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Asymptotically almost periodic solutions of limit and almost periodic linear difference systems

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Abstract. In this paper, limit periodic and almost periodic homogeneous linear difference systems are considered. We study the systems in which the coefficient matrices are taken from a given bounded group and the elements of the matrices are from an infinite field with an absolute value. We show a condition on limit periodic and almost periodic systems which ensures, that the considered systems can be transformed into new systems having certain properties. The new systems possess non-asymptotically almost periodic solutions. The transformation can be done by arbitrarily small changes.

Keywords: limit periodicity, almost periodicity, asymptotic almost periodicity, limit periodic sequences, almost periodic sequences, linear difference systems.

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1 Introduction

We consider the homogeneous linear difference systems of the form

$$x_{k+1} = A_k \cdot x_k, \tag{1.1}$$

where $A_k \in X$. We suppose, that X is a bounded group of square matrices over an infinite field. The cases, when sequences $\{A_k\}$ are limit periodic and almost periodic, are studied. We are interested in non-asymptotically almost periodic solutions of the considered systems. Our current research is motivated by the following two facts. The smallest class of systems (1.1), which can have at least one non-asymptotically almost periodic solution and which generalize the pure periodic case, is formed by the limit periodic systems. The most studied class is given by the almost periodic systems.

Our main motivation comes from papers [7,12,13,22,23,26]. Papers [22] and [23] (and also [20]) are devoted to unitary and orthogonal homogeneous linear difference systems (1.1). It is shown in [22,23], that, in any neighbourhood of any orthogonal or unitary system, there exists a system of the form (1.1) with a non-almost periodic solution. In papers [7,12,13,26],

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general systems of the form (1.1) are studied. In [7,13], it is supposed, that X is a commutative group. In [12,26], transformable groups are studied. The results of these papers say that, in an arbitrary neighbourhood of any considered system (1.1), there exists a system of the same form without any almost periodic solution other than the trivial one. Our main goal is to complement these results. We investigate more general situations and show, that the systems of the form (1.1) with non-asymptotically almost periodic solutions form a dense subset of the set of all considered systems as well. To prove this result, we improve the method based on constructions introduced in papers [24,25].

The almost periodic (and also limit periodic) systems are studied closely. There are many papers from the field of almost periodic linear systems. In this paragraph, we point out the most relevant of them. In books [4,8,10,19,29], one can find the basic properties of limit periodic and almost periodic sequences and functions. The linear almost periodic equations, with regard to the almost periodicity of their solutions, are analyzed in, e.g., [1,30]. For general difference systems, criteria of the existence of almost periodic solutions are presented in [31,32]. Concerning linear almost periodic difference systems and their almost periodic solutions, we can refer to [5,6,30] (and also [11,14]). We refer to papers [2,15,18] for other properties of (complex) almost periodic systems. The findings about the skew-Hermitian and skew-symmetric differential systems, which correspond to ones from [22,23], can be found in [25] and [27], respectively. For almost periodic solutions of these systems, we can refer to [16,17,21] as well. Further, if one considers limit periodic homogeneous linear difference systems with respect to their almost periodic solutions, then the properties of such systems can be found in [7,13,28].

This paper is organized as follows. In the next section, we introduce the notation that is used in the whole paper, and we recall some elementary properties of infinite fields with absolute values. In Section 3, we recall the definitions of limit periodicity, almost periodicity, and asymptotic almost periodicity. To define these notions, we recall the Bohr and also the Bochner concept. In the final section, we give the basic motivation explicitly and we formulate and prove the main theorem.

2 Preliminaries

Let F be an infinite field. Let $|\cdot| : \mathrm{F} \to \mathbb{R}$ be an absolute value on F. Then, the properties

(i) $|f| \geq 0$ and $|f| = 0 \Leftrightarrow f = 0$,

(ii) $|f+g| \leq |f| + |g|$,

(iii) $|f \cdot g| = |f| \cdot |g|$

hold for every $f, g \in \mathrm{F}$, where symbol 0 stands for the real number and, at the same time, for the zero element of F. Note that we will later denote also the zero vector and the zero matrix by the same symbol. Let $m \in \mathbb{N}$ be arbitrarily given. We denote the set of all square matrices of dimension m with elements in F by symbol $\mathrm{Mat}_m(\mathrm{F})$ and the set of all $m \times 1$ vectors with elements in F by symbol F^m. Using the absolute value, we can define the norms $\|\cdot\|$ on F^m, $\mathrm{Mat}_m(\mathrm{F})$ as the sums of the absolute values of the elements. We have

(i) $\|A\| \geq 0$ and $\|A\| = 0 \Leftrightarrow A = 0$,

(ii) $\|A + B\| \leq \|A\| + \|B\|$,

(iii) $\|f \cdot A\| = |f| \cdot \|A\|$

for all $f \in \mathrm{F}$ and $A, B \in \mathrm{Mat}_m(\mathrm{F})$ or $A, B \in \mathrm{F}^m$. We denote the identity matrix in $\mathrm{Mat}_m(\mathrm{F})$ as I. The absolute value on F and the norms on F^m, $\mathrm{Mat}_m(\mathrm{F})$ induce the corresponding metrics. For simplicity, each of these metrics will be denoted by symbol $\varrho(\cdot,\cdot)$. Then, we consider the δ-neighbourhoods in these metrics as

$$\mathcal{O}_\delta(A) = \{B \mid \varrho(B, A) < \delta\},$$

where $A, B \in \mathrm{F}$, F^m or $\mathrm{Mat}_m(\mathrm{F})$.

Let $X \subset \mathrm{Mat}_m(\mathrm{F})$ be a bounded group. In particular, for every matrix $A \in X$, there exists the inverse matrix $A^{-1} \in X$ and a number $H > 0$ satisfying $\|A\| \le H$ for every $A \in X$. Let us denote the set of all limit periodic and almost periodic sequences in X by symbol $\mathcal{LP}(X)$ and $\mathcal{AP}(X)$, respectively. For the notion of limit and almost periodicity, see Definitions 3.1 and 3.2 below. In $\mathcal{AP}(X)$, we consider the metric

$$\varrho\left(\{A_k\}, \{B_k\}\right) = \sup_k \|A_k - B_k\|.$$

For the reader's convenience, the δ-neighbourhoods in this set are again denoted by $\mathcal{O}_\delta$. We put $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

3 Limit, almost, and asymptotic almost periodicity

We recall the definitions of limit periodic, almost periodic, and asymptotically almost periodic sequences and we mention their properties, which we will need in the proof of the main theorem. The general metric space (M, ϱ) is considered. First, we recall the definition of limit periodicity. Note that it can be defined in another equivalent manner (see [3]).

Definition 3.1. We say that a sequence $\{\varphi_k\}_{k\in\mathbb{N}_0}$ is limit periodic if there exists a sequence of periodic sequences $\{\varphi_k^n\}_{k\in\mathbb{N}_0} \subseteq M$, $n \in \mathbb{N}$, such that $\lim_{n\to\infty} \varphi_k^n = \varphi_k$ and the convergence is uniform with respect to $k \in \mathbb{N}_0$.

Next, we recall the concept of almost periodicity. It can be also defined in several equivalent ways. As a definition, we remind the so-called Bohr concept of almost periodicity. We also recall the so-called Bochner concept in the theorem below.

