Electronic Journal of Qualitative Theory of Differential Equations
2015, No. **69**, 1–11; doi: 10.14232/ejqtde.2015.1.69 http://www.math.u-szeged.hu/ejqtde/

Monotone iterative technique for $(k, n-k)$ conjugate boundary value problems

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Received 10 July 2015, appeared 23 October 2015
Communicated by Jeff R. L. Webb

Abstract. In this paper, a comparison result for $(k, n-k)$ conjugate boundary value problems is established. By using the monotone iterative technique and the method of upper and lower solutions, we investigate the existence of extremal solutions for a nonlinear differential equation with $(k, n-k)$ conjugate boundary value problems. As an application, an example is presented to illustrate the main results.

Keywords: $(k, n-k)$ conjugate boundary value problems, monotone iterative technique, comparison result.

2010 Mathematics Subject Classification: 34B15.

1 Introduction

We consider the existence of solution of the following $(k, n-k)$ conjugate boundary value problems for nonlinear ordinary differential equations, using the method of upper and lower solutions and its associated monotone iterative technique

$$\begin{cases} (-1)^{n-k}x^{(n)}(t) = f(t, x(t)), \ \ 0 < t < 1, \ \ n \geq 2, \ \ 1 \leq k \leq n-1, \\ x^{(i)}(0) = x^{(j)}(1) = 0, \ \ 0 \leq i \leq k-1, \ \ 0 \leq j \leq n-k-1, \end{cases} \tag{1.1}$$

where $n \geq 2$ and $k \geq 1$ are fixed integers.

The subject of $(k, n-k)$ conjugate boundary value problems for nonlinear ordinary differential equations derives from its theoretical challenge, and have close relationship with oscillation theory (see [4] for more details). Recently, many people paid attention to existence result of solution of $(k, n-k)$ conjugate boundary value problems, such as [1,2,5–7,9,10,12–20], by means of some fixed point theorems.

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The method of upper and lower solutions coupled with the monotone iterative technique plays a very important role in investigating the existence of solutions to ordinary differential equation problems, for example [3, 8, 11]. However, as far as we know, there are no papers dealing with the existence of solutions for $(k, n-k)$ conjugate boundary value problems, by means of the lower and upper solutions method.

The aims of this paper are to establish comparison result for $(k, n-k)$ conjugate boundary value problems and to investigate the existence of extremal solutions of problem (1.1).

The rest of this paper is organized as follows: in Section 2, we present some useful preliminaries and lemmas. The main results are given in Section 3. In Section 4, examples are presented to illustrate the main results.

2 Preliminaries and lemmas

Let $C[0,1]$ denote the Banach space of real-valued continuous function with norm $\|x\| = \max_{t\in[0,1]} |x(t)|$.

Throughout this paper, we shall use the following notation:

$$G(t,s) = \begin{cases} \dfrac{1}{(k-1)!(n-k-1)!} \displaystyle\int_0^{t(1-s)} u^{k-1}(u+s-t)^{n-k-1}du, & 0 \le t \le s \le 1, \\ \dfrac{1}{(k-1)!(n-k-1)!} \displaystyle\int_0^{s(1-t)} u^{n-k-1}(u+t-s)^{k-1}du, & 0 \le s \le t \le 1. \end{cases}$$

It is well known from the papers [10, 17] that $G(t,s)$ is the Green's function of the following homogeneous boundary value problem:

$$\begin{cases} (-1)^{n-k}x^{(n)}(t) = 0, \quad 0 < t < 1, \quad n \ge 2, \quad 1 \le k \le n-1, \\ x^{(i)}(0) = x^{(j)}(1) = 0, \quad 0 \le i \le k-1, \quad 0 \le j \le n-k-1. \end{cases}$$

Lemma 2.1 ([14, 19])**.** *The function $G(t,s)$ defined as above has the following properties:*

$$G(t,s) \le \beta s^{n-k}(1-s)^k, \qquad 0 \le t,s \le 1,$$
$$\frac{\beta}{n-1} g(t) s^{n-k}(1-s)^k \le G(t,s) \le \alpha g(t) s^{n-k-1}(1-s)^{k-1}, \qquad 0 \le t,s \le 1,$$

where

$$\beta = \frac{1}{(k-1)!(n-k-1)!}, \qquad g(t) = t^k(1-t)^{n-k},$$
$$\alpha = \frac{1}{\min\{k, n-k\}(k-1)!(n-k-1)!}.$$

