operator $N\colon\ \mathbb{R}^n \to \mathcal{P}_{cl}(\mathbb{R}^n)$ is called:*

(a) γ-Lipschitz if there exists $\gamma > 0$ such that

$$H_d(N(x),N(y)) \leq \gamma d(x,y), \;\; \text{for each} \;\; x,\; y \in \mathbb{R}^n;$$

(b) a contraction if it is γ-Lipschitz with $\gamma < 1$.

Additional details on multi-valued maps can be found in the works of Aubin and Cellina [7], Aubin and Frankowska [8], Deimling [17], Gorniewicz [20], Hu and Papageorgiou [30], Kamenskii [32], Kisielewicz [33], and Tolstonogov [42].

2.1 Background in Geometric Topology

We begin with some elementary notions from geometric topology. For additional details, we recommend [12, 22, 28, 31]. In what follows, we let (X, d) denote a metric space. A set $A \in \mathcal{P}(X)$ is *contractible* provided there exists a continuous homotopy $h : A \times [0, 1] \to A$ such that

(i) $h(x, 0) = x$, for every $x \in A$, and

(ii) $h(x, 1) = x_0$, for every $x \in A$.

Note that if $A \in \mathcal{P}_{cv,cp}(X)$, then A is contractible. Clearly, the class of contractible sets is much larger than the class of all compact convex sets. The following concepts are needed in the sequel.

Definition 2.4 *A space X is called an absolute retract (written as $X \in AR$) provided that for every space Y, a closed subset $B \subseteq Y$, and a continuous map $f : B \to X$, there exists a continuous extension $\widetilde{f} : Y \to X$ of f over Y, i.e., $\widetilde{f}(x) = f(x)$ for every $x \in B$.*

Definition 2.5 *A space X is called an absolute neighborhood retract (written as $X \in ANR$) if for every space Y, any closed subset $B \subseteq Y$, and any continuous map $f : B \to X$, there exists a open neighborhood U of B and a continuous map $\widetilde{f} : U \to X$ such that $\widetilde{f}(x) = f(x)$ for every $x \in B$.*

Definition 2.6 *A space X is called an $R_\delta-$set if there exists a sequence of nonempty compact contractible spaces $\{X_n\}$ such that*

$$X_{n+1} \subset X_n \text{ for every } n$$

and

$$X = \bigcap_{n=1}^{\infty} X_n.$$

It is well known that any contractible set is acyclic and that the class of acyclic sets is larger than that of contractible sets. The continuity of the Čech cohomology functor yields the following lemma.

Lemma 2.7 ([22]) *Let X be a compact metric space. If X is an R_δ–set, then it is an acyclic space.*

3 Space of Solutions

Let $J_k = (t_k, t_{k+1}]$, $k = 0, 1, \dots, m$, and let y_k be the restriction of a function y to J_k. In order to define mild solutions to the problem (1)–(3), consider the space

$$PC = \{y\colon\ J \to \mathbb{R}^n \mid y_k \in C(J_k, \mathbb{R}^n),\ k = 0, 1, \dots, m,\ \text{and}$$
$$y(t_k^-)\ \text{and}\ y(t_k^+)\ \text{exist and satisfy}\ \ y(t_k^-) = y(t_k)\ \ \text{for}\ \ k = 1, 2, \dots, m\}.$$

Endowed with the norm

$$\|y\|_{PC} = \max\{\|y_k\|_\infty :\ k = 0, 1, \dots, m\},$$

this is a Banach space.

Definition 3.1 *A function $y \in PC \cap \cup_{k=0}^m AC(J_k, \mathbb{R}^n)$ is said to be a solution of problem (1)–(3) if there exists $v \in L^1(J, \mathbb{R}^n)$ such that $v(t) \in F(t, y(t))$ a.e. $t \in J$, $y'(t) - \lambda y(t) = v(t)$ for $t \in J \backslash \{t_1, \dots, t_m\}$, $y(t_k^+) - y(t_k) = I_k(y(t_k))$, $k = 1, 2, \dots, m$, and $y(0) = y(b)$.*

4 Solutions Sets

In this section, we present results about the topological structure of the set of solutions of some nonlinear functional equations due to Aronszajn [5] and further developed by Browder and Gupta in [14].

Theorem 4.1 *Let X be a space, let $(E, \|\cdot\|)$ be a Banach space, and let $f : X \to E$ be a proper map, i.e., f is continuous and for every compact $K \subset E$ the set $f^{-1}(K)$ is compact. Assume further that for each $\epsilon > 0$ a proper map $f_\epsilon : X \to E$ is given, and the following two conditions are satisfied:*

(i) $\|f_\epsilon(x) - f(x)\| < \epsilon$ for every $x \in X$;

(ii) for every $\epsilon > 0$ and $u \in E$ such that $\|u\| \leq \epsilon$, the equation $f_\epsilon(x) = u$ has exactly one solution.

Then the set $S = f^{-1}(0)$ is R_δ.

Definition 4.2 *A single valued map $f : [0; a] \times X \to Y$ is said to be measurable-locally-Lipschitz if $f(\cdot, x)$ is measurable for every $x \in X$, and for each $x \in X$ there exists a neighborhood V_x of x and an integrable function $L_x : [0, a] \to [0, \infty)$ such that*

$$\|f(t, x_1) - f(t, x_2)\| \leq L_x(t)\|x_1 - x_2\| \text{ for every } t \in [0, a] \text{ and } x_1, x_2 \in V_x.$$

The following result is know as the Lasota–Yorke Approximation Theorem.

Theorem 4.3 ([20]) *Let E be a normed space, X be a metric space, and let $f : X \to E$ be a continuous map. Then, for each $\epsilon > 0$ there is a locally Lipschitz map $f_\epsilon : X \to E$ such that*

$$\|f(x) - f_\epsilon(x)\| < \epsilon \text{ for every } x \in X.$$

We consider the impulsive problem

$$y'(t) - \lambda y(t) = f(t, y(t)), \quad \text{a.e. } t \in J\backslash\{t_1, \ldots, t_m\}, \tag{4}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1, 2, \ldots, m, \tag{5}$$

$$y(0) = y(b), \tag{6}$$

where $J = [0, b]$, $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathédory function, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b$, $\lambda \in \mathbb{R}\backslash\{0\}$, $\Delta y|_{t=t_k} = y(t_k^+) - y(t_k^-)$, and $y(t_k^+) = \lim_{h\to 0^+} y(t_k + h)$ and $y(t_k^-) = \lim_{h\to 0^+} y(t_k - h)$ represent the right and left limits of $y(t)$ at $t = t_k$, respectively.