Definition 3.2. A sequence $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq M$ is called almost periodic if, for any $\varepsilon > 0$, there exists $r(\varepsilon) \in \mathbb{N}$ such that any set consisting of $r(\varepsilon)$ consecutive integers contains at least one number $l \in \mathbb{Z}$ satisfying

$$\varrho\left(\varphi_{k+l}, \varphi_k\right) < \varepsilon, \quad k \in \mathbb{Z}.$$

Theorem 3.3. *Let $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq M$ be given. The sequence $\{\varphi_k\}_{k\in\mathbb{Z}}$ is almost periodic if and only if any sequence $\{l_n\}_{n\in\mathbb{N}_0} \subseteq \mathbb{Z}$ has a subsequence $\{\bar{l}_n\}_{n\in\mathbb{N}_0} \subseteq \{l_n\}_{n\in\mathbb{N}_0}$ such that, for any $\varepsilon > 0$, there exists $K(\varepsilon) \in \mathbb{N}$ satisfying*

$$\varrho\left(\varphi_{k+\bar{l}_i}, \varphi_{k+\bar{l}_j}\right) < \varepsilon, \quad i, j > K(\varepsilon), \quad k \in \mathbb{Z}. \tag{3.1}$$

Proof. See, e.g., [24]. □

To complete this section, we also recollect the definition of asymptotic almost periodicity.

Definition 3.4. A sequence $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq M$ is called asymptotically almost periodic if, for any $\varepsilon > 0$, there exists $r(\varepsilon) \in \mathbb{N}$ and $m(\varepsilon) \in \mathbb{N}$ such that any set consisting of $r(\varepsilon)$ consecutive positive integers contains at least one number $l \in \mathbb{N}$ satisfying

$$\varrho\left(\varphi_{k+l}, \varphi_k\right) < \varepsilon, \quad k > m(\varepsilon), \quad k \in \mathbb{N}.$$

Note that, in Banach spaces, any asymptotically almost periodic sequence is the sum of an almost periodic sequence and a sequence, which vanishes at infinity. Similarly, as in the case of almost periodicity, we remind the equivalent concept of asymptotic almost periodicity.

Theorem 3.5. *Let $\{\varphi_k\}_{k\in\mathbb{N}_0} \subseteq M$ be given. The sequence $\{\varphi_k\}_{k\in\mathbb{N}_0}$ is asymptotically almost periodic if and only if any sequence $\{l_n\}_{n\in\mathbb{N}_0} \subseteq \mathbb{Z}$, $\lim_{n\to\infty} l_n = \infty$ has a subsequence $\{\bar{l}_n\}_{n\in\mathbb{N}_0} \subseteq \{l_n\}_{n\in\mathbb{N}_0}$ such that, for any $\varepsilon > 0$, there exists $K(\varepsilon) \in \mathbb{N}$ satisfying*

$$\varrho\left(\varphi_{k+\bar{l}_i}, \varphi_{k+\bar{l}_j}\right) < \varepsilon, \quad i, j > K(\varepsilon), \quad k \in \mathbb{N}_0. \tag{3.2}$$

Proof. See [9]. □

4 Results

In the beginning of this section, we call up the most relevant known results. By doing this, one can see, how our result complements our motivations.

Theorem 4.1. *Let $\mathcal{X} \subseteq \mathrm{Mat}_m(\mathrm{F})$ be a commutative group. Let, for every non-zero vector $u \in \mathrm{F}^m$, there exist $\xi > 0$ such that, for every $\delta > 0$, there exist matrices $M_1, \ldots, M_l \in \mathcal{X}$ satisfying*

$$M_i \in \mathcal{O}_\delta(I), \quad i \in \{1, \ldots, l\}, \quad \|M_l \cdots M_1 \cdot u - u\| > \xi. \tag{4.1}$$

Let $\varepsilon > 0$ and a non-zero vector $u \in \mathrm{F}^m$ be arbitrary. For any $\{A_k\}_{k\in\mathbb{N}_0} \in \mathcal{LP}(\mathcal{X})$, there exists $\{S_k\}_{k\in\mathbb{N}_0} \in \mathcal{O}_\varepsilon(\{A_k\}_{k\in\mathbb{N}_0}) \cap \mathcal{LP}(\mathcal{X})$ such that the solution of

$$x_{k+1} = S_k \cdot x_k, \quad k \in \mathbb{N}_0, \quad x_0 = u$$

is not almost periodic.

Proof. See [13]. □

Theorem 4.2. *Let $\mathcal{X} \subseteq \mathrm{Mat}_m(\mathrm{F})$ be a commutative group. Let there exist $\xi > 0$ such that, for every $\delta > 0$, there exists $l \in \mathbb{N}$ such that, for every $u \in \mathrm{F}^m$ fulfilling $\|u\| \geq 1$, there exist matrices $M_1, \ldots, M_l \in \mathcal{X}$ with the property that (4.1) is valid. Let $\varepsilon > 0$ be arbitrary. Then, for every $\{A_k\}_{k\in\mathbb{N}_0} \in \mathcal{LP}(\mathcal{X})$ and every sequence $\{u_n\}_{n\in\mathbb{N}}$ of non-zero vectors $u_n \in \mathrm{F}^m$, there exists $\{S_k\}_{k\in\mathbb{N}_0} \in \mathcal{O}_\varepsilon(\{A_k\}_{k\in\mathbb{N}_0}) \cap \mathcal{LP}(\mathcal{X})$ such that the solution of*

$$x_{k+1} = S_k \cdot x_k, \quad k \in \mathbb{N}_0, \quad x_0 = u_n$$

is not almost periodic for any $n \in \mathbb{N}$.

Proof. See [7]. □

Theorem 4.3. *Let (F, ϱ) be separable. Let $\mathcal{X} \subseteq \mathrm{Mat}_m(\mathrm{F})$ be a bounded group. Let there exist $\xi > 0$ such that, for every $\delta > 0$, there exists $l \in \mathbb{N}$ such that, for every $u \in \mathrm{F}^m$ fulfilling $\|u\| \geq 1$, there exist matrices $M_1, \ldots, M_l \in \mathcal{X}$ with the property that*

$$M_1 \in \mathcal{O}_\delta(I), \quad M_{i+1} \in \mathcal{O}_\delta(M_i), \quad i \in \{1, \ldots, l-1\}, \quad \|M_l \cdot u - u\| > \xi. \tag{4.2}$$

Let $\varepsilon > 0$ be arbitrary. Then, for every $\{A_k\}_{k\in\mathbb{N}_0} \in \mathcal{LP}(\mathcal{X})$, there exists $\{S_k\}_{k\in\mathbb{N}_0} \in \mathcal{O}_\varepsilon(\{A_k\}_{k\in\mathbb{N}_0}) \cap \mathcal{LP}(\mathcal{X})$ such that the system

$$x_{k+1} = S_k \cdot x_k, \quad k \in \mathbb{N}_0$$

does not have any non-zero asymptotically almost periodic solution.

Proof. See [28]. □

Theorem 4.4. *Let (F, ϱ) be separable. Let $\mathcal{X} \subseteq \mathrm{Mat}_m(\mathrm{F})$ be a bounded group. Let there exist $\xi > 0$ such that, for every $\delta > 0$, there exists $l \in \mathbb{N}$ such that, for every $u \in \mathrm{F}^m$ fulfilling $\|u\| \geq 1$, there exist matrices $M_1, \ldots, M_l \in \mathcal{X}$ with the property that (4.2) is valid. Let $\varepsilon > 0$ be arbitrary. Then, for every $\{A_k\}_{k\in\mathbb{Z}} \in \mathcal{AP}(\mathcal{X})$, there exists $\{S_k\}_{k\in\mathbb{Z}} \in \mathcal{O}_\varepsilon(\{A_k\}_{k\in\mathbb{Z}})$ such that the system*

$$x_{k+1} = S_k \cdot x_k, \quad k \in \mathbb{N}_0$$

does not have any non-zero asymptotically almost periodic solution.