In the rest of this paper, we also make the following assumptions:

(H_1) $\emptyset \ne I^+ \cup I^- \subset \{0, 1, \dots, k-1\}$, where $i \in I^+$ (or $i \in I^-$) means that the following $(k, n-k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k}x^{(n)}(t) = 0, \quad 0 < t < 1, \quad n \ge 2, \quad 1 \le k \le n-1, \\ x(0) = x'(0) = \cdots = x^{(i-1)}(0) = x^{(i+1)}(0) = \cdots = x^{(k-1)}(0) = 0, \\ x^{(i)}(0) = 1, \ x^{(j)}(1) = 0, \quad 0 \le j \le n-k-1 \end{cases}$$

has a unique nonnegative (or nonpositive) solution $I_i(t)$ with $|I_i(t)| \geq \frac{t^k(1-t)^{n-k}}{n!}$, $t \in [0,1]$.

(H_2) $\emptyset \neq J^+ \cup J^- \subset \{0, 1, \dots, n-k-1\}$, where $j \in J^+$ (or $j \in J^-$) means that the following $(k, n-k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k} x^{(n)}(t) = 0, \quad 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n-1, \\ x^{(i)}(0) = 0, \quad 0 \leq i \leq k-1, \quad x^{(j)}(1) = 1, \\ x(1) = x'(1) = \cdots = x^{(j-1)}(1) = x^{(j+1)}(1) = \cdots = x^{(n-k-1)}(1) = 0 \end{cases}$$

has a unique nonnegative (nonpositive) solution $J_j(t)$ with $|J_j(t)| \geq \frac{t^k(1-t)^{n-k}}{n!}$, $t \in [0,1]$.

Remark 2.2. It follows from (H_1) and (H_2) that for any $a_i, b_j \in \mathbb{R}$ $(0 \leq i \leq k-1,\ 0 \leq j \leq n-k-1)$ such that

$$a_i = 0, \quad \text{if } i \notin I^+ \cup I^-$$

and

$$b_j = 0, \quad \text{if } j \notin J^+ \cup J^-,$$

the $(k, n-k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k} x^{(n)}(t) = 0, \quad 0 < t < 1, \quad n \geq 2, \quad 1 \leq k \leq n-1, \\ x^{(i)}(0) = a_i, \ x^{(j)}(1) = b_j, \quad 0 \leq i \leq k-1, \quad 0 \leq j \leq n-k-1 \end{cases}$$

has a unique solution $\psi(t) = \sum_{i=0}^{k-1} a_i I_i(t) + \sum_{j=0}^{n-k-1} b_j J_j(t)$, in which we may take $I_i(t) = J_j(t) \equiv 0$ for $i \notin I^+ \cup I^-$ and $j \notin J^+ \cup J^-$. Moreover, if

$$a_i \geq 0, \quad \text{if } i \in I^+; \qquad a_i \leq 0, \quad \text{if } i \in I^-$$

and

$$b_j \geq 0, \quad \text{if } j \in J^+; \qquad b_j \leq 0, \quad \text{if } j \in J^-$$

hold, $\psi(t)$ becomes a nonnegative function.

Remark 2.3. We point out from examples below that the assumptions (H_1) and (H_2) appear naturally in the study involving $(k, n-k)$ conjugate boundary value problem.