The following result of Graef and Ouahab will be used to prove our main existence theorems.

Lemma 4.4 ([26]) *The function y is the unique solution of the problem (4)–(6) if and only if*

$$y(t) = \int_0^b H(t, s) f(y(s)) ds + \sum_{k=1}^{m} H(t, t_k) I_k(y(t_k)), \tag{7}$$

where

$$H(t, s) = (e^{-\lambda b} - 1)^{-1} \begin{cases} e^{-\lambda(b+s-t)}, & \text{if } 0 \le s \le t \le b, \\ \\ e^{-\lambda(s-t)}, & \text{if } 0 \le t < s \le b. \end{cases}$$

We denote by $S(f, 0, b)$ the set of all solutions of the impulsive problem (4)–(6). Now, we are in a position to state and prove our first Aronsajn type result. For the study of this problem, we first list the following hypotheses.

($\mathcal{R}_1$) There exist functions p, $q \in L^1(J, \mathbb{R}_+)$ and $\alpha \in [0, 1)$ such that

$$|f(t, y)| \le p(t)|y|^\alpha + q(t) \quad \text{for each} \quad (t, y) \in J \times \mathbb{R}^n.$$

($\mathcal{R}_2$) There exist constants c_k^*, $b_k^* \in \mathbb{R}_+$ and $\alpha_k \in [0, 1)$ such that

$$|I_k(y)| \le c_k^* + b_k^* |y|^{\alpha_k}, \quad k = 1, 2, \ldots, m, \quad y \in \mathbb{R}^n.$$

Theorem 4.5 *Assume that conditions ($\mathcal{R}_1$)–($\mathcal{R}_2$) hold. Then $S(f, 0, b)$ is R_δ. Moreover, $S(f, 0, b)$ is an acyclic space.*

Proof. Let $G : PC \to PC$ defined by

$$G(y)(t) = \int_0^b H(t,s)f(s,y(s))ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)),\ t \in [t_0,b].$$

Thus, $FixG = S(f,0,b)$. From $(\mathcal{R}_1)$ and $(\mathcal{R}_2)$, we have $S(f,0,b) \neq \emptyset$ (see [26]) and there exists $M > 0$ such that

$$\|y\|_{PC} \leq M \text{ for every } y \in S(f,0,b).$$

Define

$$\widetilde{f}(t,y(t)) = \begin{cases} f(t,y(t)), & \text{if } |y(t)| \leq M, \\ f(t, \frac{My(t)}{|y(t)|}), & \text{if } |y(t)| \geq M, \end{cases}$$

and

$$\widetilde{I}_k(y(t)) = \begin{cases} I_k(y(t)), & \text{if } |y(t)| \leq M, \\ I_k(\frac{My(t)}{|y(t)|}), & \text{if } |y(t)| \geq M. \end{cases}$$

Since f is an L^1-Carathédory function, $\widetilde{f}$ is Carathédory and integrably bounded. We consider the following modified problem

$$y'(t) - \lambda y(t) = \widetilde{f}(t,y(t)), \quad \text{a.e. } t \in J\backslash\{t_1,\dots,t_m\}, \tag{8}$$

$$\Delta y|_{t=t_k} = \widetilde{I}_k(y(t_k^-)), \quad k = 1,2,\dots,m, \tag{9}$$

$$y(0) = y(b). \tag{10}$$

We can easily prove that $S(f,0,b) = S(\widetilde{f},0,b)$. Since $\widetilde{f}$ integrably bounded, there exists $h \in L^1(J,\mathbb{R}_+)$ such that

$$\|\widetilde{f}(t,x)\| \leq h(t) \ \text{ a.e. } t \in J \ \text{ and for all } x \in \mathbb{R}^n. \tag{11}$$

Now $S(\widetilde{f},0,b) = Fix\widetilde{G}$, where $\widetilde{G} : PC \to PC$ is defined by

$$\widetilde{G}(y)(t) = \int_0^b H(t,s)\widetilde{f}(s,y(s))ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)),\ t \in [t_0,b].$$

By inequality (11) and the continuity of the I_k, we have

$$\|\widetilde{G}(y)\|_{PC} \leq H_*\|h\|_{L^1} + H_* \sum_{k=1}^{m} [c_k^* + M^{\alpha_k} b_k^*] := r,$$

where

$$H_* = \sup\{H(t,s) \mid (t,s) \in J \times J\}.$$

By the same method used in [25, 26, 37], we can prove that $\widetilde{G} : PC \to PC$ is a compact operator, and we define the vector filed associated with $\widetilde{G}$ by $g = y - \widetilde{G}(y)$. From the compactness of $\widetilde{G}$ and the Lasota–Yorke Approximation Theorem (Theorem 4.3 above), we can easily prove that all the conditions of Theorem 4.1 are satisfied, and so $S(\widetilde{f}, 0, b)$ is R_δ. That it is acyclic follows from Lemma 2.7. □

The following definition and lemma can be found in [20, 29].

Definition 4.6 *A mapping $F : X \to \mathcal{P}(Y)$ is LL-selectionable provided there exists a measurable, locally-Lipchitzian map $f : X \to Y$ such that $f \subset F$.*

Lemma 4.7 *If $\varphi : X \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ is an u.s.c. multi-valued map, then φ is $\sigma-$LL-selectionable.*

Let $S(F, 0, b)$ denote the set of all solutions of (1)–(3). We are now going to characterize the topological structure of $S(F, 0, b)$. First, we prove the following result.

Theorem 4.8 *Let $F\colon J \times \mathbb{R}^n \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ be a Carathéodory map that is mLL-selectionable. In addition to conditions ($\mathcal{R}_1$)–($\mathcal{R}_2$), assume that:*

($\mathcal{H}_1$) *There exist constants $c_k \geq 0$ such that*

$$|I_k(u) - I_k(z)| \leq c_k|u - z|, \quad k = 1, 2, \ldots, m, \quad \textit{for each} \;\; u, z \in \mathbb{R}^n;$$

($\mathcal{H}_2$) *There exist a function $p \in L^1(J, \mathbb{R}^+)$ such that*

$$H_d(F(t, z_1), F(t, z_2)) \leq p(t)\|z_1 - z_2\| \;\; \textit{for all} \;\; z_1,\, z_2 \in \mathbb{R}^n$$

and

$$d(0, F(t, 0)) \leq p(t), \;\; t \in J.$$

If $H_* \left[\sum_{k=1}^{m} c_k + \|p\|_{L^1} \right] < 1$, then the solutions set $S(F, 0, b)$ of the problem (1)–(3) is a contractible set.