Proof. See [28]. □

For the reader's convenience (see Theorem 4.6 below), we recall the definitions of transformable and weakly transformable groups (for further informations, see, e.g., [12, 26]).

Definition 4.5. We say that an infinite set $\mathcal{X} \subseteq \mathrm{Mat}_m(\mathrm{F})$ is transformable, if it meets the following conditions:

(i) for all $A, B \in \mathcal{X}$, it holds
$$A \cdot B \in \mathcal{X}, \quad A^{-1} \in \mathcal{X};$$

(ii) for any $L \in (0, \infty)$ and $\varepsilon > 0$, there exists $p = p(L, \varepsilon) \in \mathbb{N}$ such that, for any $n \geq p$ ($n \in \mathbb{N}$) and any sequence $\{C_0, C_1, \ldots, C_n\} \subset \mathcal{X}$, $L \leq \varrho(C_i, 0)$, $i \in \{0, \ldots, n\}$, one can find a sequence $\{D_1, \ldots, D_n\} \subset \mathcal{X}$ for which
$$D_i \in \mathcal{O}_\varepsilon(C_i), \quad i \in \{1, \ldots, n\}, \quad D_n \cdots D_1 = C_0;$$

(iii) the multiplication of matrices is uniformly continuous on $\mathcal{X}$ and has the Lipschitz property on a neighbourhood of I in $\mathcal{X}$;

(iv) for any $L \in (0, \infty)$, there exists $Q = Q(L) \in (0, \infty)$ such that, for every $\varepsilon > 0$ and $C, D \in \mathcal{X} \setminus \mathcal{O}_L(0)$ satisfying $C \in \mathcal{O}_\varepsilon(D)$, it is valid that
$$C^{-1} \cdot D, D \cdot C^{-1} \in \mathcal{O}_{\varepsilon \cdot Q}(I).$$

The group $\mathcal{X}$ is weakly transformable if there exist a transformable group $\mathcal{X}_0 \subset \mathcal{X}$, matrices $X_1, \ldots, X_l \in \mathcal{X}$, and $\delta_\mathcal{X} > 0$ such that the following conditions hold:

(i) any $U \in \mathcal{X}$ can be expressed as $U = C(U) \cdot X_j$ for some $C(U) \in \mathcal{X}_0$, $j \in \{1, \ldots, l\}$;

(ii) $\varrho(C \cdot X_i, D \cdot X_j) > \delta_\mathcal{X}$ for all $C, D \in \mathcal{X}_0$, $i \neq j$, $i, j \in \{1, \ldots, l\}$.

Theorem 4.6. *Let (F, ϱ) be complete. Let $\mathcal{X} \subseteq \mathrm{Mat}_m(\mathrm{F})$ be weakly transformable. Let there exist a sequence $\{M_i\}_{n\in\mathbb{N}} \subseteq \mathcal{X}_0$ such that, for any non-zero vector $u \in \mathrm{F}^m$, one can find $i = i(u) \in \mathbb{N}$ satisfying $M_i \cdot u \neq u$. Let $\varepsilon > 0$ be arbitrary. If $\{A_k\}_{k\in\mathbb{Z}} \in \mathcal{AP}(\mathcal{X})$, then there exists $\{S_k\}_{k\in\mathbb{Z}} \in \mathcal{O}_\varepsilon(\{A_k\}_{k\in\mathbb{Z}})$ such that the system*

$$x_{k+1} = S_k \cdot x_k, \quad k \in \mathbb{Z},$$

does not possess a non-trivial almost periodic solution.

Proof. See [12]. □

Before we formulate the main result of this paper, we recall some elementary properties of the bounded group X. We use them in the proof of the main theorem.

Lemma 4.7. *Let $V_k \in X$ and $M_k \in X$, $k \in \{0, \dots, K\}$, be given matrices. Then, there exist matrices $T_k \in X$, $k \in \{0, \dots, K\}$, such that:*

(i) $M_K \cdots M_0 \cdot V_K \cdots V_0 = V_K \cdot T_K \cdots V_0 \cdot T_0$;

(ii) $M_K \cdots M_0 \cdot V_K \cdots V_0 = T_K \cdot V_K \cdots T_0 \cdot V_0$

hold. Moreover, one can assume that $T_k \in \mathcal{O}_{H^2\delta}(I)$ if $M_k \in \mathcal{O}_\delta(I)$, and $T_k = I$ if $M_k = I$.

Proof. It is seen, that the matrices

$$\begin{aligned} T_K &= V_K^{-1} \cdot M_K \cdot V_K, \\ T_{K-1} &= (V_K \cdot V_{K-1})^{-1} \cdot M_{K-1} \cdot V_K \cdot V_{K-1}, \\ &\vdots \\ T_0 &= (V_K \cdots V_0)^{-1} \cdot M_0 \cdot V_K \cdots V_0 \end{aligned}$$

satisfy the equality in the part (i). Analogously, the matrices

$$\begin{aligned} T_K &= M_K, \\ T_{K-1} &= (V_K)^{-1} \cdot M_{K-1} \cdot V_K, \\ &\vdots \\ T_0 &= (V_K \cdots V_1)^{-1} \cdot M_0 \cdot V_K \cdots V_1 \end{aligned}$$

satisfy the equality in (ii). It holds

$$\|V^{-1} \cdot M_k \cdot V - I\| \le \|V^{-1}\| \cdot \|M_k - I\| \cdot \|V\| \le H^2 \cdot \|M_k - I\|$$

for every $V \in X$, $k \in \{0, \dots, K\}$, which completes the proof. □

Remark 4.8. Let $A, B, C \in X$. If $\|A - I\| > \xi$ and $\|C - B\| < \xi/(2H)$, then $\|A \cdot B - C\| > \xi/(2H)$ holds. It can be directly verified by the simple computation

$$\begin{aligned} \xi < \|A - I\| = \|A \cdot BB^{-1} - BB^{-1}\| &\le \|A \cdot B - B + C - C\| \cdot \|B^{-1}\| \\ &\le (\|A \cdot B - C\| + \|B - C\|) \cdot H. \end{aligned}$$

Now, we can prove the announced result. We recall, that X is a bounded group.

Theorem 4.9. *Let X have the property that there exists $\xi > 0$ such that, for every $\delta > 0$, there exist matrices $M_1, \dots, M_l \in X$ with the property that*

$$M_i \in \mathcal{O}_\delta(I), \quad i \in \{1, \dots, l\}, \quad \|M_l \cdots M_1 - I\| > \xi. \tag{4.3}$$

Then, for every $\{A_k\}_{k\in\mathbb{N}_0} \in \mathcal{LP}(X)$ and an arbitrary positive number ε, there exists $\{T_k\}_{k\in\mathbb{N}_0} \in \mathcal{O}_\varepsilon(\{A_k\}_{k\in\mathbb{N}_0}) \cap \mathcal{LP}(X)$ such that the fundamental matrix $\{X_k\}_{k\in\mathbb{N}_0}$ of

$$x_{k+1} = T_k \cdot x_k, \quad k \in \mathbb{N}_0, \tag{4.4}$$

is not asymptotically almost periodic.

