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EXISTENCE OF SOLUTIONS FOR A NONLINEAR FRACTIONAL ORDER DIFFERENTIAL EQUATION

ERIC R. KAUFMANN AND KOUADIO DAVID YAO

Abstract. Let D^α denote the Riemann-Liouville fractional differential operator of order α. Let $1 < \alpha < 2$ and $0 < \beta < \alpha$. Define the operator L by $L = D^\alpha - aD^\beta$ where $a \in \mathbb{R}$. We give sufficient conditions for the existence of solutions of the nonlinear fractional boundary value problem

$$Lu(t) + f(t, u(t)) = 0, \quad 0 < t < 1,$$
$$u(0) = 0,\, u(1) = 0.$$

1. Introduction

For $u \in L^p[0, T], 1 \le p < \infty$, the Riemann-Liouville fractional integral of order $\alpha > 0$ is defined as

$$I^\alpha u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s)\, ds.$$

For $n-1 \le \alpha < n$, the Riemann-Liouville fractional derivative of order α is defined by

$$D^\alpha u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)\, ds.$$

Also, when $\alpha < 0$, we will sometimes use the notation $I^\alpha = D^{-\alpha}$. Define the operator L by $L = D^\alpha - aD^\beta$ where $a \in \mathbb{R}$. We give sufficient conditions for the existence of solutions of the nonlinear fractional boundary value problem

$$Lu(t) + f(t, u(t)) = 0, \quad 0 < t < 1, \tag{1}$$

$$u(0) = 0,\, u(1) = 0. \tag{2}$$

While much attention has focused on the Cauchy problem for fractional differential equations for both the Reimann-Liouville and Caputo differential operators, see [3, 6, 8, 9, 10, 11, 12, 13, 14] and references therein, there are few papers devoted to the study of fractional order boundary value problems, see for example [1, 2, 4, 5, 15].

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In the remainder of this section we present some fundamental results from fractional calculus that will be used later in the paper. For more information on fractional calculus we refer the reader to the manuscripts [9, 11, 12, 13]. In Section 2 we use the properties given below to find an equivalent integral operator to (1), (2). We also state the fixed point theorems that we employ to find solutions. In Section 3 we present our main results.

It is well known that if $n-1 \le \alpha < n$, then $D^\alpha t^{\alpha-k} = 0$, $k = 1, 2, \ldots, n$. Furthermore, if $u \in L^1[0,T]$ and $\alpha > 0$, then for $t \in [0,T]$, we have

$$D^\alpha I^\alpha u(t) = u(t). \tag{3}$$

The semi-group property,

$$I^\delta I^\alpha h(t) = I^{\delta+\alpha} h(t) = I^\alpha I^\delta h(t), \tag{4}$$

holds when $\alpha + \delta > 0$, $t \in [0,T]$.

If $D^\alpha u \in L^1[0,T]$, then

$$I^\alpha D^\alpha u(t) = u(t) - \sum_{m=1}^{n} \left[D^{\alpha-m} u(t) \right]_{t=0^+} \frac{t^{\alpha-m}}{\Gamma(\alpha-m+1)}, \tag{5}$$

for $t \in [0,T]$. Also, if $u \in C[0,T]$ and $n-1 \le \alpha < n$, then

$$\begin{aligned} \lim_{t\to 0^+} \left(D^{\alpha-n} u \right)(t) &= \lim_{t\to 0^+} \left(I^{n-\alpha} u \right)(t) \\ &= \lim_{t\to 0^+} \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-1-\alpha} u(s)\, ds \\ &= 0, \end{aligned} \tag{6}$$

and consequently if $u \in C^m[0,T]$, then $D^{\alpha-n+m} u(0) = 0$, $0 \le m \le n-1$.

2. Preliminary Results

Let $h \in C[0,1]$, $\alpha \in (1,2)$ and suppose that $\beta \in (0,\alpha)$. Let $u \in \{v : D^\alpha v \in C[0,1]\}$ be a solution of the problem

$$(D^\alpha - aD^\beta)u(t) + h(t) = 0, \quad 0 < t < 1, \tag{7}$$

$$u(0) = 0, \quad u(1) = 0. \tag{8}$$

Our first goal in this section is to invert the linearized equation (7), (8).