Proof. Let $f \subset F$ be measurable and locally Lipschitz. Consider the single-valued problem

$$y'(t) - \lambda y(t) = f(t, y(t)), \;\; a.e. \; t \in J\backslash\{t_1, t_2, \ldots, t_m\} \tag{12}$$

$$y(t_k^+) - y(t_k) = I_k(x(t_k^-)), \;\; k = 1, 2, \ldots, m \tag{13}$$

$$y(0) = y(b). \tag{14}$$

By the Banach fixed point theorem, we can prove that the problem (12)–(14) has exactly one solution. From Theorem 5.1 in [27], the set $S(F,0,b)$ is compact in PC. We define the homotopy $h : S(F,0,b) \times [0,1] \to S(F,0,b)$ by

$$h(y,\alpha) = \begin{cases} y, & \text{for } \alpha = 1 \text{ and } y \in S(F,0,b), \\ \overline{x}, & \text{for, } \alpha = 0, \end{cases}$$

where $\overline{x} = S(f,0,b)$ is exactly one solution of the problem (12)–(14). Note that

$$h(y,\alpha)(t) = \begin{cases} y(t), & \text{for } 0 \le t \le \alpha b, \\ \overline{x}(t), & \text{for, } \alpha b < t \le b, \end{cases}$$

Now we prove that h is a continuous homotopy. Let $(y_n, \alpha_n) \in S(F,0,b) \times [0,1]$ such that $(y_n, \alpha_n) \to (y,\alpha)$. We shall show that $h(y_n,\alpha_n) \to h(y,\alpha)$. We have

$$h(y_n,\alpha_n)(t) = \begin{cases} y_n(t), & \text{for } t \in [0,\alpha_n b], \\ \overline{x}(t), & \text{for, } t \in (\alpha_n b, b]. \end{cases}$$

If $\lim_{n\to\infty} \alpha_n = 0$, then

$$h(y,\alpha)(t) = \overline{x}(t), \qquad t \in (0,b].$$

Hence, $\|h(y_n,\alpha_n) - h(y,\alpha)\|_{PC} \to 0$ as $n \to \infty$. If $\alpha_n \neq 0$ and $0 < \lim_{n\to\infty} \alpha_n = \alpha < 1$, then

$$h(y,\alpha)(t) = \begin{cases} y(t), & \text{for } t \in [0,\alpha b], \\ \overline{x}(t), & \text{for } t \in (\alpha b, b]. \end{cases}$$

Since $y_n \in S(F,0,b)$, there exist $v_n \in S_{F,y_n}$ such that

$$y_n(t) = \int_0^b H(t,s)v_n(s)ds + \sum_{k=1}^m H_k(t,t_k)I_k(y_n(t_k)), \ t \in [0,\alpha_n b].$$

Since y_n converges to y, there exists $R > 0$ such that

$$\|y_n\|_{PC} \le R.$$

Hence, from $(\mathcal{R}_1)$, we have

$$|v_n(t)| \le p(t)R^\alpha + q(t), \ \text{a.e. } t \in J$$

which implies

$$v_n(t) \in p(t)R^\alpha + q(t)B(0,1).$$

This implies that there exists a subsequence $v_{n_m}(t)$ converge in $\mathbb{R}^n$ to $v(t)$. Since $F(t,\cdot)$ is upper semicontinuous, then for every $\epsilon > 0$, there exist $n_0 \geq 0$ such that for every $n \geq n_0$, we have

$$v_n(t) \in F(t, y_n(t)) \subset F(t, y(t)) + \epsilon B(0,1) \ \text{ a.e. } t \in [0, \alpha b].$$

Using the compactness of $F(\cdot,\cdot)$ we then have

$$v(t) \in F(t, y(t)) + \epsilon B(0,1) \quad \text{which implies} \quad v(t) \in F(t, y(t)) \ \text{ a.e. } \ t \in J.$$

From the Lebesgue Dominated Convergence Theorem, we have that $v \in L^1(J, \mathbb{R}^n)$, so $v \in S_{F,y}$. Using the continuity of I_k, we have

$$y(t) = \int_0^b H(t,s)v(s)ds + \sum_{k=1}^{m} H(t,t_k)I_k(y(t_k)) \ \text{ for } \ t \in [0, \alpha b].$$

If $t \in (\alpha_n b, b]$, then

$$h(y_n, \alpha_n)(t) = h(y, \alpha)(t).$$

Thus,

$$\|h(y_n, \alpha_n) - h(y, \alpha)\|_{PC} \to 0 \text{ as } n \to \infty.$$

In the case where $\alpha = 1$, we have

$$h(y, \alpha) = y.$$

Hence, h is a continuous function, $h(y, 0) = \overline{x}$, and $h(y, 1) = y$. Therefore, $S(F, 0, b)$ contracts to the point $\overline{x} = S(f, 0, b)$. □

4.1 σ-selectionable multivalued maps

The next two definitions and the theorem that follows can be found in [20, 29] (see also [7], p. 86).

Definition 4.9 *We say that a map $F : X \to \mathcal{P}(Y)$ is $\sigma-$Ca-selectionable if there exists a decreasing sequence of compact valued u.s.c. maps $F_n : X \to Y$ satisfying:*

(a) F_n has a Carathédory selection, for all $n \geq 0$ (F_n are called Ca-selectionable);

(b) $F(x) = \bigcap_{n\geq 0} F_n(x)$, for all $x \in X$.

Definition 4.10 *We say that a multivalued map $\phi : [0, a] \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ with closed values is upper-Scorza-Dragoni if, given $\delta > 0$, there exists a closed subset $A_\delta \subset [0, a]$ such that the measure $\mu([0, a] \setminus A_\delta) \leq \delta$ and the restriction ϕ_δ of ϕ to $A_\delta \times \mathbb{R}^n$ is u.s.c.*

Theorem 4.11 ([20, Theorem 19.19]) *Let E and E_1 be two separable Banach spaces and let $F : [a,b] \times E \to \mathcal{P}_{cp,cv}(E_1)$ be an upper-Scorza-Dragoni map. Then F is $\sigma-$Ca-selectionable, the maps $F_n : [a,b] \times E \to \mathcal{P}(E_1)$ $(n \in \mathbb{N})$ are almost upper semicontinuous, and*

$$F_n(t,e) \subset \overline{conv}\left(\cup_{x\in E} F_n(t,x)\right).$$

Moreover, if F is integrably bounded, then F is $\sigma-$mLL-selectionable.

We are now in a position to state and prove another characterization of the geometric structure of the set $S(F,0,b)$ of all solutions of the problem (1)–(3).