Example 2.4. When $n = 3$, $k = 1$, the unique solution of

$$x'''(t) = 0, \quad x(0) = a, \quad x(1) = b, \quad x'(1) = c$$

can be explicitly given by

$$\psi(t) = aI_0(t) + bJ_0(t) + cJ_1(t),$$

where

$$I_0(t) = 1 - t^2 \geq 0, \qquad J_0(t) = -t^2 + 2t \geq 0, \qquad J_1(t) = -t(1-t) \leq 0, \ t \in [0,1].$$

Example 2.5 ([15])**.** When $n = 4$, $k = 2$, the unique solution of

$$x^{(4)}(t) = 0, \quad x(0) = a, \quad x(1) = b, \quad x'(0) = c, \quad x'(1) = d$$

can be explicitly given by

$$\psi(t) = aI_0(t) + bJ_0(t) + cI_1(t) + dJ_1(t),$$

where

$$I_0(t) = 2t^3 - 3t^2 + 1 \geq 0, \qquad J_0(t) = -2t^3 + 3t^2 \geq 0,$$
$$I_1(t) = t^3 - 2t^2 + t \geq 0, \qquad J_1(t) = t^3 - t^2 \leq 0, \qquad t \in [0,1].$$

Example 2.6. When $n = 5$, $k = 3$, the unique solution of

$$x^{(5)}(t) = 0, \quad x(0) = a, \quad x(1) = b, \quad x'(0) = c, \quad x'(1) = d, \quad x''(0) = e$$

can be explicitly given by

$$\psi(t) = aI_0(t) + bJ_0(t) + cI_1(t) + dJ_1(t) + eI_2(t),$$

where

$$I_0(t) = 3t^4 - 4t^3 + 1 \geq 0, \qquad J_0(t) = -3t^4 + 4t^3 \geq 0,$$
$$I_1(t) = t(2t+1)(1-t)^2 \geq 0, \qquad J_1(t) = t^3 - t^4 \leq 0,$$
$$I_2(t) = \frac{1}{2}t^2(1-t)^2 \geq 0, \qquad t \in [0,1].$$

Remark 2.7. Under assumptions (H_1), (H_2), we give the definition of lower and upper solution for $(k, n-k)$ conjugate boundary value problem.

Definition 2.8. $u \in C^n[0,1]$ is called a lower solution of $(k, n-k)$ conjugate boundary value problem if

$$\begin{cases} (-1)^{n-k}u^{(n)}(t) \leq f(t, u(t)), \ \ 0 < t < 1, \ \ n \geq 2, \ \ 1 \leq k \leq n-1, \\ u^{(i)}(0) \leq 0, \text{ if } i \in I^+; \ u^{(i)}(0) \geq 0, \text{ if } i \in I^-; \ u^{(i)}(0) = 0, \text{ if } i \notin I^+ \cup I^-; \\ u^{(j)}(1) \leq 0, \text{ if } j \in J^+; \ u^{(j)}(1) \geq 0, \text{ if } j \in J^-; \ u^{(j)}(1) = 0, \text{ if } j \notin J^+ \cup J^-. \end{cases}$$

Analogously, $v \in C^n[0,1]$ is called an upper solutions of $(k, n-k)$ conjugate boundary value problem if the above inequalities are reversed.

For example, u is a lower solution of $(3,2)$ conjugate boundary value problem if

$$\begin{cases} u^{(5)}(t) \leq f(t, u(t)), \ \ 0 < t < 1, \\ u(0) \leq 0, \ u'(0) \leq 0, \ u''(0) \leq 0; \\ u(1) \leq 0, \ u'(1) \geq 0. \end{cases}$$

Now we consider the linear $(k, n-k)$ conjugate boundary value problem

$$\begin{cases} (-1)^{n-k}x^{(n)}(t) = -Mx(t) + \sigma(t), \ 0 < t < 1, \ n \geq 2, \ 1 \leq k \leq n-1, \\ x^{(i)}(0) = a_i, \ x^{(j)}(1) = b_j, \ \ 0 \leq i \leq k-1, \ \ 0 \leq j \leq n-k-1 \end{cases} \tag{2.1}$$

where M is a nonnegative constant and $\sigma \in C[0,1]$, $a_i, b_j \in \mathbb{R}$.