We begin by solving equation (7) for $D^\alpha u$ and applying the integral operator I^α to both sides.

$$I^\alpha D^\alpha u(t) = aI^\alpha D^\beta u(t) - I^\alpha h(t), \quad 0 < t < 1. \tag{9}$$

From (5) and (6) we have that $I^\alpha D^\alpha u(t) = u(t) + c_1 t^{\alpha-1}$ for some constant c_1. Furthermore, by (4) we see that $I^\alpha D^\beta u(t) = I^{\alpha-\beta} I^\beta D^\beta u(t)$. At this point we need to consider three cases. If $\beta < 1$, then $I^\alpha D^\beta u(t) = I^{\alpha-\beta} u(t)$ since $u \in C[0,1]$. If $\beta = 1$, then $I^\alpha D^\beta u(t) = I^{\alpha-\beta} u(t)$ since $u(0) = 0$. If $\beta > 1$, then $I^\alpha D^\beta u(t) = I^{\alpha-\beta} u(t) + c_2 t^{\alpha-1}$. In any case, equation (9) simplifies to

$$u(t) = aI^{\alpha-\beta} u(t) + ct^{\alpha-1} - I^\alpha h(t), \quad 0 < t < 1 \tag{10}$$

for some constant c.

Let $t = 1$ in (10) and apply the second boundary condition in (8) to get

$$0 = u(1) = aI^{\alpha-\beta} u(1) + c - I^\alpha h(1).$$

Thus, $c = I^\alpha h(1) - aI^{\alpha-\beta} u(1)$. Consequently, if $u \in \{v : D^\alpha v \in C[0,1]\}$ is a solution of (7), (8), then u satisfies the integral equation

$$u(t) = aI^{\alpha-\beta} u(t) - at^{\alpha-1} I^{\alpha-\beta} u(1) + t^{\alpha-1} I^\alpha h(1) - I^\alpha h(t), \quad 0 \le t \le 1. \tag{11}$$

Conversely, using (3), we see that if $u \in C[0,1]$ is a solution (11), then u satisfies the boundary value problem (7), (8). We thus have the following lemma.

Lemma 2.1. *Let $h \in C[0,1]$, $\alpha \in (1,2)$ and $\beta \in (0,\alpha)$. Then $u \in \{v : D^\alpha v \in C[0,1]\}$ is a solution of* (7), (8) *if and only if $u \in C[0,1]$ is a solution of the integral equation*

$$u(t) = \int_0^1 G(t,s)h(s)\,ds - a\int_0^1 G^*(t,s)u(s)\,ds, \quad 0 \le t \le 1,$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-1} - (t-s)^{\alpha-1}, & 0 \le s \le t, \\ t^{\alpha-1}(1-s)^{\alpha-1}, & t \le s \le 1, \end{cases}$$

and

$$G^*(t,s) = \frac{1}{\Gamma(\alpha-\beta)} \begin{cases} t^{\alpha-1}(1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta-1}, & 0 \le s \le t, \\ t^{\alpha-1}(1-s)^{\alpha-\beta-1}, & t \le s \le 1. \end{cases}$$

Remarks: While the function $G(t,s)$ satisfies $G(t,s) > 0$ for all $t, s \in (0,1)$, see [2], the function $G^*(t,s)$ is not of constant sign.

We seek a fixed point of an operator associated with (1), (2), using a Nonlinear Alternative of Leray-Schauder type and the Krasnosel'skiĭ-Zabeiko fixed point theorem [7]. For completeness we state these theorems below.

Theorem 2.2. *Let $\mathcal{B}$ be a normed linear space. Let $\mathcal{C} \subset \mathcal{B}$ be a convex set and let U be open in $\mathcal{C}$ such that $0 \in U$. Let $T : \overline{U} \to \mathcal{C}$ be a continuous and compact mapping. Then either*

(1) *the mapping T has a fixed point in $\overline{U}$, or*

(2) *there exists a $u \in \partial U$ and a $\lambda \in (0,1)$ such that $u = \lambda T u$.*

Theorem 2.3. *Let $\mathcal{B}$ be a Banach Space. Let $T : \mathcal{B} \to \mathcal{B}$ be a completely continuous mapping and let $L : \mathcal{B} \to \mathcal{B}$ be a bounded linear mapping such that $\lambda = 1$ is not an eigenvalue of L. Suppose that*

$$\lim_{\|u\| \to \infty} \frac{\|Tu - Lu\|}{\|u\|} = 0. \tag{12}$$

Then T has a fixed point in $\mathcal{B}$.

3. Main Results

Define the Banach space $\mathcal{B} = \big(C[0,1], \|\cdot\|\big)$, where $\|u\| = \max_{0 \le t \le 1} |u(t)|$, and the operator $T : \mathcal{B} \to \mathcal{B}$ by

$$Tu(t) = \int_0^1 G(t,s) f(s,u(s))\, ds - a \int_0^1 G^*(t,s) u(s)\, ds. \tag{13}$$

Note that fixed points of (13) are solutions of (1), (2) and vice versa.