Theorem 4.12 *Let $F\colon\ J \times \mathbb{R}^n \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ be a Carathéodory and mLL-selectionable multi-valued map and assume that conditions $(\mathcal{R}_1)$–$(\mathcal{R}_2)$ and $(\mathcal{H}_1)$–$(\mathcal{H}_2)$ hold with*

$$H_*\left[\|p\|_{L^1} + \sum_{k=1}^{k=m} c_k\right] < 1.$$

Then, $S(F,0,b)$ is an R_δ-set.

Proof. Since F is $\sigma-$Ca-selectionable, there exists a decreasing sequence of multivalued maps $F_k : J \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ $(k \in \mathbb{N})$ that have Carathéodory selections and satisfy

$$F_{k+1}(t,u) \subset F_k(t,x) \text{ for almost all } t \in J \text{ and all } x \in \mathbb{R}^n$$

and

$$F(t,x) = \bigcap_{k=0}^{\infty} F_k(t,x),\ x \in \mathbb{R}^n.$$

Then,

$$S(F,0,b) = \bigcap_{k=0}^{\infty} S(F_k,0,b).$$

From Theorems 5.1 and 5.2 in [27], the sets $S(F_k,0,b)$ are compact for all $k \in \mathbb{N}$. Using Theorem 4.8, the sets $S(F_k,0,b)$ are contractible for each $k \in \mathbb{N}$. Hence, $S(F,0,b)$ is an R_δ-set. □

Alternately, we have the following result.

Theorem 4.13 *Let $F\colon\ J \times \mathbb{R}^n \to \mathcal{P}_{cp,cv}(\mathbb{R}^n)$ be an upper-Scorza-Dragoni. Assume that all conditions of Theorem 4.12 are satisfied. Then the solution set $S(F,0,b)$ is an R_δ-set.*

Proof Since F is upper-Scorza-Dragoni, then from Theorem 4.11, F is a $\sigma-Ca$-selection map. Therefore $S(F,0,b)$ is an R_δ-set.

5 Filippov's Theorem

Let A be a subset of $J\times\mathbb{R}^n$. We say that A is $\mathcal{L}\otimes\mathcal{B}$ *measurable* if A belongs to the σ-algebra generated by all sets of the form $\mathcal{J}\times\mathbb{R}^n$ where $\mathcal{J}$ is Lebesgue measurable in J and D is Borel measurable in $\mathbb{R}^n$. A subset A of $L^1(J,\mathbb{R})$ is *decomposable* if for all $u,v\in A$ and measurable $\mathcal{J}\subset J$, $u\chi_{\mathcal{J}}+v\chi_{J-\mathcal{J}}\in A$, where χ stands for the characteristic function. The family of all nonempty closed and decomposable subsets of $L^1(J,\mathbb{R}^n)$ is denoted by $\mathcal{D}$.

Definition 5.1 *Let Y be a separable metric space and let $N:Y\to\mathcal{P}(L^1(J,\mathbb{R}^n))$ be a multivalued operator. We say N has property (BC) if*

1) N is lower semi-continuous (l.s.c.), and

2) N has nonempty closed and decomposable values.

Let $F:J\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ be a multivalued map with nonempty compact values. Assign to F the multivalued operator

$$\mathcal{F}:PC\to\mathcal{P}(L^1(J,\mathbb{R}^n))$$

by letting

$$\mathcal{F}(y)=\{w\in L^1(J,\mathbb{R}^n):w(t)\in F(t,y(t))\text{ for a.e. }t\in J\}.$$

The operator $\mathcal{F}$ is called the Niemytzki operator associated to F.

Definition 5.2 *Let $F:J\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ be a multivalued function with nonempty compact values. We say F is of lower semi-continuous type (l.s.c. type) if its associated Niemytzki operator $\mathcal{F}$ is lower semi-continuous and has nonempty closed and decomposable values.*

Next, we state a selection theorem due to Bressan and Colombo.

Theorem 5.3 ([13]) *Let Y be separable metric space and let $N:Y\to\mathcal{P}(L^1(J,\mathbb{R}^n))$ be a multivalued operator that has property (BC). Then N has a continuous selection, i.e., there exists a continuous (single-valued) function $\tilde{g}:Y\to L^1(J,\mathbb{R}^n)$ such that $\tilde{g}(y)\in N(y)$ for every $y\in Y$.*

The following result is due to Colombo et al.

Proposition 5.4 ([16]) *Consider a l.s.c. multivalued map $G:S\to\mathcal{D}$ and assume that $\phi:S\to L^1(J,\mathbb{R}^n)$ and $\psi:S\to L^1(J,\mathbb{R}^+)$ are continuous maps such that for every $s\in S$, the set*

$$H(s)=\overline{\{u\in G(s):|u(t)-\phi(s)(t)|<\psi(s)(t)\}}$$

is nonempty. Then the map $H:S\to\mathcal{D}$ is l.s.c. and admits a continuous selection.

Let us introduce the following hypotheses which are assumed hereafter.

$(\mathcal{H}_3)$ $F : J \times E \longrightarrow \mathcal{P}(E)$ is a nonempty compact valued multivalued map such that:
a) $(t,y) \mapsto F(t,y)$ is $\mathcal{L} \otimes \mathcal{B}$ measurable;
b) $y \mapsto F(t,y)$ is lower semi-continuous for a.e. $t \in J$.

$(\mathcal{H}_4)$ For each $q > 0$, there exists a function $h_q \in L^1(J, \mathbb{R}^+)$ such that

$$\|F(t,y)\| \leq h_q(t) \text{ for a.e. } t \in J \text{ and for } y \in \mathbb{R}^n \text{ with } \|y\| \leq q.$$

The following lemma is crucial in the proof of our main theorem.

Lemma 5.5 ([19]). *Let $F : J \times E \to \mathcal{P}(E)$ be a multivalued map with nonempty, compact values. Assume that $(\mathcal{H}_3)$ and $(\mathcal{H}_4)$ hold. Then F is of l.s.c. type.*

The following two lemmas are concerned with the measurability of multi-functions; they will be needed in this section. The first one is the well known Kuratowski-Ryll-Nardzewski selection theorem.