Lemma 2.9. *If*

$$\alpha M B(n,n) < 1, \tag{2.2}$$

where α is given in Lemma 2.1 and $B(t,s)$ denotes the Beta function, then (2.1) has a unique solution x, which can be expressed by

$$x(t) = \psi(t) + \int_0^1 Q(t,s)\psi(s)ds + \int_0^1 H(t,s)\sigma(s)ds, \tag{2.3}$$

where $\psi(t)$ is given in Remark 2.2,

$$G_1(t,s) = -MG(t,s), \qquad Q(t,s) = \sum_{m=1}^{+\infty} G_m(t,s), \tag{2.4}$$

$$G_m(t,s) = (-M)^m \int_0^1 \cdots \int_0^1 G(t,r_1)G(r_1,r_2)\cdots G(r_{m-1},s)dr_1\cdots dr_{m-1},$$

and

$$H(t,s) = G(t,s) + \int_0^1 Q(t,\tau)G(\tau,s)d\tau.$$

All functions $G_n(t,s)$, $H(t,s)$, $Q(t,s)$ are continuous on $[0,1]\times[0,1]$ and the series on the right-hand side of (2.4) converges uniformly on $[0,1]\times[0,1]$.

Proof. It follows from the paper [10] that $x \in C^n[0,1]$ is a solution of (2.1) if and only if $x \in C[0,1]$ is a solution of the following operator equation

$$x + Tx = \varphi \tag{2.5}$$

with operator $T\colon C[0,1] \to C[0,1]$ given by

$$(Tx)(t) = M\int_0^1 G(t,s)x(s)ds,$$

and

$$\varphi(t) = \psi(t) + \int_0^1 G(t,s)\sigma(s)ds. \tag{2.6}$$

We shall prove $r(T) < 1$, where $r(T)$ denotes the spectral radius of operator T. Actually, for $x \in C[0,1]$, by Lemma 2.1, we have

$$\begin{aligned}
|Tx(t)| &\le M\int_0^1 G(t,s)|x(s)|ds \\
&\le \alpha M t^k(1-t)^{n-k}\int_0^1 s^{n-k-1}(1-s)^{k-1}ds\|x\| \\
&= \alpha M B(k,n-k)\|x\| t^k(1-t)^{n-k}.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
|T^2x(t)| &\le M\int_0^1 G(t,s)|Tx(s)|ds \\
&\le \alpha^2 M^2 B(k,n-k)\|x\| t^k(1-t)^{n-k}\int_0^1 s^{n-1}(1-s)^{n-1}ds \\
&= \alpha^2 M^2 B(k,n-k)B(n,n)\|x\| t^k(1-t)^{n-k}.
\end{aligned}$$

By the induction method, we have

$$|T^m x(t)| \le \alpha^m M^m B(k, n-k) B^{m-1}(n,n) \|x\| t^k (1-t)^{n-k},$$

which implies that $\|T^m\| \le \alpha^m M^m B(k, n-k) B^{m-1}(n,n)$. It follows from $r(T) = \lim_{m\to\infty} \|T^m\|^{1/m}$ that

$$r(T) \le \alpha M B(n,n) < 1.$$

This yields that the unique solution of operator equation (2.5) is given by

$$x = (I+T)^{-1}\varphi = (I - T + T^2 + \cdots + (-1)^m T^m + \cdots)\varphi.$$

Substituting (2.6) into the above equality, we get (2.3) and the proof is complete. □

Lemma 2.10. *Suppose that $x \in C^n[0,1]$ satisfies*

$$\begin{cases} (-1)^{n-k} x^{(n)}(t) \ge -Mx(t), \ \ 0<t<1, \ \ n \ge 2, \ \ 1 \le k \le n-1, \\ x^{(i)}(0) \ge 0, \ \textit{if } i \in I^+; \ x^{(i)}(0) \le 0, \ \textit{if } i \in I^-; \ x^{(i)}(0) = 0, \ \textit{if } i \notin I^+ \cup I^-, \\ x^{(j)}(1) \ge 0, \ \textit{if } j \in J^+; \ x^{(j)}(1) \le 0, \ \textit{if } j \in J^-; \ x^{(j)}(1) = 0, \ \textit{if } j \notin J^+ \cup J^-, \end{cases}$$

where the nonnegative constant M satisfies (2.2),

$$B(k, n-k)\left[M\alpha\beta + \frac{M^3\alpha^2\beta^2 B(n,n) B(k+1, n-k+1)}{1 - M^2\alpha^2\beta^2 B^2(n,n)}\right] < \frac{\beta}{n-1}, \tag{2.7}$$