Assume that the function f satisfies the following conditions.

(H_1) $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous and $f(t,0)$ does not vanish identically on any compact subset of $[0,1]$.

(H_2) There exists positive functions $a_1, a_2 \in C[0,1]$ such that

$$|f(t,z)| \le a_1(t) + a_2(t)|z|$$

for all $t \in [0,1]$.

Theorem 3.1. *Assume (H_1) and (H_2) hold. Then the operator $T : \mathcal{B} \to \mathcal{B}$ is continuous and compact.*

Proof. It follows trivially that $T : \mathcal{B} \to \mathcal{B}$.

Let $\{v_i\} \subset \mathcal{B}$ be such that $v_i \to v$ as $i \to \infty$. By (H_2) and the continuity of f we have,

$$\begin{aligned}
\lim_{i\to\infty} |Tv_i(t) - Tv(t)| &\le \lim_{i\to\infty} \int_0^1 G(t,s)\big|f(s,v_i(s)) - f(s,v(s))\big|\,ds \\
&\quad + |a| \lim_{i\to\infty} \int_0^1 \big|G^*(t,s)\big|\big|v_i(s)) - v(s)\big|\,ds \\
&\le \int_0^1 G(t,s) \lim_{i\to\infty} \big|f(s,v_i(s)) - f(s,v(s))\big|\,ds \\
&\quad + |a| \int_0^1 \big|G^*(t,s)\big| \lim_{i\to\infty} \big|v_i(s) - v(s)\big|\,ds \to 0.
\end{aligned}$$

Hence T is continuous.

Let $V = \{v\} \subset \mathcal{B}$ be a uniformly bounded subset and let $R > 0$ be such that $\|v\| \le R$ for all $v \in V$. Then for each $v \in V$ we have

$$\begin{aligned}
|Tv(t)| &\le \int_0^1 G(t,s)|f(s,v(s))|\,ds + |a| \int_0^1 |G^*(t,s)||v(s)|\,ds \\
&\le MK_1 + |a|K_2R,
\end{aligned}$$

where $M = \max_{(t,z)\in[0,1]\times[0,1]} |f(t,z)|, K_1 = \max_{t\in[0,1]} \int_0^1 G(t,s)\,ds,$ and $K_2 = \max_{t\in[0,1]} \int_0^1 |G^*(t,s)|\,ds$. Hence TV is uniformly bounded.

Let $v \in V$ and suppose that $t_1, t_2 \in [0,1]$ are such that $t_1 \le t_2$. Then,

$$\begin{aligned}
|Tv(t_1) - Tv(t_2)| &\le \int_0^1 |G(t_1,s) - G(t_2,s)||f(s,v(s))|\,ds \\
&\quad + |a| \int_0^1 |G^*(t_1,s) - G^*(t_2,s)||v(s)|\,ds \\
&\le (M\varepsilon_1 + |a|R\varepsilon_2)|t_1 - t_2|,
\end{aligned}$$

where $\varepsilon_1 > 0$ and $\varepsilon_2 > 0$ are such that $|G(t_1,s) - G(t_2,s)| \le \varepsilon_1|t_1 - t_2|$ and $|G^*(t_1,s) - G^*(t_2,s)| \le \varepsilon_2|t_1 - t_2|$ respectively. As such, the operative T is equicontinuous. By the Arzela-Ascoli Theorem, the operator T is compact and the proof is complete. □

Define $A, B \in \mathbb{R}$ by

$$\begin{aligned} A &\equiv \max_{t\in[0,1]} \left(\int_0^1 G(t,s)a_1(s)\,ds \right) \quad \text{and} \\ B &\equiv \max_{t\in[0,1]} \left(\int_0^1 G(t,s)a_2(s) + |a\,G^*(t,s)|\,ds \right). \end{aligned}$$

These quantities will be used in our first main theorem.

Theorem 3.2. *Assume that conditions* (H_1) *and* (H_2) *hold. Suppose that* $0 < A < \infty$ *and* $0 < B < 1$. *Then there exists a solution of the boundary value problem* (1), (2).

Proof. Let $U = \{u \in \mathcal{B} : \|u\| < R\}$ where $R = \frac{A}{1-B}$. Then,

$$\begin{aligned} |Tu(t)| &\leq \int_0^1 G(t,s)\,|f(s,u(s))|\,ds + |a| \int_0^1 |G^*(t,s)|\,|u(s)|\,ds \\ &\leq \int_0^1 G(t,s)a_1(s)\,ds + \left(\int_0^1 G(t,s)a_2(s) + |a\,G^*(t,s)|\,ds \right) \|u\| \\ &= A + B\|u\|. \end{aligned}$$

Suppose there exists a $u \in \partial U$ and a $\lambda \in (0,1)$ such that $u = \lambda Tu$, then for this u and λ we have

$$R = \|u\| = \lambda \|Tu\| < A + B\|u\| = R,$$

which is a contradiction.