Lemma 5.6 ([20, Theorem 19.7]) *Let E be a separable metric space and G a multi-valued map with nonempty closed values. Then G has a measurable selection.*

Lemma 5.7 ([43]) *Let $G : J \to \mathcal{P}(E)$ be a measurable multifunction and let $g : J \to E$ be a measurable function. Then for any measurable $v : J \to \mathbb{R}_+$ there exists a measurable selection u of G such that*

$$|u(t) - g(t)| \leq d(g(t), G(t)) + v(t).$$

Corollary 5.8 *Let $G : [0,b] \to \mathcal{P}_{cp}(E)$ be a measurable multifunction and $g : [0,b] \to E$ be a measurable function. Then there exists a measurable selection u of G such that*

$$|u(t) - g(t)| \leq d(g(t), G(t)).$$

Proof Let $v_\epsilon : [0,b] \to \mathbb{R}_+$ be defined by $v_\epsilon(t) = \epsilon > 0$. Then, from Lemma 5.7, there exist a measurable selection u_ϵ of G such that

$$|u_\epsilon(t) - g(t)| \leq d(g(t), G(t)) + \epsilon.$$

We take $\epsilon = \dfrac{1}{n}$, $n \in \mathbb{N}$, hence for every $n \in \mathbb{N}$, we have

$$|u_n(t) - g(t)| \leq d(g(t), G(t)) + \frac{1}{n}.$$

Using the fact that G has compact values, we may pass to a subsequence if necessary to get that $u_n(\cdot)$ converges to a measurable function u in E. Thus,

$$|u(t) - g(t)| \leq d(g(t), G(t))$$

completing the proof of the corollary. □

In the case of both differential equations and inclusions, existence results for problem (1)–(3) can be found in [25, 26, 37]. The main result in this section is contained in the following theorem. It is a Filippov type result for problem (1)–(3).

Theorem 5.9 *In addition to* $(\mathcal{H}_1)$, $(\mathcal{H}_3)$, *and* $(\mathcal{H}_4)$, *assume that the following conditions hold.*

$(\mathcal{H}_5)$ *There exist a function* $p \in L^1(J, \mathbb{R}^+)$ *such that*

$$H_d(F(t, z_1), F(t, z_2)) < p(t)\|z_1 - z_2\| \quad \text{for all} \ \ z_1,\, z_2 \in \mathbb{R}^n.$$

$(\mathcal{H}_6)$ *There exists continuous mappings* $g(\cdot) : PC \to L^1(J, \mathbb{R}^n)$ *and* $x \in PC$ *such that*

$$\begin{cases} x'(t) - \lambda x(t) &= g(x)(t), \qquad a.e. \ \ t \in J\backslash\{t_1, \dots, t_m\}, \\ x(t_k^+) - x(t_k) &= I_k(x(t_k^-)), \quad k = 1, 2, \dots, m, \\ x(0) &= x(b). \end{cases} \tag{15}$$

If

$$H_*\|p\|_{L^1} + H_*\sum_{k=1}^{m} c_k < 1$$

and there exists $\epsilon \in (0, 1]$ *such that*

$$d(g(y_0)(t), F(t, y_0(t)) < \epsilon \quad \text{and} \quad \frac{\epsilon H_*\|p\|_{L^1}}{1 - H_*\|p\|_{L^1} - H_*\sum_{k=1}^{m} c_k} < 1,$$

then the problem (1)–(3) has at least one solution y *satisfying the estimates*

$$\|y - x\|_{PC} \ \ \leq \ \ \frac{2H_*\|p\|_{L^1}}{\left(1 - H_*\sum_{k=1}^{m} c_k\right)\left(1 - H_*\sum_{k=1}^{m} c_k - H_*\|p\|_{L^1}\right)}$$

and

$$|y'(t) - \lambda y(t) - g(x)(t)| \leq 2\widetilde{H}p(t) \ \ a.e. \ t \in J,$$

where

$$\widetilde{H} = \frac{H_*\|p\|_{L^1}}{\left(1 - H_*\sum_{k=1}^{m} c_k - H_*\|p\|_{L^1}\right)}$$

and

$$H_* = \sup\{H(t, s) \mid (t, s) \in J \times J\}.$$

Proof Let $f_0(y_0)(t) = g(x)(t)$, $t \in J$, and

$$y_0(t) = \int_0^b H(t, s)f_0(y_0)(s)ds + \sum_{k=0}^{m} H(t, t_k)I_k(x(t_k)), \qquad y_0(t_k) = x(t_k).$$

Let $G_1\colon\ PC \to \mathcal{P}(L^1(J,\mathbb{R}^n))$ be given by

$$G_1(y) = \{v \in L^1(J,\mathbb{R}^n) : v(t) \in F(t,y(t)) \text{ a.e. } t \in J\}.$$

and $\widetilde{G}_1 : PC \to \mathcal{P}(L^1(J,\mathbb{R}^n))$ be defined by

$$\widetilde{G}_1(y) = \overline{\{v \in G_1(y) : |v(t) - g(y_0)(t)| < p(t)|y(t) - y_0(t)| + \epsilon\}}.$$

Since $t \to F(t,y(t))$ is a measurable multifunction, by Corollary 5.8, there exists a function v_1 which is a measurable selection of $F(t,y(t))$ a.e. $t \in J$ such that

$$\begin{aligned} |v_1(t) - g(y_0)(t)| &\leq d(g(y_0)(t), F(t,y(t))) \\ &< p(t)|y_0(t) - y(t)| + \epsilon. \end{aligned}$$

Thus, $v_1 \in \widetilde{G}_1(y) \neq \emptyset$. By Lemma 5.5, F is of lower semi-continuous type. Then $y \to \widetilde{G}_1(y)$ is l.s.c. and has decomposable values. Thus, $y \to \widetilde{G}_1(y)$ is l.s.c with decomposable values.

By Theorem 5.3, there exists a continuous function $f_1 : PC \to L^1(J,\mathbb{R}^n)$ such that $f_1(y) \in \widetilde{G}_1(y)$ for all $y \in PC$. Consider the problem

$$y'(t) - \lambda y(t) = f_1(y)(t), \quad t \in J,\ t \neq t_k, \quad k = 1,2,\dots,m, \tag{16}$$

$$\Delta y|_{t=t_k} = I_k(y(t_k^-)), \quad k = 1,2,\dots,m, \tag{17}$$

$$y(0) = y(b). \tag{18}$$

From [10, 26], the problem (16)–(18) has at least one solution which we denote by y_1. Consider

$$y_1(t) = \int_0^b H(t,s)f_1(y_1)(s)ds + \sum_{k=0}^{m} H(t,t_k)I_k(y_1(t_k)), \qquad t \in J,$$

where y_1 is a solution of the problem (16)–(18). For every $t \in J$, we have

$$\begin{aligned} |y_1(t) - y_0(t)| &\leq \int_0^b |H(t,s)||f_1(y_1)(s) - f_0(y_0)(s)|\,ds \\ &\quad + \sum_{k=1}^{m} |H(t,s)||I_k(y_1(t_k)) - I_k(y_0(t_k))| \\ &\leq H_*\|p\|_{L^1}\|y_1 - y_0\|_{PC} + H_*\|p\|_{L^1}\epsilon + H_*\sum_{k=1}^{m} c_k|y_1(t_k) - y_0(t_k)|. \end{aligned}$$