Proof. Let $\varepsilon > 0$ be arbitrary. We denote $\zeta = \xi/(2H)$. We use the following construction. In the first step of the construction, for

$$\delta_1 = \frac{1}{2} \cdot \frac{\varepsilon}{H^3}, \tag{4.5}$$

there exist matrices $M_1^{(1)}, M_2^{(1)}, \dots, M_{l(\delta_1)}^{(1)} \in \mathcal{O}_{\delta_1}(I)$ (taken from (4.3)). Denote $r_1 = 2 \cdot l(\delta_1)$, $i(1,1) = 0$, $p(1,1) = r_1$. Let us consider the matrices $M_0^{(1,1)} = I, M_1^{(1,1)} = M_1^{(1)}, \dots, M_{p(1,1)-2}^{(1,1)} = I$, $M_{p(1,1)-1}^{(1,1)} = M_{l(\delta_1)}^{(1)}$. Then, there exist matrices $\tilde{T}_j^{(1,1)}$, $j \in \{0, \dots, p(1,1) - 1\}$, satisfying (see Lemma 4.7)

$$M_{p(1,1)-1}^{(1,1)} \cdots M_0^{(1,1)} \cdot A_{p(1,1)-1} \cdots A_0 = A_{p(1,1)-1} \cdot \tilde{T}_{p(1,1)-1}^{(1,1)} \cdots A_0 \cdot \tilde{T}_0^{(1,1)}$$

and $\tilde{T}_0^{(1,1)} = I, \tilde{T}_1^{(1,1)} \in \mathcal{O}_{H^2\delta_1}(I), \dots, \tilde{T}_{p(1,1)-2}^{(1,1)} = I, \tilde{T}_{p(1,1)-1}^{(1,1)} \in \mathcal{O}_{H^2\delta_1}(I)$. We define the periodic sequence $\{T_k^{(1,1)}\}_{k\in\mathbb{N}_0}$ with the period $p(1,1)$ in the following way. If $\|A_{i(1,1)}\| > 1$ and $\|A_{i(1,1)+r_1-1} \cdots A_0 - I\| < \zeta$, then we define $T_j^{(1,1)} = \tilde{T}_j^{(1,1)}$, $j \in \{0, \dots, p(1,1) - 1\}$. In the other cases, we define $T_0^{(1,1)} = \cdots = T_{p(1,1)-1}^{(1,1)} = I$. We denote $T_k^1 = T_k^{(1,1)}$ and $V_k^1 = A_k \cdot T_k^1$ for $k \in \mathbb{N}_0$.

In the second step, there exists a positive integer $i(2,1)$ divisible by 4 satisfying $i(2,1) > p(1,1)$. For

$$\delta_2 = \frac{1}{4} \cdot \frac{\varepsilon}{H^3},$$

there exist matrices (see (4.3))

$$M_1^{(2)}, M_2^{(2)}, \dots, M_{l(\delta_2)}^{(2)} \in \mathcal{O}_{\delta_2}(I).$$

Without loss of generality, we can assume that $l(\delta_2) \geq l(\delta_1)$. Denote $r_2 = 16 \cdot l(\delta_2) \cdot l(\delta_1)$, $p(2,1) = [i(2,1) + r_2] \cdot p(1,1)$. We consider the matrices

$$\begin{gathered}
M_0^{(2,1)} = \cdots = M_{i(2,1)-1}^{(2,1)} = I, \\
M_{i(2,1)}^{(2,1)} = I, \quad M_{i(2,1)+1}^{(2,1)} = I, \quad M_{i(2,1)+2}^{(2,1)} = M_1^{(2)}, \\
M_{i(2,1)+3}^{(2,1)} = I, \quad M_{i(2,1)+4}^{(2,1)} = I, \quad M_{i(2,1)+5}^{(2,1)} = I, \quad M_{i(2,1)+6}^{(2,1)} = M_2^{(2)}, \\
\vdots \\
M_{i(2,1)+4(l(\delta_2)-1)}^{(2,1)} = I, \quad M_{i(2,1)+1+4(l(\delta_2)-1)}^{(2,1)} = I, \quad M_{i(2,1)+2+4(l(\delta_2)-1)}^{(2,1)} = M_{l(\delta_2)}^{(2)}, \\
M_{i(2,1)+3+4(l(\delta_2)-1)}^{(2,1)} = \cdots = M_{p(2,1)-1}^{(2,1)} = I.
\end{gathered}$$

Then, there exist matrices $\tilde{T}_j^{(2,1)}$, $j \in \{0, \dots, p(2,1)-1\}$, satisfying (see Lemma 4.7)

$$M_{p(2,1)-1}^{(2,1)} \cdots M_0^{(2,1)} \cdot V_{p(2,1)-1}^1 \cdots V_0^1 = V_{p(2,1)-1}^1 \cdot \tilde{T}_{p(2,1)-1}^{(2,1)} \cdots V_0^1 \cdot \tilde{T}_0^{(2,1)}$$

and

$$\begin{gathered} \tilde{T}_0^{(2,1)} = \cdots = \tilde{T}_{i(2,1)-1}^{(2,1)} = I, \\ \tilde{T}_{i(2,1)}^{(2,1)} = I, \quad \tilde{T}_{i(2,1)+1}^{(2,1)} = I, \quad \tilde{T}_{i(2,1)+2}^{(2,1)} \in \mathcal{O}_{H^2\delta_2}(I), \\ \tilde{T}_{i(2,1)+3}^{(2,1)} = I, \quad \tilde{T}_{i(2,1)+4}^{(2,1)} = I, \quad \tilde{T}_{i(2,1)+5}^{(2,1)} = I, \quad \tilde{T}_{i(2,1)+6}^{(2,1)} \in \mathcal{O}_{H^2\delta_2}(I), \\ \vdots \\ \tilde{T}_{i(2,1)+4(l(\delta_2)-1)}^{(2,1)} = I, \quad \tilde{T}_{i(2,1)+1+4(l(\delta_2)-1)}^{(2,1)} = I, \quad \tilde{T}_{i(2,1)+2+4(l(\delta_2)-1)}^{(2,1)} \in \mathcal{O}_{H^2\delta_2}(I), \\ \tilde{T}_{i(2,1)+3+4(l(\delta_2)-1)}^{(2,1)} = \cdots = \tilde{T}_{p(2,1)-1}^{(2,1)} = I. \end{gathered}$$

We define the periodic sequence $\{T_k^{(2,1)}\}_{k\in\mathbb{N}_0}$ with the period $p(2,1)$ in the following way. If $\|V_{i(2,1)}^1\| > 1/4$ and $\|V_{i(2,1)+r_2-1}^1 \cdots V_0^1 - V_{i(2,1)-1}^1 \cdots V_0^1\| < \zeta$, then we put $T_j^{(2,1)} = \tilde{T}_j^{(2,1)}$, $j \in \{0, \dots, p(2,1)-1\}$. Otherwise, we define $T_0^{(2,1)} = \cdots = T_{p(2,1)-1}^{(2,1)} = I$. We put $V_k^{(2,1)} = V_k^1 \cdot T_k^{(2,1)}$, $k \in \mathbb{N}_0$.