$$MN\alpha + \frac{NM^3\alpha^2\beta B(n,n)B(k,n-k)}{1 - M^2\alpha^2\beta^2 B^2(n,n)} < \frac{1}{n!}, \tag{2.8}$$

in which

$$N = \max\left\{\int_0^1 s^{n-k-1}(1-s)^{k-1} y(s)ds : \ y \in \{|I_i|, \ i \in I^+ \cup I^-\} \cup \{|J_j|, \ j \in J^+ \cup J^-\}\right\}.$$

Then $x(t) \ge 0$ for $t \in [0,1]$.

Proof. Let $\sigma(t) = (-1)^{n-k} x^{(n)}(t) + Mx(t)$ and

$$a_i = x^{(i)}(0), \quad 0 \le i \le k-1; \quad b_j = x^{(j)}(1), \quad 0 \le j \le n-k-1.$$

Then $\sigma(t) \ge 0$ and

$$\begin{cases} a_i \ge 0, \ \text{if } i \in I^+; \ a_i \le 0, \ \text{if } i \in I^-; \ a_i = 0, \ \text{if } i \notin I^+ \cup I^-; \\ b_j \ge 0, \ \text{if } j \in J^+; \ b_j \le 0, \ \text{if } j \in J^-; \ b_j = 0, \ \text{if } j \notin J^+ \cup J^-. \end{cases}$$

By Lemma 2.9, (2.3) holds in which $\psi(t) \ge 0$ for $t \in [0,1]$. It follows from the expression of $G_m(t,s)$ that $G_m(t,s) \le 0$ when m is odd and $G_m(t,s) \ge 0$ when m is even. Thus, we obtain for $m = 3, 5, \ldots$, by using Lemma 2.1,

$$\begin{aligned}
G_m(t,s) = & -M^m \int_0^1 \cdots \int_0^1 G(t,r_1)G(r_1,r_2)\cdots G(r_{m-2},r_{m-1})G(r_{m-1},s)dr_1\cdots dr_{m-1} \\
\geq & -M^m \int_0^1 \cdots \int_0^1 \left(\alpha g(t) r_1^{n-k-1}(1-r_1)^{k-1}\right)\cdot\left(\alpha r_1^k (1-r_1)^{n-k} r_2^{n-k-1}(1-r_2)^{k-1}\right)\cdots \\
& \times \left(\alpha r_{m-2}^k (1-r_{m-2})^{n-k} r_{m-1}^{n-k-1}(1-r_{m-1})^{k-1}\right)\cdot\left(\beta s^{n-k}(1-s)^k\right) dr_1\cdots dr_{m-1} \\
= & -M^m \alpha^{m-1}\beta g(t) s^{n-k}(1-s)^k \int_0^1 r_1^{n-1}(1-r_1)^{n-1}dr_1 \\
& \times \int_0^1 r_2^{n-1}(1-r_2)^{n-1}dr_2 \cdots \int_0^1 r_{m-2}^{n-1}(1-r_{m-2})^{n-1}dr_{m-2} \\
& \times \int_0^1 r_{m-1}^{n-k-1}(1-r_{m-1})^{k-1}dr_{m-1} \\
= & -M^m \alpha^{m-1}\beta g(t) s^{n-k}(1-s)^k B^{m-2}(n,n)B(k,n-k).
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
H(t,s) &= G(t,s) + \int_0^1 Q(t,\tau)G(\tau,s)d\tau = G(t,s) + \sum_{m=1}^{+\infty}\int_0^1 G_m(t,\tau)G(\tau,s)d\tau \\
&\geq G(t,s) - M\int_0^1 G(t,\tau)G(\tau,s)d\tau + \sum_{m=1}^{+\infty}\int_0^1 G_{2m+1}(t,\tau)G(\tau,s)d\tau \\
&\geq \frac{\beta}{n-1} g(t)s^{n-k}(1-s)^k - M\alpha\beta g(t)s^{n-k}(1-s)^k\int_0^1 \tau^{n-k-1}(1-\tau)^{k-1}d\tau \\
&\quad - \sum_{m=1}^{+\infty} M^{2m+1}\alpha^{2m}\beta^2 g(t)s^{n-k}(1-s)^k B^{2m-1}(n,n)B(k,n-k)\int_0^1 \tau^{n-k}(1-\tau)^k d\tau \\
&= g(t)s^{n-k}(1-s)^k\Bigg[\frac{\beta}{n-1} - M\alpha\beta B(k,n-k) \\
&\quad - \sum_{m=1}^{+\infty} M^{2m+1}\alpha^{2m}\beta^2 B^{2m-1}(n,n)B(k,n-k)B(k+1,n-k+1)\Bigg].
\end{aligned}$$