Then,

$$\|y_1 - y_0\|_{PC} \leq \frac{H_*\|p\|_{L^1}\epsilon}{1 - H_*\|p\|_{L^1} - H_*\displaystyle\sum_{k=1}^{m} c_k}.$$

Define the set-valued map $G_2 : PC \to \mathcal{P}(L^1(J, \mathbb{R}^n)$ by

$$G_2(y) = \{v \in L^1(J, \mathbb{R}^n) : v(t) \in F(t, y(t)), \text{ a.e. } t \in J\}$$

and

$$\widetilde{G}_2(y) = \overline{\{v \in G_2(y) : |v(t) - f_1(y_1)(t)| < p(t)|y(t) - y_1(t)| + \epsilon p(t)|y_1(t) - y_0(t)|\}}.$$

Since $t \to F(t, y(t)$ is measurable, by Corollary 5.8, there exists a function $v_2 \in \widetilde{G}_2$ which is a measurable selection of $F(t, y_1(t))$ a.e. $t \in J$ such that

$$\begin{aligned} |v_2(t) - f_1(y_1)(t)| &\leq d(f_1(y_1)(t), F(t, y(t))) \\ &\leq H_d(f_1(y_1(t)), F(t, y(t))) \\ &\leq p(t)|y_1(t) - y(t)| \\ &< p(t)|y_1(t) - y(t)| + \epsilon p(t)|y_1(t) - y_0(t)|. \end{aligned}$$

Then, $v_2 \in \widetilde{G}_2(y) \neq \emptyset$. Using the above method, we can prove that $\widetilde{G}_2$ has at least one continuous selection denoted by f_2. Thus, there exists $y_2 \in PC$ such that

$$y_2(t) = \int_0^b H(t, s) f_2(y_2)(s) ds + \sum_{k=1}^{m} H(t, t_k) I_k(y_2(t_k)), \qquad t \in J,$$

and y_2 is a solution of the problem

$$\left\{ \begin{array}{rcll} y'(t) - \lambda y(t) &=& f_2(y)(t), & \text{a.e. } t \in J \backslash \{t_1, \ldots, t_m\}, \\ y(t_k^+) - y(t_k) &=& I_k(y(t_k^-)), & k = 1, 2, \ldots, m, \\ y(0) &=& y(b). & \end{array} \right. \tag{19}$$

We then have

$$\begin{aligned} |y_2(t) - y_1(t)| &\leq H_* \int_0^b |f_2(y_2)(s) - f_1(y_1)(s)|\, ds + H_* \sum_{k=1}^{m} c_k |y_2(t_k) - y_1(t_k)| \\ &\leq H_* \int_0^b p(s)|y_2(s) - y_1(s)|\, ds + H_* \int_0^b \epsilon\, p(s)|y_1(s) - y_0(s)|\, ds \\ &+ H_* \sum_{k=1}^{m} c_k |y_2(t_k) - y_1(t_k)| \\ &\leq H_* \|p\| \int_0^b |y_2(s) - y_1(s)|\, ds + \epsilon H_* \|p\|_{L^1} \|y_1 - y_0\|_{PC} \\ &+ H_* \sum_{k=1}^{m} c_k |y_2(t_k) - y_1(t_k)|. \end{aligned}$$

Thus,

$$\|y_2 - y_1\|_{PC} \le \frac{\epsilon^2 H_*^2 \|p\|_{L^1}^2}{\left(1 - H_*\|p\|_{L^1} - H_* \sum_{k=1}^{m} c_k\right)^2}.$$

Let

$$G_3(y) = \{v \in L^1(J, \mathbb{R}^n) : v(t) \in F(t, y(t)) \text{ a.e. } t \in J\}$$

and

$$\widetilde{G}_3(y) = \overline{\{v \in G_3(y) : |v(t) - f_2(y_2)(t)| < p(t)|y(t) - y_2(t)| + \epsilon p(t)|y_2(t) - y_1(t)|\}}.$$

Arguing as we did for $\widetilde{G}_2$ shows that $\widetilde{G}_3$ is a l.s.c. type multi-valued map with nonempty decomposable values, so there exists a continuous selection $f_3(y) \in \widetilde{G}_3(y)$ for all $y \in PC$. Consider

$$y_3(t) = \int_0^b H(t,s) f_3(y_3)(s) ds + \sum_{k=1}^{m} I_k(y_3(t_k)), \qquad t \in J,$$

where y_3 is a solution of the problem

$$\left\{\begin{array}{rcl} y'(t) - \lambda y(t) &=& f_3(y)(t), \quad \text{a.e. } t \in J\backslash\{t_1, \dots, t_m\}, \\ y(t_k^+) - y(t_k) &=& I_k(y(t_k^-)), \quad k = 1, 2, \dots, m, \\ y(0) &=& y(b). \end{array}\right. \tag{20}$$

We have

$$|y_3(t) - y_2(t)| \;\; \le \;\; H_* \int_0^t |f_3(s) - f_2(s)|\, ds + H_* \sum_{k=1}^{m} c_k |y_3(t_k) - y_2(t_k)|.$$

Hence, from the estimates above, we have

$$\|y_3 - y_2\|_{PC} \le \frac{\epsilon^3 H_*^3 \|p\|_{L^1}^3}{\left(1 - H_*\|p\|_{L^1} - H_* \sum_{k=1}^{m} c_k\right)^3}.$$

Repeating the process for $n = 1, 2, \dots,$ we arrive at the bound

$$\|y_n - y_{n-1}\|_{PC} \le \frac{\epsilon^n H_*^n \|p\|_{L^1}^n}{\left(1 - H_*\|p\|_{L^1} - H_* \sum_{k=1}^{m} c_k\right)^n}. \tag{21}$$

By induction, suppose that (21) holds for some n. Let

$$\widetilde{G}_{n+1}(y) = \overline{\{v \in G_{n+1}(y) : |v(t) - f_n(y_n)(t)| < p(t)|y(t) - y_n(t)| + \epsilon p(t)|y_n(t) - y_{n-1}(t)|\}}.$$

Since again $\widetilde{G}_{n+1}$ is a l.s.c type multifunction, there exists a continuous function $f_{n+1}(y) \in \widetilde{G}_{n+1}(y)$ that allows us to define

$$y_{n+1}(t) = \int_0^b H(t,s) f_{n+1}(y_{n+1})(s)ds + \sum_{k=1}^{m} H(t,t_k) I_k(y_{n+1}(t_k)), \qquad t \in J. \tag{22}$$