There exists a positive integer $i(2,2)$ divisible by 8 satisfying $i(2,2) > p(2,1)$. We define the periodic sequence $\{T_k^{(2,2)}\}_{k\in\mathbb{N}_0}$ with the period $p(2,2) = [i(2,2) + r_2 - r_1] \cdot p(2,1)$ in the following way. Let us consider the matrices

$$M_0^{(2,2)} = \cdots = M_{i(2,2)-1}^{(2,2)} = I,$$

$$\begin{gathered} M_{i(2,2)}^{(2,2)} = \cdots = M_{i(2,2)+3}^{(2,2)} = I, \quad M_{i(2,2)+4}^{(2,2)} = M_1^{(2)}, \\ M_{i(2,1)+5}^{(2,1)} = \cdots = M_{i(2,2)+11}^{(2,2)} = I, \quad M_{i(2,2)+12}^{(2,2)} = M_2^{(2)}, \\ \vdots \\ M_{i(2,2)-3+8(l(\delta_2)-1)}^{(2,2)} = \cdots = M_{i(2,2)+3+8(l(\delta_2)-1)}^{(2,2)} = I, \quad M_{i(2,2)+4+8(l(\delta_2)-1)}^{(2,2)} = M_{l(\delta_2)}^{(2)}, \\ M_{i(2,2)+5+8(l(\delta_2)-1)}^{(2,2)} = \cdots = M_{p(2,2)-1}^{(2,2)} = I. \end{gathered}$$

We know that there exist matrices $\tilde{T}_j^{(2,2)}$, $j \in \{0, \dots, p(2,2)-1\}$, satisfying (see Lemma 4.7)

$$M_{p(2,2)-1}^{(2,2)} \cdots M_0^{(2,2)} \cdot V_{p(2,2)-1}^{(2,1)} \cdots V_0^{(2,1)} = V_{p(2,2)-1}^{(2,1)} \cdot \tilde{T}_{p(2,2)-1}^{(2,2)} \cdots V_0^{(2,1)} \cdot \tilde{T}_0^{(2,2)}$$

and

$$\tilde{T}_0^{(2,2)} = \cdots = \tilde{T}_{i(2,2)-1}^{(2,2)} = I,$$

$$\begin{gathered} \tilde{T}_{i(2,2)}^{(2,2)} = \cdots = \tilde{T}_{i(2,2)+3}^{(2,2)} = I, \quad \tilde{T}_{i(2,2)+4}^{(2,2)} \in \mathcal{O}_{H^2\delta_2}(I), \\ \tilde{T}_{i(2,1)+5}^{(2,2)} = \cdots = \tilde{T}_{i(2,2)+11}^{(2,2)} = I, \quad \tilde{T}_{i(2,2)+12}^{(2,2)} \in \mathcal{O}_{H^2\delta_2}(I), \\ \vdots \end{gathered}$$

$$\tilde{T}^{(2,2)}_{i(2,2)-3+8(l(\delta_2)-1)} = \cdots = \tilde{T}^{(2,2)}_{i(2,2)+3+8(l(\delta_2)-1)} = I, \quad \tilde{T}^{(2,2)}_{i(2,2)+4+8(l(\delta_2)-1)} \in \mathcal{O}_{H^2\delta_2}(I),$$
$$\tilde{T}^{(2,2)}_{i(2,2)+5+8(l(\delta_2)-1)} = \cdots = \tilde{T}^{(2,2)}_{p(2,2)-1} = I.$$

If $\|V^{(2,1)}_{i(2,2)}\| > 1/4$ and $\|V^{(2,1)}_{i(2,2)+r_2-r_1-1} \cdots V^{(2,1)}_0 - V^{(2,1)}_{i(2,2)-1} \cdots V^{(2,1)}_0\| < \zeta$, then we put $T^{(2,2)}_j = \tilde{T}^{(2,2)}_j$, $j \in \{0, \dots, p(2,2)-1\}$. We define $T^{(2,2)}_0 = \cdots = T^{(2,2)}_{p(2,2)-1} = I$ in the other cases. We put $T^2_k = T^{(2,1)}_k \cdot T^{(2,2)}_k$, $V^{(2,2)}_k = V^{(2,1)}_k \cdot T^{(2,2)}_k$, $V^2_k = V^{(2,2)}_k$, $k \in \mathbb{N}_0$.

We continue in the construction in the same way. Before the n-th step, we have $\{V^{n-1}_k\}_{k\in\mathbb{N}_0} \equiv \{A_k \cdot T^1_k \cdot T^2_k \cdots T^{n-1}_k\}_{k\in\mathbb{N}_0}$, where the sequence $\{T^1_k \cdot T^2_k \cdots T^{n-1}_k\}_{k\in\mathbb{N}_0}$ has the period
$$p(n-1,n-1) = [i(n-1,n-1) + r_{n-1} - r_{n-2}] \cdot p(n-1,n-2).$$

We denote

$$\alpha(x,y) = 2^{\frac{(x-1)x}{2}+y}, \quad x \in \mathbb{N}, \quad y \in \{1,2,\dots,x\}, \tag{4.6}$$
$$\delta_j = \frac{1}{2} \cdot \frac{1}{j} \cdot \frac{\varepsilon}{H^3}, \quad j \in \mathbb{N}, \tag{4.7}$$
$$\delta_{jj} = H^2 \cdot \delta_j, \quad j \in \mathbb{N}, \tag{4.8}$$
$$r_j = \prod_{s=1}^{j} \alpha(s,s) \cdot l(\delta_s), \quad j \in \mathbb{N}. \tag{4.9}$$

For the n-th step, there exists $i(n,1) \in \mathbb{N}$ divisible by $\alpha(n,1)$ such that $i(n,1) > p(n-1,n-1)$. Taken from (4.3), for δ_n, there exist matrices

$$M^{(n)}_1, M^{(n)}_2, \dots, M^{(n)}_{l(\delta_n)} \in \mathcal{O}_{\delta_n}(I), \tag{4.10}$$

where $l(\delta_n)$ can be taken in such a way that $l(\delta_n) \geq l(\delta_{n-1})$. We denote

$$p(n,1) = [i(n,1) + r_n] \cdot p(n-1,n-1).$$

We consider the matrices

$$M^{(n,1)}_0 = \cdots = M^{(n,1)}_{i(n,1)-1} = I,$$
$$M^{(n,1)}_{i(n,1)} = \cdots = M^{(n,1)}_{i(n,1)+\alpha(n,1)/2-1} = I,$$
$$M^{(n,1)}_{i(n,1)+\alpha(n,1)/2} = M^{(n)}_1,$$
$$M^{(n,1)}_{i(n,1)+\alpha(n,1)/2+1} = \cdots = M^{(n,1)}_{i(n,1)+\alpha(n,1)+\alpha(n,1)/2-1} = I,$$
$$M^{(n,1)}_{i(n,1)+\alpha(n,1)+\alpha(n,1)/2} = M^{(n)}_2,$$
$$\vdots$$
$$M^{(n,1)}_{i(n,1)-\alpha(n,1)/2+1+\alpha(n,1)(l(\delta_n)-1)} = \cdots = M^{(n,1)}_{i(n,1)+\alpha(n,1)/2-1+\alpha(n,1)(l(\delta_n)-1)} = I,$$
$$M^{(n,1)}_{i(n,1)+\alpha(n,1)/2+\alpha(n,1)(l(\delta_n)-1)} = M^{(n)}_{l(\delta_n)},$$
$$M^{(n,1)}_{i(n,1)+\alpha(n,1)/2+1+\alpha(n,1)(l(\delta_n)-1)} = \cdots = M^{(n,1)}_{p(n,1)-1} = I.$$