By Theorem 2.2, there exists a fixed point $u \in \overline{U}$ of T. This fixed point is a solution of (1), (2) and the proof is complete. □

In our next theorem we replace condition (H_2) with the following condition.

(H_3) $\lim_{|u|\to\infty} \frac{f(t,u)}{|u|} = \varphi(t)$ uniformly in $[0,1]$, where $\varphi \in C[0,T]$.

We use Theorem 2.3 to establish a fixed point for the operator T.

Theorem 3.3. *Suppose that* (H_1) *and* (H_3) *hold and that*

$$0 < \int_0^1 G(t,s)\varphi(s) + |a\,G^*(t,s)|\,ds < 1. \tag{14}$$

Then there exists a solution of (1), (2).

Proof. We use the same operator defined in (13) and note that under condition (H_3) standard arguments can be used to show that T is compact.

Define an operator $L : \mathcal{B} \to \mathcal{B}$ by

$$Lu(t) = \int_0^1 G(t,s)\varphi(s)u(s)\,ds - a\int_0^1 G^*(t,s)u(s)\,ds.$$

Then L is a bounded linear mapping. Furthermore, if u is such that $u = Lu$ and $u \not\equiv 0$, then by (14)

$$\begin{aligned} \|u\| &= \|Lu(t)\| \\ &\le \left(\int_0^1 G(t,s)\varphi(s) + |a\,G^*(t,s)|\,ds\right)\|u\| \\ &< \|u\|, \end{aligned}$$

which is a contradiction. Consequently, $\lambda = 1$ is not an eigenvalue of L.

Fix $\varepsilon > 0$. By condition (H_3), there exists an $N > 0$ such that if $|z| > N$ then

$$\left|\frac{f(s,z)}{|z|} - \varphi(s)\right| < \varepsilon \tag{15}$$

for all $s \in [0,1]$. Set

$$B = \max\{f(s,z) : s \in [0,1],\ |z| \in [0,N]\}$$

and note that if $|u(s)| \le N$ then $\big|f(s,u(s)) - \varphi(s)|u(s)|\big| \le B + \|\varphi\|N$.

Pick $M > N$ such that $B + \|\varphi\|N < \varepsilon M$ and let $u \in \mathcal{B}$ be such that $\|u\| \ge M$. If $s \in [0,1]$ is such that $|u(s)| \le N$ then

$$\Big|f(s,u(s)) - \varphi(s)|u(s)|\Big| \le B + \|\varphi\|N \le \|u\|\varepsilon.$$

If $s \in [0,1]$ is such that $|u(s)| > N$ then by (15),

$$\Big|f(s,u(s)) - \varphi(s)|u(s)|\Big| \le \|u\|\varepsilon.$$

Hence for all $s \in [0,1]$,

$$\Big|f(s,u(s)) - \varphi(s)|u(s)|\Big| \le \|u\|\varepsilon.$$

For $\|u\| > M$ we have

$$\begin{aligned} |Tu(t) - Lu(t)| &\leq \int_0^1 G(t,s)\Big|f(s,u(s)) - \varphi(s)|u(s)|\Big|\,ds \\ &\leq \left(\max_{t\in[0,1]}\int_0^1 G(t,s)\,ds\right)\|u\|\varepsilon. \end{aligned}$$

That is, for some constant C, $\|Tu-Lu\| \leq C\|u\|\varepsilon$. Hence condition (12) of Theorem 2.3 is valid. Since all the conditions of Theorem 2.3 hold, there exists a fixed point u of T. This solution u is a solution of the boundary value problem (1), (2) and the proof is complete. □

Remark: With slight modifications, the results of Theorems 3.2 and 3.3 can be extended to the boundary value problem

$$\begin{aligned} &D^\alpha u(t) - \sum_{k=1}^m a_k D^{\beta_k} u(t) + f(t,u(t)) = 0,\ 0 < t < 1, \\ &u(0) = u(1) = 0, \end{aligned}$$

where $\alpha \in (1,2)$, $0 < \beta_1 < \beta_2 < \cdots < \beta_k < \alpha$, and $a_k \in \mathbb{R}, k = 0,1,\ldots,m$.

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Department of Mathematics & Statistics, University of Arkansas at Little Rock, Little Rock, AR 72204, USA

E-mail address: erkaufmann@ualr.edu

E-mail address: kdyao@ualr.edu