Therefore,

$$|y_{n+1}(t) - y_n(t)| \leq H_* \int_0^b |f_{n+1}(y_{n+1})(s) - f_n(y_n)(s)|\, ds + H_* \sum_{k=1}^{m} c_k |y_n(t_k) - y_{n+1}(t_k)|.$$

Thus, we arrive at

$$\|y_{n+1} - y_n\|_{PC} \leq \frac{\epsilon^{n+1} H_*^{n+1} \|p\|_{L^1}^{n+1}}{\left(1 - H_* \|p\|_{L^1} - H_* \sum_{k=1}^{m} c_k\right)^{n+1}}. \tag{23}$$

Hence, (21) holds for all $n \in \mathbb{N}$, and so $\{y_n\}$ is a Cauchy sequence in PC, converging uniformly to a function $y \in PC$. Moreover, from the definition of $G_n(y)$, $n \in \mathbb{N}$,

$$|f_{n+1}(y_{n+1})(t) - f_n(y_n)(t)| \leq p(t)|y_n(t) - y_{n-1}(t)| \quad \text{for a.e. } t \in J.$$

Therefore, for almost every $t \in J$, $\{f_n(y_n)(t) : n \in \mathbb{N}\}$ is also a Cauchy sequence in $\mathbb{R}^n$ and so converges almost everywhere to some measurable function $f(\cdot)$ in $\mathbb{R}^n$. Moreover, since $f_0 = g$, we have

$$\begin{aligned}
|f_n(y_n)(t)| &\leq |f_n(y_n)(t) - f_{n-1}(y_{n-1})(t)| + |f_{n-1}(y_{n-1})(t) - f_{n-2}(y_{n-2})(t)| + \ldots \\
&\quad + |f_2(y_2)(t) - f_1(y_1)(t)| + |f_1(y_1)(t) - f_0(y_0)(t)| + |f_0(y_0)(t)| \\
&\leq \sum_{k=1}^{n} p(t)|y_k(t) - y_{k-1}(t)| + |f_0(y_0)(t)| \\
&\leq p(t) \sum_{k=1}^{\infty} |y_k(t) - y_{k-1}(t)| + |g(x)(t)| \\
&\leq 2\widetilde{H} p(t) + |g(x)(t)|.
\end{aligned}$$

Then, for all $n \in \mathbb{N}$,

$$|f_n(y_n)(t)| \leq 2\widetilde{H} p(t) + g(x)(t) \text{ a.e. } t \in J. \tag{24}$$

From (24) and the Lebesgue Dominated Convergence Theorem, we conclude that $f_n(y_n)$ converges to $f(y)$ in $L^1(J, \mathbb{R}^n)$. Passing to the limit in (22), the function

$$y(t) = \int_0^b H(t,s) f(y)(s)ds + \sum_{k=1}^{m} H(t,t_k) I_k(y(t_k)), \quad t \in J,$$

is a solution to the problem (1)–(3).

Next, we give estimates for $|y'(t)-\lambda y(t)-g(x)(t)|$ and $|x(t)-y(t)|$. We have

$$\begin{aligned}|y'(t)-\lambda y(t)-g(x)(t)| &= |f(y)(t)-f_0(x)(t)|\\ &\le |f(y)(t)-f_n(y_n)(t)|+|f_n(y_n)(t)-f_0(x)(t)|\\ &\le |f(y)(t)-f_n(y_n)(t)|+\sum_{k=1}^{n}|f_k(y_k)(t)-f_{k-1}(y_{k-1})(t)|\\ &\le |f(y)(t)-f_n(y_n)(t)|+2\sum_{k=1}^{n}p(t)|y_k(t)-y_{k-1}(t)|.\end{aligned}$$

Using (23) and passing to the limit as $n\to+\infty$, we obtain

$$\begin{aligned}|y'(t)-\lambda y(t)-g(x)(t)| &\le 2p(t)\sum_{k=1}^{\infty}|y_{k-1}(t)-y_{k-1}(t)|\\ &\le 2p(t)\sum_{k=1}^{\infty}\frac{H_*^k\|p\|_{L^1}^k}{\left(1-H_*\|p\|_{L^1}-H_*\sum_{i=1}^{m}c_i\right)^k},\end{aligned}$$

so

$$|y'(t)-\lambda y(t)-g(x)(t)|\le 2\widetilde{H}p(t),\quad t\in J.$$

Similarly,

$$\begin{aligned}|x(t)-y(t)| &= \Big|\int_0^b H(t,s)g(x)(s)ds+\sum_{k=1}^{m}H(t,t_k)I_k(x(t_k))\\ &\quad -\int_0^b H(t,s)f(y)(s)ds-\sum_{k=1}^{m}H(t,t_k)I_k(y(t_k))\Big|\\ &\le H_*\int_0^b|f(y)(s)-f_0(y_0)(s)|ds+H_*\sum_{k=1}^{m}c_k|x(t_k)-y(t_k)|\\ &\le H_*\int_0^b|f(y)(s)-f_n(y_n)(s)|ds+H_*\int_0^b|f_n(y_n)(s)-f_0(y_0)(s)|ds\\ &\quad +H_*\sum_{k=1}^{m}c_k|x(t_k)-y(t_k)|.\end{aligned}$$

As $n\to\infty$, we arrive at

$$\|x-y\|_{PC}\le\frac{2H_*\|p\|_{L^1}}{(1-H_*\sum_{k=1}^{m}c_k)(1-H_*\sum_{k=1}^{m}c_k-H_*\|p\|_{L^1})},$$

completing the proof of the theorem.

References

[1] Z. Agur, L. Cojocaru, G. Mazaur, R. M. Anderson, and Y. L. Danon, Pulse mass measles vaccination across age cohorts, *Proc. Nat. Acad. Sci. USA.* **90** (1993), 11698–11702.

[2] J. Andres, G. Gabor, and L. Górniewicz, Boundary value problems on infinite intervals, *Trans. Amer. Math. Soc.* **351** (1999), 4861-4903.

[3] J. Andres, G. Gabor, and L. Górniewicz, Topological structure of solution sets to multivalued asymptotic problems, *Z. Anal. Anwendungen* **18**, (1999), 1–20.

[4] J. Andres, G. Gabor, and L. Górniewicz, Acyclicity of solutions sets to functional inclusions, *Nonlinear Anal.*, **49** (2002), 671–688.