There exist matrices $\tilde{T}_j^{(n,1)}$, $j \in \{0, \dots, p(n,1)-1\}$, satisfying (see Lemma 4.7)

$$M_{p(n,1)-1}^{(n,1)} \cdots M_0^{(n,1)} \cdot V_{p(n,1)-1}^{n-1} \cdots V_0^{n-1} = V_{p(n,1)-1}^{n-1} \cdot \tilde{T}_{p(n,1)-1}^{(n,1)} \cdots V_0^{n-1} \cdot \tilde{T}_0^{(n,1)} \tag{4.11}$$

and

$$\begin{gathered}
\tilde{T}_0^{(n,1)} = \cdots = \tilde{T}_{i(n,1)-1}^{(n,1)} = I, \\
\tilde{T}_{i(n,1)}^{(n,1)} = \cdots = \tilde{T}_{i(n,1)+\alpha(n,1)/2-1}^{(n,1)} = I, \\
\tilde{T}_{i(n,1)+\alpha(n,1)/2}^{(n,1)} \in \mathcal{O}_{\delta_{nn}}(I), \\
\tilde{T}_{i(n,1)+\alpha(n,1)/2+1}^{(n,1)} = \cdots = \tilde{T}_{i(n,1)+\alpha(n,1)+\alpha(n,1)/2-1}^{(n,1)} = I, \\
\tilde{T}_{i(n,1)+\alpha(n,1)+\alpha(n,1)/2}^{(n,1)} \in \mathcal{O}_{\delta_{nn}}(I), \\
\vdots \\
\tilde{T}_{i(n,1)-\alpha(n,1)/2+1+\alpha(n,1)(l(\delta_n)-1)}^{(n,1)} = \cdots = \tilde{T}_{i(n,1)+\alpha(n,1)/2-1+\alpha(n,1)(l(\delta_n)-1)}^{(n,1)} = I, \\
\tilde{T}_{i(n,1)+\alpha(n,1)/2+\alpha(n,1)(l(\delta_n)-1)}^{(n,1)} \in \mathcal{O}_{\delta_{nn}}(I), \\
\tilde{T}_{i(n,1)+\alpha(n,1)/2+1+\alpha(n,1)(l(\delta_n)-1)}^{(n,1)} = \cdots = \tilde{T}_{p(n,1)-1}^{(n,1)} = I.
\end{gathered}$$

Next, we define the periodic sequence $\{T_k^{(n,1)}\}_{k\in\mathbb{N}_0}$ with the period $p(n,1)$. If $\|V_{i(n,1)}^{n-1}\| > 1/n^2$ and $\|V_{i(n,1)+r_n-1}^{n-1} \cdots V_0^{n-1} - V_{i(n,1)-1}^{n-1} \cdots V_0^{n-1}\| < \zeta$, then we put $T_j^{(n,1)} = \tilde{T}_j^{(n,1)}$, $j \in \{0, \dots, p(n,1)-1\}$. Otherwise, we define

$$T_0^{(n,1)} = \cdots = T_{p(n,1)-1}^{(n,1)} = I. \tag{4.12}$$

We put $V_k^{(n,1)} = V_k^{n-1} \cdot T_k^{(n,1)}$, $k \in \mathbb{N}_0$.

We continue in the n-th step in the same way. There exists $i(n,n) \in \mathbb{N}$ divisible by $\alpha(n,n)$ such that $i(n,n) > p(n,n-1)$. Let us denote

$$p(n,n) = [i(n,n) + r_n - r_{n-1}] \cdot p(n,n-1).$$

We consider the matrices

$$\begin{gathered}
M_0^{(n,n)} = \cdots = M_{i(n,n)-1}^{(n,n)} = I, \\
M_{i(n,n)}^{(n,n)} = \cdots = M_{i(n,n)+\alpha(n,n)/2-1}^{(n,n)} = I, \\
M_{i(n,n)+\alpha(n,n)/2}^{(n,n)} = M_1^{(n)}, \\
M_{i(n,n)+\alpha(n,n)/2+1}^{(n,n)} = \cdots = M_{i(n,n)+\alpha(n,n)+\alpha(n,n)/2-1}^{(n,n)} = I, \\
M_{i(n,n)+\alpha(n,n)+\alpha(n,n)/2}^{(n,n)} = M_2^{(n)}, \\
\vdots \\
M_{i(n,n)-\alpha(n,n)/2+1+\alpha(n,n)(l(\delta_n)-1)}^{(n,n)} = \cdots = M_{i(n,n)+\alpha(n,n)/2-1+\alpha(n,n)(l(\delta_n)-1)}^{(n,n)} = I, \\
M_{i(n,n)+\alpha(n,n)/2+\alpha(n,n)(l(\delta_n)-1)}^{(n,n)} = M_{l(\delta_n)}^{(n)}, \\
M_{i(n,n)+\alpha(n,n)/2+1+\alpha(n,n)(l(\delta_n)-1)}^{(n,n)} = \cdots = M_{p(n,n)-1}^{(n,n)} = I.
\end{gathered}$$

According to Lemma 4.7, there exist matrices $\tilde{T}_j^{(n,n)}$, $j \in \{0, \dots, p(n,n) - 1\}$, satisfying

$$M_{p(n,n)-1}^{(n,n)} \cdots M_0^{(n,n)} \cdot V_{p(n,n)-1}^{(n,n-1)} \cdots V_0^{(n,n-1)} = V_{p(n,n)-1}^{(n,n-1)} \cdot \tilde{T}_{p(n,n)-1}^{(n,n)} \cdots V_0^{(n,n-1)} \cdot \tilde{T}_0^{(n,n)} \tag{4.13}$$

and

$$\tilde{T}_0^{(n,n)} = \cdots = \tilde{T}_{i(n,n)-1}^{(n,n)} = I,$$

$$\tilde{T}_{i(n,n)}^{(n,n)} = \cdots = \tilde{T}_{i(n,n)+\alpha(n,n)/2-1}^{(n,n)} = I,$$

$$\tilde{T}_{i(n,n)+\alpha(n,n)/2}^{(n,n)} \in \mathcal{O}_{\delta_{nn}}(I),$$

$$\tilde{T}_{i(n,n)+\alpha(n,n)/2+1}^{(n,n)} = \cdots = \tilde{T}_{i(n,n)+\alpha(n,n)+\alpha(n,n)/2-1}^{(n,n)} = I,$$

$$\tilde{T}_{i(n,n)+\alpha(n,n)+\alpha(n,n)/2}^{(n,n)} \in \mathcal{O}_{\delta_{nn}}(I),$$

$$\vdots$$

$$\tilde{T}_{i(n,n)-\alpha(n,n)/2+1+\alpha(n,n)(l(\delta_n)-1)}^{(n,n)} = \cdots = \tilde{T}_{i(n,n)+\alpha(n,n)/2-1+\alpha(n,n)(l(\delta_n)-1)}^{(n,n)} = I,$$

$$\tilde{T}_{i(n,n)+\alpha(n,n)/2+\alpha(n,n)(l(\delta_n)-1)}^{(n,n)} \in \mathcal{O}_{\delta_{nn}}(I),$$

$$\tilde{T}_{i(n,n)+\alpha(n,n)/2+1+\alpha(n,n)(l(\delta_n)-1)}^{(n,n)} = \cdots = \tilde{T}_{p(n,n)-1}^{(n,n)} = I.$$

We define the periodic sequence $\{T_k^{(n,n)}\}_{k\in\mathbb{N}_0}$ with the period $p(n,n)$ as follows. If $\|V_{i(n,n)}^{(n,n-1)}\| > 1/n^2$ and $\|V_{i(n,n)+r_n-r_{n-1}-1}^{(n,n-1)} \cdots V_0^{(n,n-1)} - V_{i(n,n)-1}^{(n,n-1)} \cdots V_0^{(n,n-1)}\| < \zeta$, then we put $T_j^{(n,n)} = \tilde{T}_j^{(n,n)}$ for $j \in \{0, \dots, p(n,n) - 1\}$. In the other cases, we define

$$T_0^{(n,n)} = \cdots = T_{p(n,n)-1}^{(n,n)} = I. \tag{4.14}$$

We put $T_k^n = T_k^{(n,1)} \cdots T_k^{(n,n)}$, $V_k^{(n,n)} = V_k^{(n,n-1)} \cdot T_k^{(n,n)}$, $V_k^n = V_k^{(n,n)}$, $k \in \mathbb{N}_0$. We continue in the same manner.