and for $y \in \{I_i,\ i\in I^+\}\cup\{-I_i,\ i\in I^-\}\cup\{J_j,\ j\in J^+\}\cup\{-J_j, j\in J^-\}$,

$$\begin{aligned}
& y(t) + \int_0^1 Q(t,s)y(s)ds \\
&\quad \geq y(t) - M\int_0^1 G(t,s)y(s)ds + \sum_{m=1}^{+\infty}\int_0^1 G_{2m+1}(t,s)y(s)ds \\
&\quad \geq \frac{g(t)}{n!} - M\alpha g(t)\int_0^1 s^{n-k-1}(1-s)^{k-1}y(s)ds + \sum_{m=1}^{+\infty}\int_0^1 G_{2m+1}(t,s)y(s)ds \\
&\quad \geq \frac{g(t)}{n!} - M\alpha g(t)\int_0^1 s^{n-k-1}(1-s)^{k-1}y(s)ds \\
&\qquad - \sum_{m=1}^{+\infty} M^{2m+1}\alpha^{2m}\beta B^{2m-1}(n,n)B(k,n-k)g(t)\int_0^1 s^{n-k}(1-s)^k y(s)ds \\
&\quad \geq \frac{g(t)}{n!} - MN\alpha g(t) - N\sum_{m=1}^{+\infty} M^{2m+1}\alpha^{2m}\beta B^{2m-1}(n,n)B(k,n-k)g(t) \\
&\quad = g(t)\left[\frac{1}{n!} - MN\alpha - N\sum_{m=1}^{+\infty} M^{2m+1}\alpha^{2m}\beta B^{2m-1}(n,n)B(k,n-k)\right].
\end{aligned}$$

Thus, by (2.8), we have that $x(t) \geq 0$ for $t \in [0,1]$, and the lemma is proved. □

3 Main results

In this section, we prove the existence of extremal solutions of differential equation (1.1).

Theorem 3.1. *Let $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$; v_0, w_0 be lower and upper solutions of (1.1) such that $v_0(t) \leq w_0(t)$ on $[0,1]$. Suppose further that there exists $M > 0$ such that*

$$f(t,x) - f(t,y) \geq -M(x-y), \tag{3.1}$$

whenever $v_0(t) \leq y \leq x \leq w_0(t)$ and M satisfies (2.2), (2.7) and (2.8). Then there exist monotone sequences $\{v_m(t)\}, \{w_m(t)\}$ which converge uniformly on $[0,1]$ to the extremal solutions of problem (1.1) in the order interval $[v_0, w_0] = \{u \in C[0,1] : v_0(t) \leq u(t) \leq w_0(t),\ t \in [0,1]\}$.

Proof. For any $\eta \in [v_0, w_0]$, we consider the linear differential equation

$$\begin{cases} (-1)^{n-k} x^{(n)}(t) = -Mx(t) + f(t, \eta(t)) + M\eta(t),\ 0 < t < 1,\ n \geq 2,\ 1 \leq k \leq n-1, \\ x^{(i)}(0) = \ x^{(j)}(1) = 0,\ \ 0 \leq i \leq k-1,\ \ 0 \leq j \leq n-k-1. \end{cases} \tag{3.2}$$

By Lemma 2.9, (3.2) has a unique solution $x(t) = \int_0^1 H(t,s)[f(s,\eta(s)) + M\eta(s)]ds$ in $C[0,1]$. Define the mapping A by $A\eta = x$ with operator $A : [v_0, w_0] \to C[0,1]$ given by

$$(A\eta)(t) = \int_0^1 H(t,s)[f(s,\eta(s)) + M\eta(s)]ds$$

and use it to construct the sequences $\{v_m(t)\}, \{w_m(t)\}$. Let us prove that

(i) $v_0 \leq Av_0$, $Aw_0 \leq w_0$;

(ii) A is a monotone operator on $[v_0, w_0]$.