[5] N. Aronszajn, Le correspondant topologique de l'unicité dans la théorie des équations différentielles, *Ann. Math.* **43** (1942), 730–738.

[6] J. P. Aubin, *Impulse differential inclusions and hybrid systems: a viability approach*, Lecture Notes, Université Paris-Dauphine, 2002.

[7] J. P. Aubin, and A. Cellina, *Differential Inclusions,* Springer-Verlag, Berlin-Heidelberg, New York, 1984.

[8] J. P. Aubin, and H. Frankowska, *Set-Valued Analysis,* Birkhauser, Boston, 1990.

[9] D. D. Bainov and P. S. Simeonov, *Systems with Impulse Effect*, Ellis Horwood Ltd., Chichister, 1989.

[10] M. Benchohra, A. Boucherif, and A. Ouahab, On nonresonance impulsive functional differential inclusions with nonconvex valued right hand side, *J. Math. Anal. Appl.*, **282** (2003), 85–94.

[11] M. Benchohra, J. Henderson, and S. K. Ntouyas, *Impulsive Differential Equations and Inclusions*, Contemporay Mathematics and Its Applications, Hindawi Publishing, New York, 2006.

[12] R. Bielawski, L. Górniewicz, and S. Plaskacz, Topological approach to differential inclusions on closed subset of $\mathbb{R}^n$, *Dynamics Reported: Expositions in Dynamical Systems* (N.S.) **1** (1992), 225–250.

[13] A. Bressan and G. Colombo, Extensions and selections of maps with decomposable values, *Studia Math.* **90** (1988), 70–85.

[14] F. E. Browder and G. P. Gupta, Topological degree and nonlinear mappings of analytic type in Banach spaces, *J. Math. Anal. Appl.*, **26**, (1969), 730-738.

[15] C. Castaing, and M. Valadier, *Convex Analysis and Measurable Multifunctions*, Lecture Notes in Mathematics **580**, Springer-Verlag, Berlin-Heidelberg-New York, 1977.

[16] R. M. Colombo, A. Fryszkowski, T. Rzežuchowski, and V. Staticu, Continuous selection of solution sets of Lipschitzean differential inclusions, *Funkcial. Ekvac.* **34** (1991), 321–330.

[17] K. Deimling, *Multivalued Differential Equations*, de Gruyter, Berlin-New York, 1992.

[18] S. Djebali, L. Górniewicz, and A. Ouahab, Filippov's theorem and solution sets for first order impulsive semilinear functional differential inclusions, *Topol. Meth. Nonlin. Anal.*, **32**, (2008), 261-312.

[19] M. Frigon and A. Granas, Théorèmes d'existence pour des inclusions différentielles sans convexité, *C. R. Acad. Sci. Paris, Ser. I* **310** (1990), 819-822.

[20] L. Górniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Mathematics and Its Applications, **495**, Kluwer, Dordrecht, 1999.

[21] L. Górniewicz, On the solution sets of differential inclusions, *J. Math. Anal. Appl.* **113** (1986) 235–244.

[22] L. Górniewicz, *Homological methods in fixed point theory of multivalued maps*, Dissertations Math. **129** (1976), 1–71.

[23] L. Górniewicz, Topological structure of solutions sets: current results, CDDE 2000 Proceedings (Brno), *Arch. Math. (Brno)* **36** (2000), suppl., 343–382.

[24] L. Górniewicz, P. Nistri, and V. Obukhovskii, Differential inclusions on proximate retracts of Hilbert spaces, *Int. J. Nonlinear Diff. Equs.*, TMA **3** (1995), 13–26.

[25] J. R. Graef and A. Ouahab, Nonresonance impulsive functional dynamic boundary value inclusions on time scales, *Nonlinear Studies*, **15** (2008), 339-354.

[26] J. R. Graef and A. Ouahab, Nonresonance impulsive functional dynamic equations on times scales, *Int. J. Appl. Math. Sci.* **2** (2005), 65–80.

[27] J. R. Graef and A. Ouahab, First order impulsive differential inclusions with periodic condition, *Electron. J. Qual. Theory Differ. Equ.* **31** (2008), 1–40.

[28] A. Granas and J. Dugundji, *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[29] G. Haddad and J. M. Lasry, Periodic solutions of functional differential inclusions and fixed points of σ-selectionable correspondences, *J. Math. Anal. Appl.* **96** (1983), 295–312.

[30] Sh. Hu and N. Papageorgiou, *Handbook of Multivalued Analysis, Volume I: Theory*, Kluwer, Dordrecht, 1997.

[31] D. M. Hyman, On decreasing sequences of compact absolute retracts, *Fund. Math.* **64** (1969) 91–97.

[32] M. Kamenskii, V. Obukhovskii, and P. Zecca, *Condensing Multi-valued Maps and Semilinear Differential Inclusions in Banach Spaces*, de Gruyter, Berlin, 2001.

[33] M. Kisielewicz, *Differential Inclusions and Optimal Control*, Kluwer, Dordrecht, The Netherlands, 1991.

[34] E. Kruger-Thiemr, Formal theory of drug dosage regiments. I. *J. Theoret. Biol.*, **13** (1966), 212–235.

[35] V. Lakshmikantham, D. D. Bainov, and P. S. Simeonov, *Theory of Impulsive Differential Equations*, World Scientific, Singapore, 1989.

[36] J. M. Lasry and R. Robert, Analvse non linéaire multivoaque. Cahier de Math de la déicision No. 761 I. Université de Paris-Dauphine.

[37] A. Ouahab, *Some Contributions in Impulsive Differential Equations and Inclusions with Fixed and Variable Times*, *Ph.D. Dissertation,* University of Sidi-Bel-Abbès (Algeria), 2006.

[38] S. G. Pandit and S. G. Deo, *Differential Systems Involving Impulses*, Lecture Notes in Mathematics Vol 954, Springer-Verlag, 1982.

[39] G. Peano, Démonstration de l'integrabilite des équations differentielles ordinaires, *Mat. Annalen* **37** (1890), 182–238.

[40] G. Peano, Sull'integrabilità delle equazioni differenziali del primo ordine, *Atti. della Reale Accad. dell Scienze di Torino* **21** (1886), 677–685.

[41] A. M. Samoilenko and N. A. Perestyuk, *Impulsive Differential Equations*, World Scientific, Singapore, 1995.

[42] A. A. Tolstonogov, *Differential Inclusions in a Banach Space*, Kluwer, Dordrecht, 2000.

[43] Q. J. Zhu, On the solution set of differential inclusions in Banach Space, *J. Differential Equations* **93** (1991), 213–237.

(Received February 10, 2009)