Let us define the sequence

$$T_k = A_k \cdot T_k^1 \cdots T_k^n \cdots, \quad k \in \mathbb{N}_0.$$

It follows directly from the construction that, for every $k \in \mathbb{N}_0$, there exists $q(k) \in \mathbb{N}$ such that

$$T_k^l = I, \quad l \neq q(k), \quad l \in \mathbb{N}. \tag{4.15}$$

In other words, $T_k = A_k \cdot T_k^{q(k)}$, $k \in \mathbb{N}_0$. Especially, $T_k \in X$ for all $k \in \mathbb{N}_0$. One can also see that

$$T_k^n \in \mathcal{O}_{\delta_{nn}}(I), \quad k \in \mathbb{N}_0, \quad n \in \mathbb{N}. \tag{4.16}$$

Moreover, $\{A_k\}_{k\in\mathbb{N}_0}$ is limit periodic. Thus, we know that there exist periodic sequences $\{B_k^n\}_{k\in\mathbb{N}_0} \subseteq X$ satisfying (see Definition 3.1)

$$\|A_k - B_k^n\| < \frac{1}{n}, \quad k \in \mathbb{N}_0, \quad n \in \mathbb{N}. \tag{4.17}$$

We denote the period of $\{T_k^n\}_{k\in\mathbb{N}_0}$ as p_n^T and the period of $\{B_k^n\}_{k\in\mathbb{N}_0}$ as p_n^B for any $n \in \mathbb{N}$. Then, the sequence $\{B_k^n \cdot T_k^1 \cdots T_k^n\}_{k\in\mathbb{N}_0}$ is periodic with the period $p_n^B \cdot p_1^T \cdots p_n^T$, $n \in \mathbb{N}$. It holds that

$$\begin{aligned}
&\left\| T_k - B_k^n \cdot T_k^1 \cdots T_k^n \right\| \\
&\quad \le \left\| T_k - B_k^n \cdot T_k^1 \cdots T_k^n \cdots \right\| + \left\| B_k^n \cdot T_k^1 \cdots T_k^n \cdots - B_k^n \cdot T_k^1 \cdots T_k^n \right\| \\
&\quad \le \| A_k - B_k^n \| \cdot \left\| T_k^1 \cdots T_k^n \cdots \right\| + \left\| B_k^n \cdot T_k^1 \cdots T_k^n \right\| \cdot \left\| T_k^{n+1} \cdots T_k^{n+j} \cdots - I \right\|,
\end{aligned} \tag{4.18}$$

where $j \in \mathbb{N}$, and

$$\left\| T_k^1 \cdots T_k^n \cdots \right\| \le H, \qquad \left\| B_k^n \cdot T_k^1 \cdots T_k^n \right\| \le H. \tag{4.19}$$

Now, from inequalities (4.17), (4.18), and (4.19), we get (see also (4.15), (4.16))

$$\| T_k - B_k^n \cdot T_k^1 \cdots T_k^n \| \le \frac{1}{n} \cdot H + H \cdot \delta_{(n+1)(n+1)}$$

for all $k \in \mathbb{N}_0$, $n \in \mathbb{N}$. From it follows (consider $\lim_{j\to\infty} \delta_{jj} = 0$ or see directly (4.7), (4.8)) that $\{T_k\}_{k\in\mathbb{N}_0}$ is the uniform limit of the sequence of periodic sequences $\{B_k^n \cdot T_k^1 \cdots T_k^n\}_{k\in\mathbb{N}_0}$, $n \in \mathbb{N}$. It means that $\{T_k\}_{k\in\mathbb{N}_0} \in \mathcal{LP}(X)$. Moreover (see (4.16) and also (4.8)), $T_k^n \in \mathcal{O}_{\delta_{11}}(I)$ for every $k \in \mathbb{N}_0$, $n \in \mathbb{N}$. From (4.15), we get (see (4.5), (4.8))

$$\| A_k - T_k \| \le \| A_k \| \cdot \left\| I - T_k^{q(k)} \right\| \le H \cdot \delta_{11} = \frac{\varepsilon}{2}$$

for all $k \in \mathbb{N}_0$. Hence,

$$\sup_{k\in\mathbb{N}_0} \| A_k - T_k \| < \varepsilon,$$

i.e., $\{T_k\}_{k\in\mathbb{N}_0} \in \mathcal{O}_\varepsilon(\{A_k\}_{k\in\mathbb{N}_0})$.

Since X is bounded, we know that $\inf_{k\in\mathbb{N}_0} \|X_k\| > 0$. We show that the fundamental matrix $\{X_k\}_{k\in\mathbb{N}_0}$, $X_0 = I$ of the system

$$x_{k+1} = T_k \cdot x_k, \quad k \in \mathbb{N}_0, \tag{4.20}$$

is not asymptotically almost periodic. By contradiction, we suppose that the fundamental matrix is asymptotically almost periodic. From $\inf_{k\in\mathbb{N}_0} \|X_k\| > 0$, it follows that, there exists $b \in \mathbb{N}$ satisfying $\|X_k\| > 1/b^2$, $k \in \mathbb{N}_0$.

Considering the construction in the steps $b, b+1, \dots$, we get (see (4.15))

$$\begin{aligned}
&\left\| X_{i(b,1)+r_b} - X_{i(b,1)} \right\| \\
&\quad = \left\| T_{i(b,1)+r_b-1} \cdots T_0 - T_{i(b,1)-1} \cdots T_0 \right\| \\
&\quad = \left\| V_{i(b,1)+r_b-1}^{b-1} \cdot T_{i(b,1)+r_b-1}^{(b,1)} \cdots V_0^{b-1} \cdot T_0^{(b,1)} - V_{i(b,1)-1}^{b-1} \cdot T_{i(b,1)-1}^{(b,1)} \cdots V_0^{b-1} \cdot T_0^{(b,1)} \right\| \\
&\quad = \left\| W_{i(b,1)+r_b-1}^{(b,1)} \cdots W_0^{(b,1)} \cdot V_{i(b,1)+r_b-1}^{b-1} \cdots V_0^{b-1} - W_{i(b,1)-1}^{(b,1)} \cdots W_0^{(b,1)} \cdot V_{i(b,1)}^{b-1} \cdots V_0^{b-1} \right\|,
\end{aligned}$$

where the matrices $W_j^{(b,1)}$, $j \in \{0, \dots, i(b,1) + r_b - 1\}$, satisfy (see Lemma 4.7) the identities

$$\begin{aligned}
V_{i(b,1)+r_b-1}^{b-1} \cdot T_{i(b,1)+r_b-1}^{(b,1)} \cdots V_0^{b-1} \cdot T_0^{(b,1)} &= W_{i(b,1)+r_b-1}^{(b,1)} \cdots W_0^{(b,1)} \cdot V_{i(b,1)+r_b-1}^{b-1} \cdots V_0^{b-1}, \\
V_{i(b,1)-1}^{b-1} \cdot T_{i(b,1)-1}^{(b,1)} \cdots V_0^{b-1} \cdot T_0^{(b,1)} &= W_{i(b,1)-1}^{(b,1)} \cdots W_0^{(b,1)} \cdot V_{i(b,1)-1}^{b-1} \cdots V_0^{b-1}.
\end{aligned}$$

If
$$\left\| V^{b-1}_{i(b,1)+r_b-1} \cdots V^{b-1}_0 - V^{b-1}_{i(b,1)-1} \cdots V^{b-1}_0 \right\| \geq \zeta,$$
then
$$W_j^{(b,1)} = T_j^{(b,1)} = I, \quad j \in \{0, \dots, i(b,1) + r_b - 1\},$$
and
$$\left\| X_{i(b,1)+r_b} - X_{i(b,1)} \right\| \geq \zeta > 0. \tag{4.21}$$
If
$$\left\| V^{b-1}_{i(b,1)+r_b-1} \cdots V^{b-1}_0 - V^{b-1}_{i(b,1)-1} \cdots V^{b-1}_0 \right\| < \zeta,$$
then $W^{(b,1)}_{i(b,1)-1} \cdots W^{(b,1)}_0 = I$ and (see (4.11) and Lemma 4.7)
$$W^{(b,1)}_{i(b,1)+r_b-1} \cdots W^{(b,1)}_0 = M^{(b)}_{l(\delta_b)} \cdots M^{(b)}_1.$$
Considering Remark 4.8, we have
$$\left\| X_{i(b,1)+r_b} - X_{i(b,1)} \right\| \geq \zeta > 0 \tag{4.22}$$
in this case as well. Thus, we have (see (4.21) and (4.22))
$$\left\| X_{i(b,1)+r_b} - X_{i(b,1)} \right\| \geq \zeta > 0 \tag{4.23}$$
in the both cases.