To prove (i), set $Av_0 = v_1$, where v_1 is the unique solution of (3.2) with $\eta = v_0$. Setting $p = v_1 - v_0$, we see that

$$\begin{cases} (-1)^{n-k} p^{(n)}(t) \geq -Mp(t),\ \ 0 < t < 1,\ \ n \geq 2,\ \ 1 \leq k \leq n-1, \\ p^{(i)}(0) \geq 0,\ \text{if } i \in I^+;\ p^{(i)}(0) \leq 0,\ \text{if } i \in I^-;\ p^{(i)}(0) = 0,\ \text{if } i \notin I^+ \cup I^-, \\ p^{(j)}(1) \geq 0,\ \text{if } j \in J^+;\ p^{(j)}(1) \leq 0,\ \text{if } j \in J^-;\ p^{(j)}(1) = 0,\ \text{if } j \notin J^+ \cup J^-. \end{cases}$$

This shows, by Lemma 2.10, that $p(t) \geq 0$ on [0,1] and hence $v_0 \leq Av_0$. Similarly, we can show that $Aw_0 \leq w_0$.

To prove (ii), let $\eta_1, \eta_2 \in [v_0, w_0]$ such that $\eta_1 \leq \eta_2$. Suppose that $x_1 = A\eta_1$, and $x_2 = A\eta_2$. Set $p = x_2 - x_1$ so that

$$\begin{cases} (-1)^{n-k} p^{(n)}(t) \geq -Mp(t),\ \ 0 < t < 1,\ \ n \geq 2,\ \ 1 \leq k \leq n-1, \\ p^{(i)}(0) = p^{(j)}(1) = 0,\ \ 0 \leq i \leq k-1,\ \ 0 \leq j \leq n-k-1, \end{cases} \tag{3.3}$$

here we have used the condition (3.1). By Lemma 2.10, (3.3) implies that $A\eta_1 \leq A\eta_2$ proving (ii).

Now let $v_m = Av_{m-1}$, $w_m = Aw_{m-1}$, $m = 1, 2, \dots$. From the foregoing arguments, we conclude that

$$v_0 \le v_1 \le \cdots \le v_m \le \cdots \le \cdots w_m \le \cdots \le w_1 \le w_0. \tag{3.4}$$

Obviously the sequences $\{v_m\}$, $\{w_m\}$ are uniformly bounded on $[0,1]$, and by (3.1), we have

$$\begin{aligned} f(t, v_0(t)) + Mv_0(t) &\le f(t, v_m(t)) + Mv_m(t) \\ &\le f(t, w_m(t)) + Mw_m(t) \le f(t, w_0(t)) + Mw_0(t), \qquad m \in \mathbb{N},\ t \in [0,1]. \end{aligned}$$

This together with the continuity of $H(t,s)$ on $[0,1] \times [0,1]$ imply that $\{v_m\}_{m=2}^{\infty} = \{Av_m\}_{m=1}^{\infty}$ and $\{w_m\}_{m=2}^{\infty} = \{Aw_m\}_{m=1}^{\infty}$ are two sequentially compact sets. As a result, there exist subsequences $\{v_{m_j}\}$, $\{w_{m_j}\}$ that converge uniformly on $[0,1]$. In view of (3.4), it also follows that the entire sequences $\{v_m\}$, $\{w_m\}$ converge uniformly and monotonically to their limit functions $v^*(t)$, $w^*(t)$ respectively, that is,

$$\lim_{m\to\infty} v_m(t) = v^*(t), \qquad \lim_{m\to\infty} w_m(t) = w^*(t), \quad \text{uniformly on } [0,1].$$

It is now easy to show that v^*, w^* are solutions of conjugate boundary value problem (1.1), using the corresponding integral equation

$$x(t) = (A\eta)(t) = \int_0^1 H(t,s)[f(s, \eta(s)) + M\eta(s)]ds$$

for (3.2).