We continue in the same manner. It holds
$$\begin{aligned}
\left\| X_{i(b,b)+r_b-r_{b-1}} - X_{i(b,b)} \right\| &= \left\| T_{i(b,b)+r_b-r_{b-1}-1} \cdots T_0 - T_{i(b,b)-1} \cdots T_0 \right\| \\
&= \Big\| V^{(b,b-1)}_{i(b,b)+r_b-r_{b-1}-1} \cdot T^{(b,b)}_{i(b,b)+r_b-r_{b-1}-1} \cdots V^{(b,b-1)}_0 \cdot T^{(b,b)}_0 \\
&\qquad - V^{(b,b-1)}_{i(b,b)-1} \cdot T^{(b,b)}_{i(b,b)-1} \cdots V^{(b,b-1)}_0 \cdot T^{(b,b)}_0 \Big\| \\
&= \Big\| W^{(b,b)}_{i(b,b)+r_b-r_{b-1}-1} \cdots W^{(b,b)}_0 \cdot V^{(b,b-1)}_{i(b,b)+r_b-r_{b-1}-1} \cdots V^{(b,b-1)}_0 \\
&\qquad - W^{(b,b)}_{i(b,b)-1} \cdots W^{(b,b)}_0 \cdot V^{(b,b-1)}_{i(b,b)-1} \cdots V^{(b,b-1)}_0 \Big\| .
\end{aligned}$$

The matrices $W^{(b,b)}_j$, $j \in \{0, \dots, i(b,b) + r_b - r_{b-1} - 1\}$, are taken in a such way, that (see Lemma 4.7)
$$\begin{aligned}
&V^{(b,b-1)}_{i(b,b)+r_b-r_{b-1}-1} \cdot T^{(b,b)}_{i(b,b)+r_b-r_{b-1}-1} \cdots V^{(b,b-1)}_0 \cdot T^{(b,b)}_0 \\
&\qquad = W^{(b,b)}_{i(b,b)+r_b-r_{b-1}-1} \cdots W^{(b,b)}_0 \cdot V^{(b,b-1)}_{i(b,b)+r_b-r_{b-1}-1} \cdots V^{(b,b-1)}_0,
\end{aligned}$$
$$V^{(b,b-1)}_{i(b,b)-1} \cdot T^{(b,b)}_{i(b,b)-1} \cdots V^{(b,b-1)}_0 \cdot T^{(b,b)}_0 = W^{(b,b)}_{i(b,b)-1} \cdots W^{(b,b)}_0 \cdot V^{(b,b-1)}_{i(b,b)-1} \cdots V^{(b,b-1)}_0.$$
If
$$\left\| V^{(b,b-1)}_{i(b,b)+r_b-r_{b-1}-1} \cdots V^{(b,b-1)}_0 - V^{(b,b-1)}_{i(b,b)-1} \cdots V^{(b,b-1)}_0 \right\| \geq \zeta,$$
then
$$W_j^{(b,b)} = T_j^{(b,b)} = I, \quad j \in \{0, \dots, i(b,b) + r_b - r_{b-1} - 1\},$$

and

$$\left\| X_{i(b,b)+r_b-r_{b-1}} - X_{i(b,b)} \right\| \geq \zeta > 0. \tag{4.24}$$

If

$$\left\| V^{(b,b-1)}_{i(b,b)+r_b-r_{b-1}-1} \cdots V^{(b,b-1)}_0 - V^{(b,b-1)}_{i(b,b)-1} \cdots V^{(b,b-1)}_0 \right\| < \zeta, \tag{4.25}$$

then $W^{(b,b)}_{i(b,b)-1} \cdots W^{(b,b)}_0 = I$ and (see (4.13) and Lemma 4.7)

$$W^{(b,b)}_{i(b,b)+r_b-r_{b-1}-1} \cdots W^{(b,b)}_0 = M^{(b)}_{l(\delta_b)} \cdots M^{(b)}_1.$$

From Remark 4.8, we obtain

$$\left\| X_{i(b,b)+r_b-r_{b-1}} - X_{i(b,b)} \right\| \geq \zeta > 0, \tag{4.26}$$

if (4.25) is valid. Again, in the both cases, we have (see (4.24) and (4.26))

$$\left\| X_{i(b,b)+r_b-r_{b-1}} - X_{i(b,b)} \right\| \geq \zeta > 0. \tag{4.27}$$

We can continue in the same way, when we obtain

$$\left\| X_{i(b+n,1)+r_{b+n}} - X_{i(b+n,1)} \right\| \geq \zeta > 0, \tag{4.28}$$

$$\vdots$$

$$\left\| X_{i(b+n,b+n)+r_{b+n}-r_{b+n-1}} - X_{i(b+n,b+n)} \right\| \geq \zeta > 0 \tag{4.29}$$

for any $n \in \mathbb{N}$.

Now we use Theorem 3.5, where we put $l_0 = 0, \ldots, l_n = r_n, \ldots$ Considering the previous inequalities (see (4.23), (4.27), and (4.28)–(4.29)), it is seen that, for all large $i, j \in \mathbb{N}$, $i \neq j$, there exists $l \in \mathbb{N}$ such that

$$\|X_{l+l_i} - X_{l+l_j}\| \geq \zeta > 0,$$

which is a contradiction with (3.2). Hence, the fundamental matrix of (4.20) is not asymptotically almost periodic. □

The next theorem is the almost periodic version of Theorem 4.9. Note that, Theorem 4.10 is not a corollary of Theorem 4.9. It is known that, there exist almost periodic systems, which are not limit periodic. For example $\{A_k\} = \{e^{ik}\}$, $k \in \mathbb{Z}$, for the unitary group of dimension one $X = U(1)$. The system has a neighbourhood (in $\mathcal{AP}(X)$) without limit periodic systems.

Theorem 4.10. *Let X have the property that there exists $\xi > 0$ such that, for every $\delta > 0$, there exist matrices $M_1, \ldots, M_l \in X$ with the property that (4.3) is valid. Then, for every $\{A_k\}_{k\in\mathbb{Z}} \in \mathcal{AP}(X)$ and $\varepsilon > 0$, there exists $\{T_k\}_{k\in\mathbb{Z}} \in \mathcal{O}_\varepsilon(\{A_k\}_{k\in\mathbb{Z}})$ such that the fundamental matrix $\{X_k\}_{k\in\mathbb{N}_0}$ of*

$$x_{k+1} = T_k \cdot x_k, \quad k \in \mathbb{N}_0,$$

is not asymptotically almost periodic.

Proof. The proof of this theorem can be lead analogously as the proof of Theorem 4.9. In particular, the same construction can be used. □

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