Next, we prove that v^*, w^* are extremal solutions of (1.1) in $[v_0, w_0]$. In fact, we assume that x is any solution of (1.1). That is,

$$\begin{cases} (-1)^{n-k} x^{(n)}(t) = f(t, x(t)),\ 0 < t < 1,\ n \ge 2,\ 1 \le k \le n-1, \\ x^{(i)}(0) = \ x^{(j)}(1) = 0,\ \ 0 \le i \le k-1,\ \ 0 \le j \le n-k-1. \end{cases}$$

By (3.1) and Lemma 2.10, it is easy by induction to show that

$$v_m \le x \le w_m, \qquad m = 1, 2, 3 \dots. \tag{3.5}$$

Now, letting $m \to \infty$ in (3.5), we have $v^* \le x \le w^*$. That is, v^* and w^* are extremal solutions of (1.1) in $[v_0, w_0]$. □

4 Examples

Consider the following $(2,2)$ conjugate boundary value problems:

$$\begin{cases} x^{(4)}(t) = \dfrac{1}{5}(t^2 - x(t))^3 - \dfrac{1}{5}t^9,\ \ 0 < t < 1, \\ x(0) = x'(0) = x(1) = x'(1) = 0. \end{cases} \tag{4.1}$$

Let $f(t,x) = \frac{1}{5}(t^2 - x)^3 - \frac{1}{5}t^9$. Obviously, $f \in C([0,1] \times \mathbb{R}, \mathbb{R})$. Take $w_0(t) = t^2 - 3t^3/4$, $v_0(t) = 0$, then $v_0(t) \le w_0(t)$ for $t \in [0,1]$ and we have

$$\begin{cases} w_0^{(4)}(t) = 0 \ge -\dfrac{37}{320}t^9 = \dfrac{1}{5}(t^2 - w_0(t))]^3 - \dfrac{1}{5}t^9, & 0 < t < 1, \\ w_0(0) = w_0'(0) = 0, \ w_0(1) = \dfrac{1}{4} \ge 0, \ w_0'(1) = -\dfrac{1}{4} \le 0, \end{cases}$$

$$\begin{cases} v_0^{(4)}(t) = 0 \le \dfrac{t^6 - t^9}{5} = \dfrac{1}{5}(t^2 - v_0(t))^3 - \dfrac{1}{5}t^9, & 0 < t < 1, \\ v_0(0) = v_0'(0) = v_0(1) = v_0'(1) = 0. \end{cases}$$

Consequently, by Definition 2.8 and Example 2.5, v_0, w_0 are lower and upper solutions of (4.1) respectively. If $v_0(t) \le v \le u \le w_0(t)$, we have

$$f(t,u) - f(t,v) = \frac{1}{5}(t^2 - u)^3 - \frac{1}{5}v(t^2 - v)^3 \ge -\frac{3}{5}(u - v).$$

It is clear that $M = \frac{3}{5}$, $\alpha = \frac{1}{2}$, $\beta = 1$, $n = 4$, $k = 2$,

$$N = \max\left\{ \int_0^1 s(1-s)y(s)ds : \ y \in \{2t^3 - 3t^2 + 1, -2t^3 + 3t^2, t^3 - 2t^2 + t, t^2 - t^3\} \right\} = \frac{1}{12},$$

and so, it is easy to show that inequalities (2.2), (2.7) and (2.8) are satisfied.

By Theorem 3.1, problem (4.1) has extremal solutions in $[v_0, w_0]$.

Acknowledgements

The authors sincerely thank the reviewer for careful reading and useful comments that have led to the present improved version of the original paper. The Project Supported by NNSF of China (11371221, 11571207), the Specialized Research Foundation for the Doctoral Program of Higher Education of China (20123705110001), the Program for Scientific Research Innovation Team in Colleges, a project of Shandong province Higher Educational Science and Technology Program (J15LI16) and SDNSF(ZR2015AL002),

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